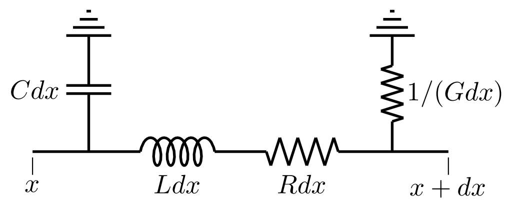
Derivation of the Telegraph Equation

Model an infinitesmal piece of telegraph wire as an electrical circuit which consists of a resistor of resistance Rdx and a coil of inductance Ldx. If $i(x,t)$ is the current through the wire, the voltage across the resistor is $iRdx$ while that across the coil is $\frac{\partial i}{\partial t}Ldx$. Denoting by $u(x,t)$ the voltage at position x and time t, we have that the change in voltage between the ends of the piece of wire is

$$du = -iRdx - \frac{\partial i}{\partial t}Ldx$$

Suppose further that current can escape from the wire to ground, either through a resistor of conductance Gdx or through a capacitor of capacitance Cdx. The amount that escapes through the resistor is $uGdx$.



Because the charge on the capacitor is $q = uCdx$, the amount that escapes from the capacitor is $q_t = u_tCdx$. In total

$$di = -uGdx - u_tCdx$$

Dividing by dx and taking the limit $dx \searrow 0$ we get the differential equations

$$u_x + Ri + Li_t = 0 \tag{1}$$

$$Cu_t + Gu + i_x = 0 \tag{2}$$

Solving (2) for i_x gives $i_x = -Cu_t - Gu$. Substituting this and its consequence $i_{xt} = -Cu_{tt} - Gu_t$ into $\frac{\partial}{\partial x}(1)$, which is $u_{xx} + Ri_x + Li_{xt} = 0$, gives

$$u_{xx} + R(-Cu_t - Gu) + L(-Cu_{tt} - Gu_t) = 0$$

Dividing by LC and moving some terms to the other side of the equation gives

$$\frac{1}{LC}u_{xx} = u_{tt} + \left(\frac{R}{L} + \frac{G}{C}\right)u_t + \frac{GR}{LC}u$$

Renaming some constants, we get the *telegraph equation*

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx}$$

where

$$c^2 = \frac{1}{LC} \qquad \alpha = \frac{G}{C} \qquad \beta = \frac{R}{L}$$