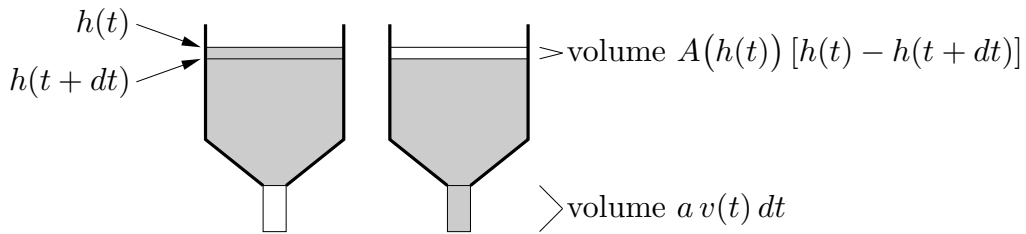


Flow from a Tank

Consider water flowing from a tank with water through a hole in its bottom. Denote by $h(t)$ the height of water in the tank at time t , $v(t)$ the speed of the water leaving through the hole at time t , $A(h)$ the cross-sectional area of the tank at height h and a the cross-sectional area of the hole. In an infinitesimal time interval dt the height changes from $h(t)$ to $h(t + dt) = h(t) + h'(t)dt$. So the volume of water that has left the tank in this time interval



is essentially $A(h(t))[h(t) - h(t + dt)] = -A(h(t))h'(t)dt$. This same volume of water has flowed, during a time interval of length dt , through a hole of cross-sectional area a at speed $v(t)$. So

$$-A(h(t))h'(t)dt = av(t)dt \implies v(t) = -\frac{1}{a}A(h(t))h'(t) \quad (1)$$

We now have one differential equation in two unknown functions – $h(t)$ and $v(t)$. We need a second equation. It is supplied by the law of conservation of energy. We assume that it is possible to ignore loss of energy through friction. We also assume that the water in the tank itself is essentially stationary, so that at time t all of the energy of the water in the tank is in the form of potential energy. At time $t + dt$, the potential energy of the system is lower because a volume $\Delta V = A(h(t))[h(t) - h(t + dt)]$ has gone from height $h(t)$ to the height of the hole. We use a coordinate system in which the hole is at height zero. Recall that the potential energy of a mass m at height h is mgh , where g is the gravitational constant. During the time interval dt the potential energy of the system has decreased by $\rho \Delta V gh(t)$, where ρ is the density of the water. This potential energy has been converted into the kinetic energy of the water flowing out of the hole, which is, $\frac{1}{2}\rho \Delta V v(t)^2$. So our second equation is

$$\rho \Delta V gh(t) = \frac{1}{2}\rho \Delta V v(t)^2 \implies v(t) = \sqrt{2gh(t)} \quad (2)$$

This is called **Torricelli's Law**. Using (2) to eliminate the $v(t)$ in (1) gives

$$h'(t) = -\frac{a}{A(h(t))} \sqrt{2gh(t)}$$

Suppose now that, in addition, water is being add to the tank at a rate of $F(t)$ units of volume per unit time. In one unit of time, the additional flow makes the height of the water $F(t)/A(h(t))$ higher than it would otherwise be, so the equation becomes

$$h'(t) = \frac{F(t) - a\sqrt{2gh(t)}}{A(h(t))}$$