Saturation Number of Ramsey-Minimal Families

Mike Ferrara [1] Jaehoon Kim [2] Elyse Yeager [2]

[1]University of Colorado-Denver

[2]University of Illinois at Urbana-Champaign

yeager2@illinois.edu

MIGHTY
University of Detroit Mercy

29 March 2014

Graph Saturation

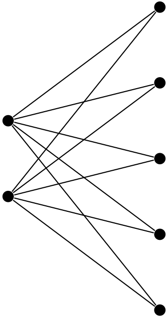
Definitions

Given a forbidden graph H, a graph G is H-**saturated** if H is not a subgraph of G, but for every $e \in \overline{G}$, H is a subgraph of $G + e$.

Graph Saturation

Definitions

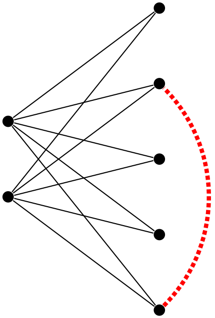
Given a forbidden graph H, a graph G is H-**saturated** if H is not a subgraph of G, but for every $e \in \overline{G}$, H is a subgraph of $G + e$.



Graph Saturation

Definitions

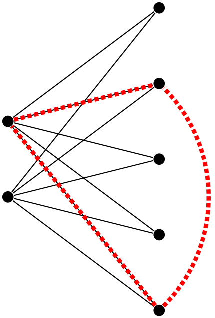
Given a forbidden graph H, a graph G is H-**saturated** if H is not a subgraph of G, but for every $e \in \overline{G}$, H is a subgraph of $G + e$.



Graph Saturation

Definitions

Given a forbidden graph H, a graph G is H-**saturated** if H is not a subgraph of G, but for every $e \in \overline{G}$, H is a subgraph of $G + e$.



Graph Saturation

Definitions

Given a forbidden graph H, a graph G is H-**saturated** if H is not a subgraph of G, but for every $e \in \overline{G}$, H is a subgraph of $G + e$.

Definitions

The **saturation number sat(n;H)** of a forbidden graph H is the smallest number of edges over all n-vertex graphs that are H-saturated.

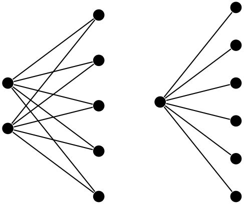
Graph Saturation

Definitions

Given a forbidden graph H, a graph G is H-**saturated** if H is not a subgraph of G, but for every $e \in \overline{G}$, H is a subgraph of $G + e$.

Definitions

The **saturation number sat(**n**;**H**)** of a forbidden graph H is the smallest number of edges over all n-vertex graphs that are H-saturated.



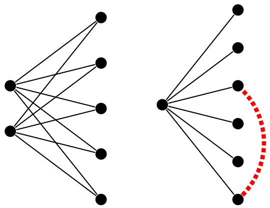
Graph Saturation

Definitions

Given a forbidden graph H, a graph G is H-**saturated** if H is not a subgraph of G, but for every $e \in \overline{G}$, H is a subgraph of $G + e$.

Definitions

The **saturation number sat**$(n;H)$ of a forbidden graph H is the smallest number of edges over all n-vertex graphs that are H-saturated.



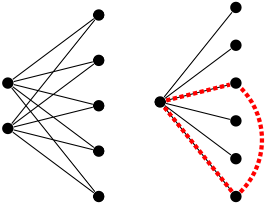
Graph Saturation

Definitions

Given a forbidden graph H, a graph G is H-**saturated** if H is not a subgraph of G, but for every $e \in \overline{G}$, H is a subgraph of $G + e$.

Definitions

The **saturation number sat(n;H)** of a forbidden graph H is the smallest number of edges over all n-vertex graphs that are H-saturated.



Graph Saturation

Definitions

Given a forbidden graph H, a graph G is H-**saturated** if H is not a subgraph of G, but for every $e \in \overline{G}$, H is a subgraph of $G + e$.

Definitions

The **saturation number sat(n;H)** of a forbidden graph H is the smallest number of edges over all n-vertex graphs that are H-saturated.

Definitions

Given a forbidden family of graphs $\mathcal{F}$, a graph G is $\mathcal{F}$-**saturated** if no member of $\mathcal{F}$ is a subgraph of G, but for every $e \in \overline{G}$, some member of $\mathcal{F}$ is a subgraph of $G + e$.
The **saturation number sat(n;$\mathcal{F}$)** is the smallest number of edges over all n-vertex graphs that are $\mathcal{F}$-saturated.

Ramsey-Minimal Families

Definitions

Given "forbidden" graphs $H_1, \ldots, H_k$, and any graph G, we write $\mathbf{G} \rightarrow (\mathbf{H_1}, \ldots, \mathbf{H_k})$ if any k coloring of $E(G)$ contains a monochromatic copy of H_i in color i, for some i.

Ramsey-Minimal Families

Definitions

Given "forbidden" graphs $H_1, \ldots, H_k$, and any graph G, we write $\mathbf{G} \to (\mathbf{H_1}, \ldots, \mathbf{H_k})$ if any k coloring of $E(G)$ contains a monochromatic copy of H_i in color i, for some i.

Famous Example: $K_6 \to (K_3, K_3)$, but $K_5 \not\to (K_3, K_3)$

Ramsey-Minimal Families

Definitions

Given "forbidden" graphs $H_1, \ldots, H_k$, and any graph G, we write $\mathbf{G \to (H_1, \ldots, H_k)}$ if any k coloring of $E(G)$ contains a monochromatic copy of H_i in color i, for some i.

Famous Example: $K_6 \to (K_3, K_3)$, but $K_5 \not\to (K_3, K_3)$

Definitions

A graph G is $(H_1, \ldots, H_k)$**-Ramsey minimal** if $G \to (H_1, \ldots, H_k)$ but for any $e \in E(G)$, $G - e \not\to (H_1, \ldots, H_k)$.

Ramsey-Minimal Families

Definitions

Given "forbidden" graphs $H_1, \ldots, H_k$, and any graph G, we write $\mathbf{G} \to (\mathbf{H_1}, \ldots, \mathbf{H_k})$ if any k coloring of $E(G)$ contains a monochromatic copy of H_i in color i, for some i.

Famous Example: $K_6 \to (K_3, K_3)$, but $K_5 \not\to (K_3, K_3)$

Definitions

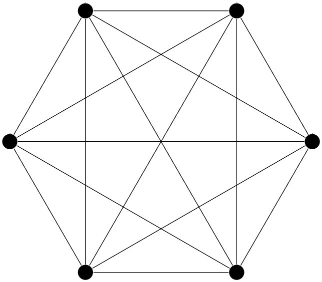
A graph G is **$(H_1, \ldots, H_k)$-Ramsey minimal** if $G \to (H_1, \ldots, H_k)$ but for any $e \in E(G)$, $G - e \not\to (H_1, \ldots, H_k)$.

Less Famous Example: K_6 is (K_3, K_3)-Ramsey Minimal.

Definitions

A graph G is $\mathbf{(H_1, \ldots, H_k)}$**-Ramsey minimal** if $G \to (H_1, \ldots, H_k)$ but for any $e \in E(G)$, $G - e \not\to (H_1, \ldots, H_k)$.

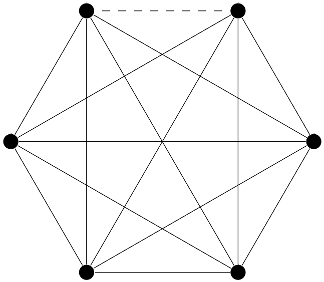
Less Famous Example: K_6 is (K_3, K_3)-Ramsey Minimal.



Definitions

A graph G is **$(H_1, \ldots, H_k)$-Ramsey minimal** if $G \to (H_1, \ldots, H_k)$ but for any $e \in E(G)$, $G - e \not\to (H_1, \ldots, H_k)$.

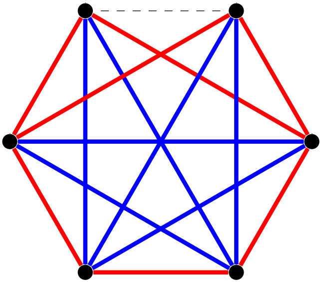
Less Famous Example: K_6 is (K_3, K_3)-Ramsey Minimal.



Definitions

A graph G is **$(H_1, \ldots, H_k)$-Ramsey minimal** if $G \to (H_1, \ldots, H_k)$ but for any $e \in E(G)$, $G - e \not\to (H_1, \ldots, H_k)$.

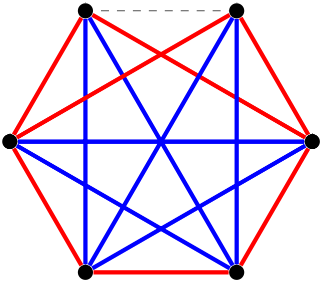
Less Famous Example: K_6 is (K_3, K_3)-Ramsey Minimal.



Definitions

A graph G is **$(H_1, \ldots, H_k)$-Ramsey minimal** if $G \to (H_1, \ldots, H_k)$ but for any $e \in E(G)$, $G - e \not\to (H_1, \ldots, H_k)$.

Less Famous Example: K_6 is (K_3, K_3)-Ramsey Minimal.



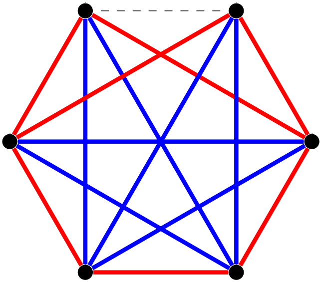
Definitions

$\mathcal{R}_{\mathbf{min}}(\mathbf{H_1}, \ldots, \mathbf{H_k}) = \mathcal{R}_{min} = \{G : G \text{ is } (H_1, \ldots, H_k)\text{-Ramsey minimal}\}$

Definitions

A graph G is **$(H_1, \ldots, H_k)$-Ramsey minimal** if $G \to (H_1, \ldots, H_k)$ but for any $e \in E(G)$, $G - e \not\to (H_1, \ldots, H_k)$.

Less Famous Example: K_6 is (K_3, K_3)-Ramsey Minimal.



$K_6 \in \mathcal{R}_{min}(K_3, K_3)$

Definitions

$\mathcal{R}_{\mathbf{min}}(\mathbf{H_1}, \ldots, \mathbf{H_k}) = \mathcal{R}_{min} = \{G : G \text{ is } (H_1, \ldots, H_k)\text{-Ramsey minimal}\}$

Saturation of Ramsey-Minimal Families

Saturation of Ramsey-Minimal Families

$\mathcal{R}_{min}(H_1, \ldots, H_k)$ Saturation

Suppose G is $\mathcal{R}_{min}(H_1, \ldots, H_k)$ saturated.

$\mathcal{R}_{min}(H_1, \ldots, H_k)$ Saturation

Suppose G is $\mathcal{R}_{min}(H_1, \ldots, H_k)$ saturated.

- G has no subgraph that is $(H_1, \ldots, H_k)$-Ramsey minimal

Saturation of Ramsey-Minimal Families

$\mathcal{R}_{min}(H_1, \ldots, H_k)$ Saturation

Suppose G is $\mathcal{R}_{min}(H_1, \ldots, H_k)$ saturated.

- G has no subgraph that is $(H_1, \ldots, H_k)$-Ramsey minimal
- Adding any edge to G creates a subgraph that is $(H_1, \ldots, H_k)$-Ramsey minimal

Saturation of Ramsey-Minimal Families

$\mathcal{R}_{min}(H_1, \ldots, H_k)$ Saturation

Suppose G is $\mathcal{R}_{min}(H_1, \ldots, H_k)$ saturated.

- G has no subgraph that is $(H_1, \ldots, H_k)$-Ramsey minimal
- Adding any edge to G creates a subgraph that is $(H_1, \ldots, H_k)$-Ramsey minimal
  - For any $e \in E(\overline{G})$, $G + e \to (H_1, \ldots, H_k)$

Saturation of Ramsey-Minimal Families

$\mathcal{R}_{min}(H_1, \ldots, H_k)$ Saturation

Suppose G is $\mathcal{R}_{min}(H_1, \ldots, H_k)$ saturated.

- G has no subgraph that is $(H_1, \ldots, H_k)$-Ramsey minimal
  - $G \not\to (H_1, \ldots, H_k)$
- Adding any edge to G creates a subgraph that is $(H_1, \ldots, H_k)$-Ramsey minimal
  - For any $e \in E(\overline{G})$, $G + e \to (H_1, \ldots, H_k)$

Saturation of Ramsey-Minimal Families

$\mathcal{R}_{min}(H_1, \ldots, H_k)$ Saturation

Suppose G is $\mathcal{R}_{min}(H_1, \ldots, H_k)$ saturated.

- G has no subgraph that is $(H_1, \ldots, H_k)$-Ramsey minimal
  - $G \not\to (H_1, \ldots, H_k)$
    Pf: If $G \to (H_1, \ldots, H_k)$, we delete edges as long as the deletion does not cause an admissible coloring to exist
- Adding any edge to G creates a subgraph that is $(H_1, \ldots, H_k)$-Ramsey minimal
  - For any $e \in E(\overline{G})$, $G + e \to (H_1, \ldots, H_k)$

Saturation of Ramsey-Minimal Families

$\mathcal{R}_{min}(H_1, \ldots, H_k)$ Saturation

Suppose G is $\mathcal{R}_{min}(H_1, \ldots, H_k)$ saturated.

- G has no subgraph that is $(H_1, \ldots, H_k)$-Ramsey minimal
  - $G \not\to (H_1, \ldots, H_k)$
    Pf: If $G \to (H_1, \ldots, H_k)$, we delete edges as long as the deletion does not cause an admissible coloring to exist
- Adding any edge to G creates a subgraph that is $(H_1, \ldots, H_k)$-Ramsey minimal
  - For any $e \in E(\overline{G})$, $G + e \to (H_1, \ldots, H_k)$

$\mathcal{R}_{min}(H_1, \ldots, H_k)$ Saturation

G is $\mathcal{R}_{min}(H_1, \ldots, H_k)$ saturated iff

- $G \not\to (H_1, \ldots, H_k)$
- For any $e \in E(\overline{G})$, $G + e \to (H_1, \ldots, H_k)$

Saturation of $\mathcal{R}_{min}(K_{k_1}, \dots, K_{k_t})$

Example

Let $r := r(k_1, \dots, k_t)$ be the Ramsey number of $(K_{k_1}, \dots, K_{k_t})$. Then

$$K_{r-2} \vee \overline{K_s}$$

is $\mathcal{R}_{min}(K_{k_1} \dots, K_{k_t})$ saturated.

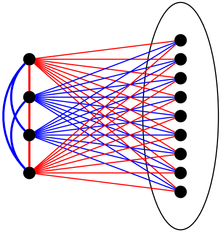
Saturation of $\mathcal{R}_{min}(K_{k_1}, \ldots, K_{k_t})$

Example

Let $r := r(k_1, \ldots, k_t)$ be the Ramsey number of $(K_{k_1}, \ldots, K_{k_t})$. Then

$$K_{r-2} \vee \overline{K_s}$$

is $\mathcal{R}_{min}(K_{k_1} \ldots, K_{k_t})$ saturated.



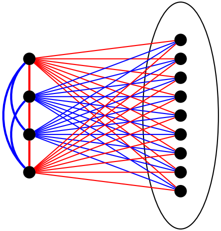
Saturation of $\mathcal{R}_{min}(K_{k_1}, \ldots, K_{k_t})$

Example

Let $r := r(k_1, \ldots, k_t)$ be the Ramsey number of $(K_{k_1}, \ldots, K_{k_t})$. Then

$$K_{r-2} \vee \overline{K_s}$$

is $\mathcal{R}_{min}(K_{k_1} \ldots, K_{k_t})$ saturated.



- $K_{r-2} \vee \overline{K_s} \not\to (K_{k_1}, \ldots, K_{k_t})$

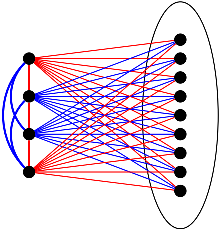
Saturation of $\mathcal{R}_{min}(K_{k_1}, \ldots, K_{k_t})$

Example

Let $r := r(k_1, \ldots, k_t)$ be the Ramsey number of $(K_{k_1}, \ldots, K_{k_t})$. Then

$$K_{r-2} \vee \overline{K_s}$$

is $\mathcal{R}_{min}(K_{k_1} \ldots, K_{k_t})$ saturated.



- $K_{r-2} \vee \overline{K_s} \not\to (K_{k_1}, \ldots, K_{k_t})$
- $K_{r-2} \vee \overline{K_s} + e \to (K_{k_1}, \ldots, K_{k_t})$

Saturation of $\mathcal{R}_{min}(K_{k_1}, \ldots, K_{k_t})$

Example

Let $r := r(k_1, \ldots, k_t)$ be the Ramsey number of $(K_{k_1}, \ldots, K_{k_t})$. Then

$$K_{r-2} \vee \overline{K_s}$$

is $\mathcal{R}_{min}(K_{k_1} \ldots, K_{k_t})$ saturated.

Corollary

$\mathrm{sat}(n; \mathcal{R}_{min}(K_{k_1}, \ldots, K_{k_t})) \leq \binom{r-2}{2} + (r-2)(n-r+2)$ when $n \geq r$

Saturation of $\mathcal{R}_{min}(K_{k_1}, \ldots, K_{k_t})$

Example

Let $r := r(k_1, \ldots, k_t)$ be the Ramsey number of $(K_{k_1}, \ldots, K_{k_t})$. Then

$$K_{r-2} \vee \overline{K_s}$$

is $\mathcal{R}_{min}(K_{k_1} \ldots, K_{k_t})$ saturated.

Corollary

$\mathrm{sat}(n; \mathcal{R}_{min}(K_{k_1}, \ldots, K_{k_t})) \leq \binom{r-2}{2} + (r-2)(n-r+2)$ when $n \geq r$

Hanson-Toft Conjecture, 1987

$$sat(n; \mathcal{R}_{min}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2} & n < r \\ \binom{r-2}{2} + (r-2)(n-r+2) & n \geq r \end{cases}$$

Hanson-Toft

Hanson-Toft Conjecture

$$sat(n; \mathcal{R}_{min}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2} & n < r \\ \binom{r-2}{2} + (r-2)(n-r+2) & n \geq r \end{cases}$$

Hanson-Toft

Hanson-Toft Conjecture

$$sat(n;\mathcal{R}_{min}(K_{k_1},\ldots,K_{k_t})) = \begin{cases} \binom{n}{2} & n < r \\ \binom{r-2}{2} + (r-2)(n-r+2) & n \geq r \end{cases}$$

Chen, Ferrara, Gould, Magnant, Schmitt; 2011

$$sat(n;\mathcal{R}_{min}(K_3,K_3)) = \begin{cases} \binom{n}{2} & n < 6 = r \\ 4n-10 & n \geq 56 \end{cases}$$

Hanson-Toft

Hanson-Toft Conjecture

$$sat(n; \mathcal{R}_{min}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2} & n < r \\ \binom{r-2}{2} + (r-2)(n-r+2) & n \geq r \end{cases}$$

Chen, Ferrara, Gould, Magnant, Schmitt; 2011

$$sat(n; \mathcal{R}_{min}(K_3, K_3)) = \begin{cases} \binom{n}{2} & n < 6 = r \\ 4n - 10 & n \geq 56 \end{cases}$$

Matchings

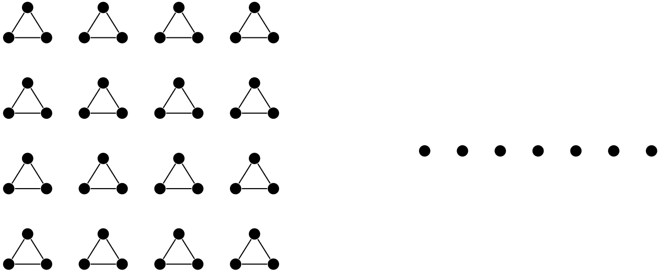
Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1 K_2, \ldots, k_t K_2)$ saturated.

Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1K_2, \ldots, k_tK_2)$ saturated.

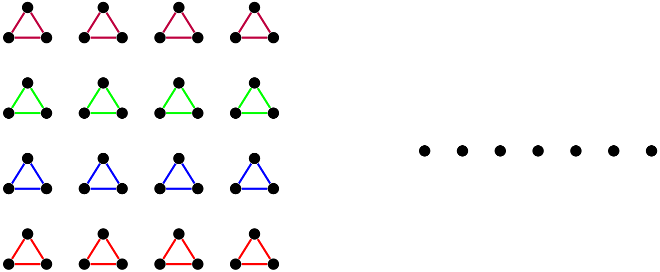


$(5K_2, 5K_2, 5K_2, 5K_2)$

Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1 K_2, \ldots, k_t K_2)$ saturated.

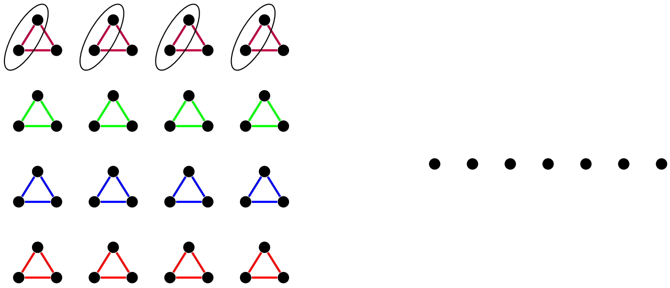


$(5K_2, 5K_2, 5K_2, 5K_2)$

Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1K_2, \ldots, k_tK_2)$ saturated.

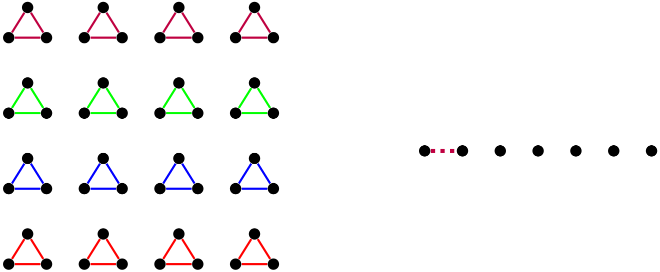


$(5K_2, 5K_2, 5K_2, 5K_2)$

Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1 K_2, \ldots, k_t K_2)$ saturated.

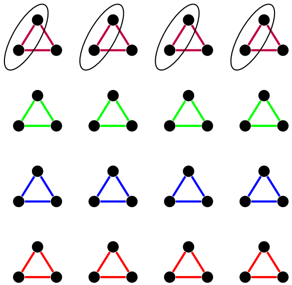


$(5K_2, 5K_2, 5K_2, 5K_2)$

Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1 K_2, \ldots, k_t K_2)$ saturated.

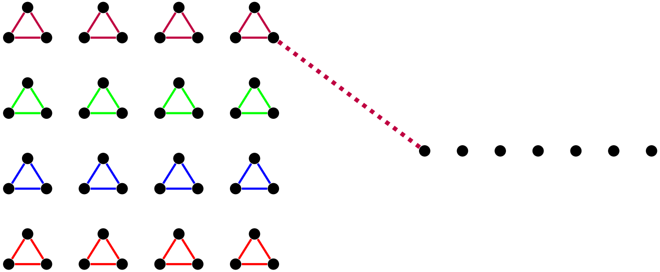


$(5K_2, 5K_2, 5K_2, 5K_2)$

Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1K_2, \ldots, k_tK_2)$ saturated.

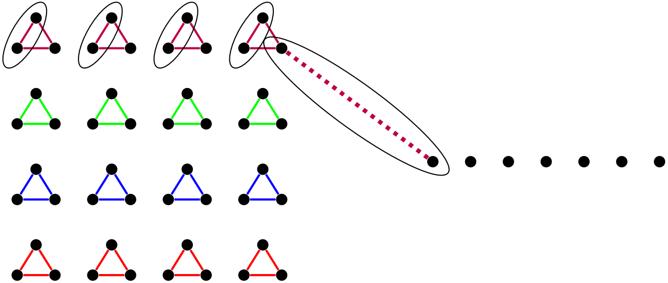


$(5K_2, 5K_2, 5K_2, 5K_2)$

Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1K_2, \ldots, k_tK_2)$ saturated.

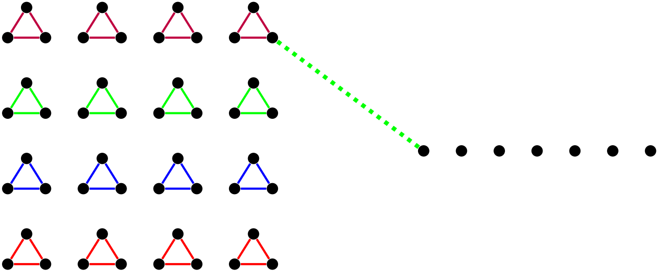


$(5K_2, 5K_2, 5K_2, 5K_2)$

Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1K_2, \ldots, k_tK_2)$ saturated.

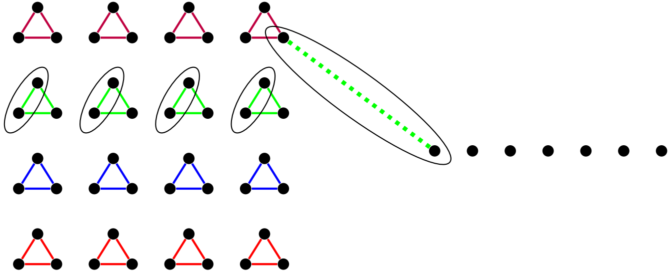


$(5K_2, 5K_2, 5K_2, 5K_2)$

Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1K_2, \ldots, k_tK_2)$ saturated.

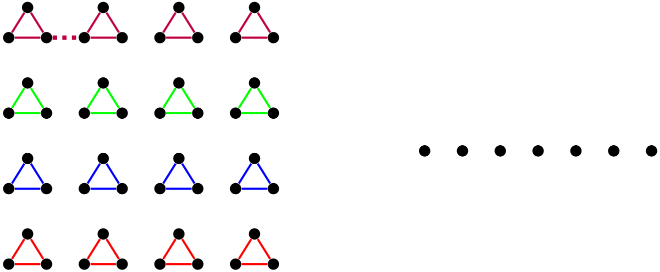


$(5K_2, 5K_2, 5K_2, 5K_2)$

Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1K_2, \ldots, k_tK_2)$ saturated.

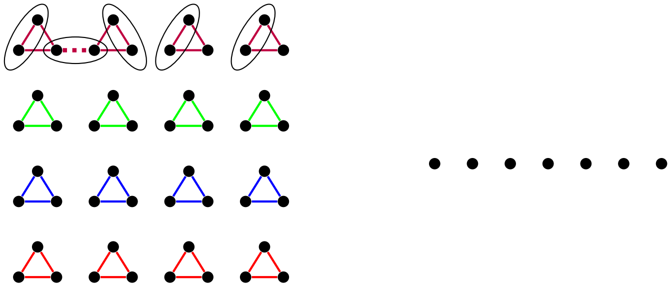


$(5K_2, 5K_2, 5K_2, 5K_2)$

Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1K_2, \ldots, k_tK_2)$ saturated.

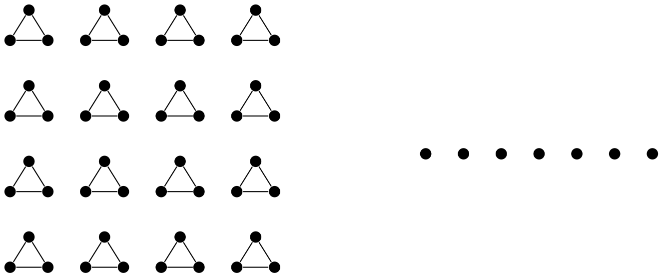


$(5K_2, 5K_2, 5K_2, 5K_2)$

Sat Number of Ramsey-Minimal Families of Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1K_2, \ldots, k_tK_2)$ saturated.



Sat Number of Ramsey-Minimal Families of Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1K_2, \ldots, k_tK_2)$ saturated.

Corollary

$\mathrm{sat}(n; \mathcal{R}_{min}(k_1K_2 + \cdots + k_tK_2)) \leq 3(k_1 + \cdots + k_t - t)$
when $n \geq 3(k_1 + \cdots + k_t - t)$

Sat Number of Ramsey-Minimal Families of Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1K_2, \ldots, k_tK_2)$ saturated.

Corollary

$\text{sat}(n; \mathcal{R}_{min}(k_1K_2 + \cdots + k_tK_2)) \leq 3(k_1 + \cdots + k_t - t)$
when $n \geq 3(k_1 + \cdots + k_t - t)$

Ferrara, Kim, Y.; 2014

$\text{sat}(n; \mathcal{R}_{min}(k_1K_2 + \cdots + k_tK_2)) = 3(k_1 + \cdots + k_t - t)$
when $n > 3(k_1 + \cdots + k_t - t)$

Sat Number of Ramsey-Minimal Families of Matchings

Example

$(k_1 + \cdots + k_t - t)K_3 + \overline{K_s}$ is $\mathcal{R}_{min}(k_1 K_2, \ldots, k_t K_2)$ saturated.

Corollary

$\text{sat}(n; \mathcal{R}_{min}(k_1 K_2 + \cdots + k_t K_2)) \leq 3(k_1 + \cdots + k_t - t)$
when $n \geq 3(k_1 + \cdots + k_t - t)$

Ferrara, Kim, Y.; 2014

$\text{sat}(n; \mathcal{R}_{min}(k_1 K_2 + \cdots + k_t K_2)) = 3(k_1 + \cdots + k_t - t)$
when $n > 3(k_1 + \cdots + k_t - t)$

Construction is generally unique: vertex-disjoint triangles with isolates.

Useful Observation

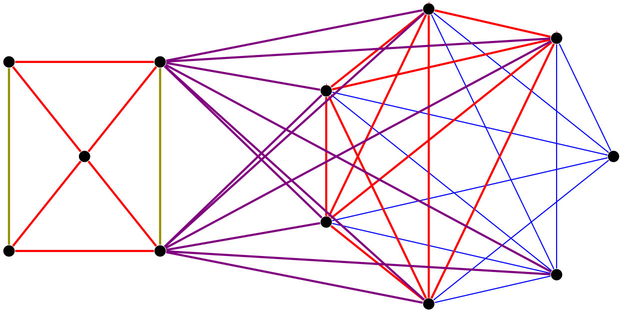
Ferrara, Kim, Y.; 2014

Looking (cleverly) at color i allows us to use results from graph saturation of the forbidden subgraph H_i.

Useful Observation

Ferrara, Kim, Y.; 2014

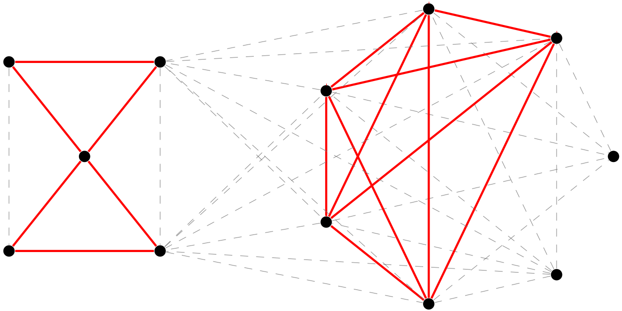
Looking (cleverly) at color i allows us to use results from graph saturation of the forbidden subgraph H_i.



Useful Observation

Ferrara, Kim, Y.; 2014

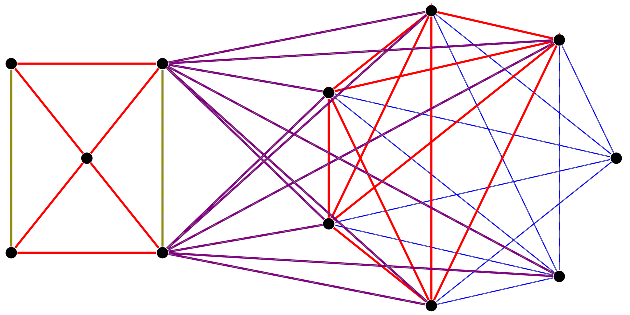
Looking (cleverly) at color i allows us to use results from graph saturation of the forbidden subgraph H_i.



Useful Observation

Ferrara, Kim, Y.; 2014

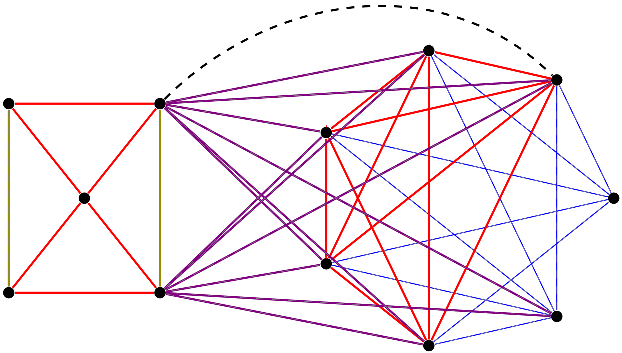
Looking (cleverly) at color i allows us to use results from graph saturation of the forbidden subgraph H_i.



Useful Observation

Ferrara, Kim, Y.; 2014

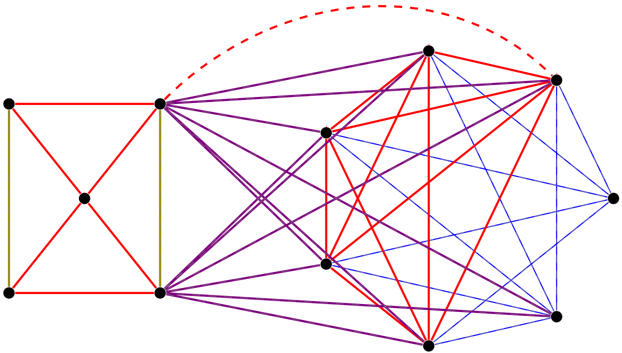
Looking (cleverly) at color i allows us to use results from graph saturation of the forbidden subgraph H_i.



Useful Observation

Ferrara, Kim, Y.; 2014

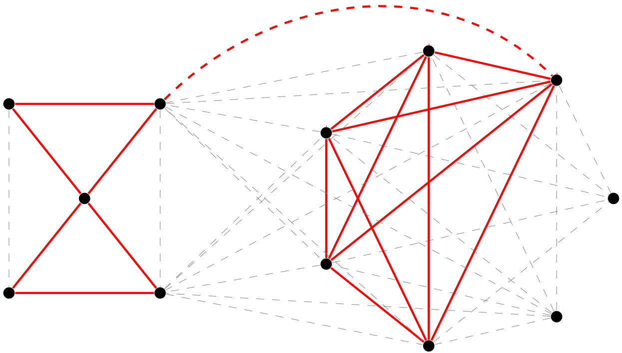
Looking (cleverly) at color i allows us to use results from graph saturation of the forbidden subgraph H_i.



Useful Observation

Ferrara, Kim, Y.; 2014

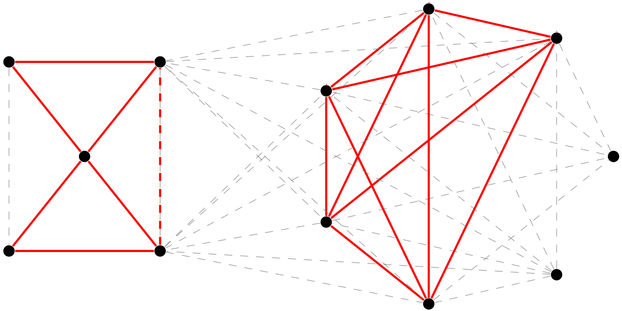
Looking (cleverly) at color i allows us to use results from graph saturation of the forbidden subgraph H_i.



Useful Observation

Ferrara, Kim, Y.; 2014

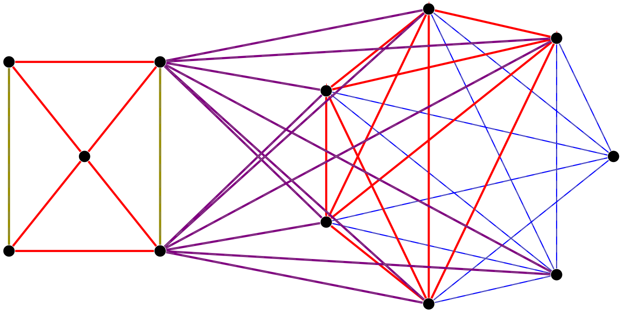
Looking (cleverly) at color i allows us to use results from graph saturation of the forbidden subgraph H_i.



Useful Observation

Ferrara, Kim, Y.; 2014

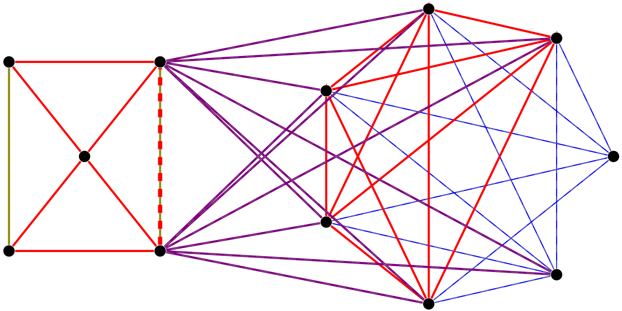
Looking (cleverly) at color i allows us to use results from graph saturation of the forbidden subgraph H_i.



Useful Observation

Ferrara, Kim, Y.; 2014

Looking (cleverly) at color i allows us to use results from graph saturation of the forbidden subgraph H_i.



Useful Observation

Ferrara, Kim, Y.; 2014

Looking (cleverly) at color i allows us to use results from graph saturation of the forbidden subgraph H_i.

Corollary

If G is $\mathcal{R}_{min}(H_1, \ldots, H_k)$ saturated, then $G = G_1 \cup \cdots \cup G_k$, where G_i is H_i saturated and all G_i share the same vertex set.

Thanks for Listening!

- G. Chen, M. Ferrara, R. Gould, C. Magnant, J. Schmitt, **Saturation numbers for families of Ramsey-minimal graphs**, J. Combin. 2 (2011) 435-455.
- M. Ferrara, J. Kim, E. Yeager, **Ramsey-minimal saturation numbers for matchings**, Discrete Math. 322 (2014) 26-30.
- A. Galluccio, M. Simonovits, G. Simonyi, **On the structure of co-critical graphs**, In: Graph Theory, Combinatorics and Algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992). Wiley-Intersci. Publ., Wiley, New York, 1053-1071.
- T. Szabo, **On nearly regular co-critical graphs**, Discrete Math. 160 (1996) 279-281.