Planetary Motion

with Corrections from General Relativity

Let $\mathbf{r}(t)$ be the position at time t of a planet (approximated by a point mass, m) in orbit around a sun (also approximated by a point mass, M) whose position is fixed at the origin. According to Newton's law of gravity

$$m\mathbf{r}''(t) = -\tfrac{GMm}{|\mathbf{r}|^3}\mathbf{r} \tag{1}$$

where G is the usual gravitational constant.

It is possible to considerably simplify (1). The first simplication is a consequence of the fact that (1) is a central force law. That is, the force $-\tfrac{GMm}{|\mathbf{r}|^3}\mathbf{r}$ is always parallel to the radius vector $\mathbf{r}$. For all solutions $\mathbf{r}(t)$ of all central force laws, $m\mathbf{r}'' = f(\mathbf{r})\mathbf{r}$, the angular momentum $\mathbf{a}(t) = m\mathbf{r}(t) \times \mathbf{r}'(t)$ is independent of t. To see this, it suffices to observe that the time derivative

$$\frac{d\mathbf{a}}{dt} = \frac{d}{dt} m\mathbf{r} \times \mathbf{r}' = m\mathbf{r}' \times \mathbf{r}' + m\mathbf{r} \times \mathbf{r}'' = m\mathbf{r}' \times \mathbf{r}' + mf(\mathbf{r})\mathbf{r} \times \mathbf{r}$$

is always zero, because $\mathbf{v}\times\mathbf{v} = 0$ for all vectors $\mathbf{v}$. Consequently, for all t, $\mathbf{r}(t)$ is perpendicular to the fixed vector $\mathbf{a}$. In other words $\mathbf{r}(t)$ lies in a fixed plane, for all t. We may as well choose our coordinate system so that it is the x–y plane. That is the first simplification.

The second simplification is achieved by switching to polar coordinates and writing

$$\begin{aligned}
\mathbf{r}(t) &= r(t)\big(\cos\theta(t), \sin\theta(t)\big)\\
\mathbf{r}'(t) &= r'(t)\big(\cos\theta(t), \sin\theta(t)\big) + r(t)\theta'(t)\big(-\sin\theta(t), \cos\theta(t)\big)\\
\mathbf{r}''(t) &= [r''(t) - r(t)\theta'(t)^2]\big(\cos\theta(t), \sin\theta(t)\big) + [2r'(t)\theta'(t) + r(t)\theta''(t)]\big(-\sin\theta(t), \cos\theta(t)\big)
\end{aligned} \tag{2}$$

Substituting (2) into (1) gives

$$m[r'' - r\theta'^2]\big(\cos\theta, \sin\theta\big) + [2r'\theta' + r\theta'']\big(-\sin\theta, \cos\theta\big) = -\tfrac{GMm}{r^2}\big(\cos\theta, \sin\theta\big)$$

matching coefficients of $\big(\cos\theta, \sin\theta\big)$ on the left and right hand sides and then matching coefficients of $\big(-\sin\theta(t), \cos\theta(t)\big)$ on the left and right hand sides gives

$$m[r'' - r\theta'^2] = -\tfrac{GMm}{r^2} \tag{3a}$$

$$2r'\theta' + r\theta'' = 0 \tag{3b}$$

In fact (3b) is redundant with conservation of angular mometum. Since $\big(\cos\theta(t),\sin\theta(t)\big)\times\big(\cos\theta(t),\sin\theta(t)\big)=0$ and $\big(\cos\theta(t),\sin\theta(t)\big)\times\big(-\sin\theta(t),\cos\theta(t)\big)$ is the unit vector $\widehat{\mathbf{k}}$ along the z–axis, the angular momentum

$$\mathbf{a}(t)=m\mathbf{r}(t)\times\mathbf{r}'(t)=mr(t)^2\theta'(t)\widehat{\mathbf{k}}$$

and conservation of angular momentum implies that

$$r(t)^2\theta'(t)=\tfrac{l}{m} \tag{4}$$

where l is the constant magnitude of the angular momentum vector $\mathbf{a}$. Differentiating (4) with respect to t and multiplying by r gives (3b). We can use (4) to eliminate the θ' in (3a)

$$r''-\tfrac{l^2}{m^2r^3}=-\tfrac{GM}{r^2} \tag{5}$$

In general relativity (see Misner, Thorne and Wheeler, *Gravitation*, page 656) this is modified to

$$r''-\tfrac{l^2}{m^2r^3}=-\tfrac{GM}{r^2}\Big(1+\tfrac{3l^2}{m^2c^2|\mathbf{r}|^2}\Big) \tag{6}$$

assuming that the planet is moving slowly compared to the speed c of light.

The final simplification is another change of variables. Replace r by $u=\frac{1}{r}$ and think of u as being a function of θ, which in turn is a function of t. That is

$$\begin{aligned}
r(t)&=\tfrac{1}{u(\theta(t))}\\
r'(t)&=-\tfrac{1}{u(\theta(t))^2}\tfrac{du}{d\theta}(\theta(t))\theta'(t)=-r(t)^2\theta'(t)\tfrac{du}{d\theta}(\theta(t))=-\tfrac{l}{m}\tfrac{du}{d\theta}(\theta(t)) \qquad \text{by (4)}\\
r''(t)&=-\tfrac{l}{m}\tfrac{d^2u}{d\theta^2}(\theta(t))\theta'(t)=-\tfrac{l}{m}\tfrac{l}{mr^2}\tfrac{d^2u}{d\theta^2}=-\tfrac{l^2}{m^2}u^2\tfrac{d^2u}{d\theta^2}
\end{aligned}$$

Substituting this into (6) gives

$$-\tfrac{l^2}{m^2}u^2\tfrac{d^2u}{d\theta^2}-\tfrac{l^2}{m^2}u^3=-GMu^2\Big(1+\tfrac{3l^2}{m^2c^2}u^2\Big)$$

Multiplying through by $-\frac{m^2}{l^2u^2}$ gives

$$\boxed{\tfrac{d^2u}{d\theta^2}+u=\tfrac{GMm^2}{l^2}+\tfrac{3GM}{c^2}u^2}$$