Math 225 · 21 November 2013 · Discussion Questions

Problem 1. Calculate Q^TQ. What does this product tell you about the norms and dot products of the columns of Q?

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$$

$$Q^TQ = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, we see that the columns form an orthonormal set. Their norms are given by the diagonal entries, which are all 1: the columns all have norm 1. The dot products between two different columns give us the non-diagonal entries, which are all zero: the columns form an orthogonal set.

Problem 2. Calculate Q^TQ. What does this product tell you about the norms and dot products of the columns of Q?

$$Q = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$$Q^TQ = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Since all the entries off the main diagonal are 0, the columns of Q are an orthogonal set. The norms of the columns are 2, 3, 6, 1, and 1, respectively.

Problem 3. An *orthogonal matrix* is a square matrix whose columns all have norm 1, and are pairwise orthogonal. Explain, using this definition, why $Q^TQ = I$ for any orthogonal matrix Q. The entry in row i and column j of Q^TQ is found by dotting the ith row of Q^T with the jth column of Q. This is the same as dotting the ith column of Q with the jth column of Q. If $i = j$, then we get one, so the diagonals of the product are all 1. If $i \neq j$, then the vectors are orthogonal, so all entries off the diagonal are 0.

Problem 4. Suppose the columns of Q are pairwise orthogonal, and the norm of the ith column is given by c_i. Find Q^TQ. The entry in the ith row and jth column of the product is given by the dot product of the ith row of Q^T and the jth column of Q, which is the same as saying the dot product of the ith and jth columns in Q. If $i \neq j$, then this dot product is zero, because the columns are orthogonal, so the product is 0 everywhere off the main diagonal. Along the main diagonal, in the entry in row and column i, it's c_i^2.

Exercise 5. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthogonal basis for $\mathbb{R}^k$. Simplify the expression

$$(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k)\mathbf{v}_i.$$

Note: this is where Theorem 5.2 comes from. $(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k)v_i = c_1\mathbf{v}_1\mathbf{v}_i + \cdots + c_k\mathbf{v}_k\mathbf{v}_i$ by the distributive property. Since $\mathbf{v}_i$ is orthogonal to every vector listed except itself, this simplifies to $c_i\mathbf{v}_i\mathbf{v}_i = c_i\|\mathbf{v}_i\|^2$.

Exercise 6. Note that

$$\mathcal{B} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is an **orthonormal** basis for $\mathbb{R}^3$. Represent the vector $\begin{bmatrix}5\\6\\7\end{bmatrix}$ as a linear combination of vectors in $\mathcal{B}$. If you don't remember the formula, you're in luck: you can figure it out!

We want to find constants c_1, c_2, and c_3 so that

$$\begin{bmatrix}5\\6\\7\end{bmatrix} = c_1 \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} + c_2 \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} + c_3 \begin{bmatrix}0\\1\\0\end{bmatrix}$$

Taking advantage of their orthogonality, we dot both sides by the first vector in the basis:

$$\begin{aligned}\begin{bmatrix}5\\6\\7\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} &= \left(c_1 \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} + c_2 \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} + c_3 \begin{bmatrix}0\\1\\0\end{bmatrix}\right) \cdot \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}\\
&= c_1 \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} + c_2 \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} + c_3 \begin{bmatrix}0\\1\\0\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}\\
&= c_1(1) + c_2(0) + c_3(0)\\
&= c_1\end{aligned}$$

So, $c_1 = \begin{bmatrix}5\\6\\7\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} = \dfrac{-2}{\sqrt{2}} = -\sqrt{2}$.

Similarly, $c_2 = \begin{bmatrix}5\\6\\7\end{bmatrix} \cdot \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} = 6\sqrt{2}$, and

Similarly, $c_3 = \begin{bmatrix}5\\6\\7\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} = 6$.

This yields our answer:

$$\begin{bmatrix}5\\6\\7\end{bmatrix} = -\sqrt{2} \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix} + 6\sqrt{2} \begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix} + 6 \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

Exercise 7. Note that

$$\mathcal{B} = \left\{ \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}3\\-12\\3\end{bmatrix} \right\}$$

is an **orthogonal** basis for $\mathbb{R}^3$. Represent the vector $\begin{bmatrix}5\\6\\7\end{bmatrix}$ as a linear combination of vectors in $\mathcal{B}$. If you don't remember the formula, you're in luck: you can figure it out!

We want to find constants c_1, c_2, and c_3 so that

$$\begin{bmatrix}5\\6\\7\end{bmatrix} = c_1 \begin{bmatrix}2\\1\\2\end{bmatrix} + c_2 \begin{bmatrix}-1\\0\\1\end{bmatrix} + c_3 \begin{bmatrix}3\\-12\\3\end{bmatrix}$$

Taking advantage of their orthogonality, we dot both sides by the first vector in the basis:

$$\begin{aligned}
\begin{bmatrix}5\\6\\7\end{bmatrix}\cdot\begin{bmatrix}2\\1\\2\end{bmatrix} &= \left(c_1\begin{bmatrix}2\\1\\2\end{bmatrix}+c_2\begin{bmatrix}-1\\0\\1\end{bmatrix}+c_3\begin{bmatrix}3\\-12\\3\end{bmatrix}\right)\cdot\begin{bmatrix}2\\1\\2\end{bmatrix}\\
&= c_1\begin{bmatrix}2\\1\\2\end{bmatrix}\cdot\begin{bmatrix}2\\1\\2\end{bmatrix}+c_2\begin{bmatrix}-1\\0\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\\2\end{bmatrix}+c_3\begin{bmatrix}3\\-12\\3\end{bmatrix}\cdot\begin{bmatrix}2\\1\\2\end{bmatrix}\\
&= c_1\left\|\begin{bmatrix}2\\1\\2\end{bmatrix}\right\|^2+c_2(0)+c_3(0)\\
&= c_1\left\|\begin{bmatrix}2\\1\\2\end{bmatrix}\right\|^2.
\end{aligned}$$

We see

$$c_1=\frac{\begin{bmatrix}5\\6\\7\end{bmatrix}\cdot\begin{bmatrix}2\\1\\2\end{bmatrix}}{\left\|\begin{bmatrix}2\\1\\2\end{bmatrix}\right\|^2}=\frac{10+6+14}{4+1+4}=\frac{10}{3}.$$

Similarly,

$$c_2=\frac{\begin{bmatrix}5\\6\\7\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\1\end{bmatrix}}{\left\|\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\|^2}=\frac{-5+0+7}{1+0+1}=1,$$

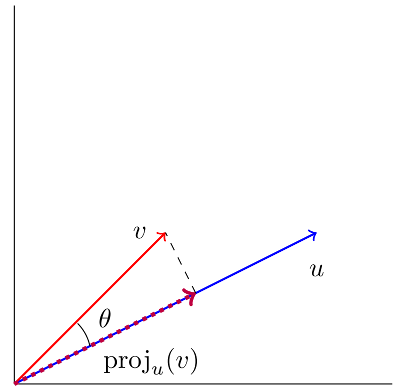
and

$$c_3=\frac{\begin{bmatrix}5\\6\\7\end{bmatrix}\cdot\begin{bmatrix}3\\-12\\3\end{bmatrix}}{\left\|\begin{bmatrix}3\\-12\\3\end{bmatrix}\right\|^2}=\frac{15-72+21}{9+144+9}=-\frac{2}{9}.$$

Thus, our sought-after representation is

$$\begin{bmatrix}5\\6\\7\end{bmatrix}=\frac{10}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}+\begin{bmatrix}-1\\0\\1\end{bmatrix}-\frac{2}{9}\begin{bmatrix}3\\-12\\3\end{bmatrix}$$

Problem 8.



The picture above represents the projection of v onto u. Fill in the blanks.

(a) $\cos(\theta) = \dfrac{\|\mathrm{proj}_u(v)\|}{\|v\|}$ (use the picture and SOHCAHTOA)

(b) $\cos(\theta) = \dfrac{(u)\cdot(v)}{\|u\|\|v\|}$ (use a formula for $\cos(\theta)$ from the first weeks of class)

(c) $\dfrac{\| \qquad \|}{\| \qquad \|} = \dfrac{(\quad)\cdot(\quad)}{\| \qquad \|\| \qquad \|}$ (from parts **??** and **??**)

(d) $\|\mathrm{proj}_u(v)\| = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}$ (from part **??**)

(e) Give the unit vector in the direction of u. $\dfrac{\mathbf{u}}{\|\mathbf{u}\|}$

(f) Using **??** and **??**, find the equation for the vector $\mathrm{proj}_u(v)$. Recall this vector is in the direction of u, and you found its magnitude. $\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\right)\mathbf{u}$

Exercise 9. If $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1/2 \end{bmatrix}$, find $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$. Using the formula:

$$\left(\frac{\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1/2 \end{bmatrix}}{\left\|\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right\|^2}\right)\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \left(\frac{4+0+3/2}{4+1+9}\right)\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11/14 \\ -11/28 \\ 33/28 \end{bmatrix}$$

Now, find the projection of $\mathbf{v}$ onto $2\mathbf{u} = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$. We can use the formula again:

$$\left(\frac{\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1/2 \end{bmatrix}}{\left\| \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \right\|^2} \right) \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = \left(\frac{8+0+3}{16+4+36} \right) \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 11/14 \\ -11/28 \\ 33/28 \end{bmatrix}$$

In retrospect, we can see that this is the same as before! In the projection, u is only there to be a platform onto which we project v; its magnitude doesn't matter, only its direction.

Problem 10. Without using any formulas, explain what you expect $\text{proj}_{\mathbf{u}}(\mathbf{v})$ to be if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. The 0 vector: the shadow is a point.

Problem 11. Without using any formulas, explain what you expect $\text{proj}_{\mathbf{u}}(\mathbf{v})$ to be if $\mathbf{u}$ and $\mathbf{v}$ are equal. $\mathbf{u}$: the shadow is the whole thing.

Problem 12. Without using any formulas, explain what you expect $\text{proj}_{\mathbf{u}}(\mathbf{v})$ to be if $\mathbf{u}$ is a scalar multiple of $\mathbf{v}$. $\mathbf{v}$: the shadow is still the whole thing.