Bloch Theory for Periodic Block Spin Transformations

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December 1, 2019

Abstract

Block spin renormalization group is the main tool used in our program to see symmetry breaking in a weakly interacting many Boson system on a three dimensional lattice at low temperature. It generates operators, like the fluctuation integral covariance, that act on some lattice but are translation invariant only with respect to a proper sublattice. This paper constructs a Bloch/Floquet framework that is appropriate for bounding such operators.

*Research supported in part by the Natural Sciences and Engineering Research Council of Canada and the Forschungsinstitut für Mathematik, ETH Zürich.

Contents

1 Introduction

One standard implementation of the renormalization group philosophy [12] uses block spin transformations. See [10, 1, 9, 2, 8, 5]. Concretely, suppose we are to control a functional integral on a finite[1] lattice $\mathcal{X}_-$ of the form

$$\int \prod_{x\in\mathcal{X}_-} \tfrac{d\phi^*(x)d\phi(x)}{2\pi i}\, e^{A(\alpha_1,\cdots,\alpha_s;\phi^*,\phi)} \tag{1}$$

with an action $A(\alpha_1,\cdots,\alpha_s;\phi_*,\phi)$ that is a function of external complex valued fields α_1, $\cdots$, α_s, and the two[2] complex fields ϕ_*,ϕ on $\mathcal{X}_-$. This scenario occurs in [6, 7], where we use block spin renormalization group maps to exhibit the formation of a potential well, signalling the onset of symmetry breaking in a many particle system of weakly interacting Bosons in three space dimensions. (For an overview, see [3].)

Under the renormalization group approach to controlling integrals like (1) one successively "integrates out" lower and lower energy degrees of freedom. In the block spin formalism this is implemented by considering a decreasing sequence of sublattices of $\mathcal{X}_-$. The formalism produces, for each such sublattice, a representation of the integral (1) that is a functional integral whose integration variables are indexed by that sublattice. To pass from the representation associated with one sublattice $\mathcal{X}\subset\mathcal{X}_-$, with integration variables $\psi(x)$, $x\in\mathcal{X}$, to the representation associated to the next coarser sublattice $\mathcal{X}_+\subset\mathcal{X}$, with integration variables $\theta(y)$, $y\in\mathcal{X}_+$, one

- paves $\mathcal{X}$ by rectangles centered at the points of $\mathcal{X}_+$ and then,
- for each $y\in\mathcal{X}_+$ integrates out all values of ψ whose "average value" over the rectangle centered at y is equal to $\theta(y)$. The precise "average value" used is determined by an averaging profile q. As in (12), one uses this profile to define an averaging operator Q from the space $\mathcal{H}$ of fields on $\mathcal{X}$ to the space $\mathcal{H}_+$ of fields on $\mathcal{X}_+$. One then implements the "integrating out" by first, inserting, into the integrand, 1 expressed as a constant times the Gaussian integral

$$\int \prod_{y\in\mathcal{X}_+} \tfrac{d\theta^*(y)d\theta(y)}{2\pi i} e^{-b\langle\theta^*-Q\,\psi_*\,,\,\theta-Q\,\psi\rangle}$$

with some constant $b>0$, and then interchanging the order of the θ and ψ integrals.

For example, in [3, 6, 7] the model is initially formulated as a functional integral with integration variables indexed by a lattice[3] $\big(\mathbb{Z}/L_{\rm tp}\mathbb{Z}\big)\times\big(\mathbb{Z}^3/L_{\rm sp}\mathbb{Z}^3\big)$. After n

[1] Usually, the finite lattice is a "volume cutoff" infinite lattice and one wants to get bounds that are uniform in the size of the volume cutoff.

[2] In the actions, we treat ϕ and its complex conjugate ϕ^* as independent variables.

[3] The volume cutoff is determined by $L_{\rm tp}$ and $L_{\rm sp}$.

renormalization group steps this lattice is scaled down to $\mathcal{X}_n = \left(\frac{1}{L^{2n}}\mathbb{Z}\Big/\frac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\right) \times \left(\frac{1}{L^n}\mathbb{Z}^3\Big/\frac{L_{\rm sp}}{L^n}\mathbb{Z}^3\right)$. The decreasing family of sublattices is $\mathcal{X}_j^{(n-j)} = \left(\frac{1}{L^{2j}}\mathbb{Z}\Big/\frac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\right) \times \left(\frac{1}{L^j}\mathbb{Z}^3\Big/\frac{L_{\rm sp}}{L^n}\mathbb{Z}^3\right)$, $j = n,\, n-1,\, \cdots$. The abstract lattices $\mathcal{X}_-$, $\mathcal{X}$, $\mathcal{X}_+$ in the above framework correspond to $\mathcal{X}_n$, $\mathcal{X}_0^{(n)}$ and $\mathcal{X}_{-1}^{(n+1)}$, respectively.

In this framework there are a good number of linear operators that act on functions defined on a finite lattice and that are translation invariant with respect to a sub–lattice. For example the block spin averaging operator Q above (which is an abstraction of the operator Q of [6, Definition 1.1.a] and [4, (2.1)]) acts on functions defined on the lattice $\mathcal{X}$, but is translation invariant only with respect to the sublattice $\mathcal{X}_+$. Similarly the operator Q_- of [5, (2)] (which is an abstraction of the operator Q_n of [6, Definition 1.5.a] and [4, (2.2)]) acts on functions defined on the lattice $\mathcal{X}_-$, but is translation invariant only with respect to the sublattice $\mathcal{X}$. As another example, the fluctuation integral covariance C of [5, (14)] (which is an abstraction of the operator $C^{(n)}$ of [6, (1.15)] and [4, §4]) acts on functions defined on the lattice $\mathcal{X}$, but is translation invariant only with respect to the sublattice $\mathcal{X}_+$. In this paper, we use the Bloch/Floquet theory (see, for example, [11]) approach to develop some general machinery for bounding such linear operators. In [6, 7] the operators of interest tend to be periodizations of operators acting on L^2 of an infinite lattice. An important example is the "differential" operator D_n. See [4, Remark 3.1.a]. We also develop general machinery for bounding such periodizations. In [4] we use the results of this paper to bound many of the operators appearing in [6, 7].

2 Periodic Operators in "Position Space" and "Momentum Space" Environments

We start by setting up a general environment consisting of a "fine" lattice and a "coarse" sub–lattice. We shall consider operators that act on functions defined on the former and that are translation invariant with respect to the latter. Let $\varepsilon_T, \varepsilon_X > 0$, $L_T, L_X \in \mathbb{N}$ and $\mathcal{L}_T \in L_T\mathbb{N}$, $\mathcal{L}_X \in L_X\mathbb{N}$ and define the (finite) $(\mathrm{d}+1)$–dimensional lattices

$$\mathcal{X}_{\rm fin} = \left(\varepsilon_T\mathbb{Z}/\varepsilon_T\mathcal{L}_T\mathbb{Z}\right) \times \left(\varepsilon_X\mathbb{Z}^{\rm d}/\varepsilon_X\mathcal{L}_X\mathbb{Z}^{\rm d}\right)$$
$$\mathcal{X}_{\rm crs} = \left(L_T\varepsilon_T\mathbb{Z}/\varepsilon_T\mathcal{L}_T\mathbb{Z}\right) \times \left(L_X\varepsilon_X\mathbb{Z}^{\rm d}/\varepsilon_X\mathcal{L}_X\mathbb{Z}^{\rm d}\right)$$

and the corresponding Hilbert spaces

$$\mathcal{H}_f = L^2(\mathcal{X}_{\rm fin}) \qquad \langle \phi_1^*, \phi_2 \rangle_f = {\rm vol}_f \sum_{u \in \mathcal{X}_{\rm fin}} \phi_1(u)^* \phi_2(u)$$
$$\mathcal{H}_c = L^2(\mathcal{X}_{\rm crs}) \qquad \langle \psi_1^*, \psi_2 \rangle_c = {\rm vol}_c \sum_{x \in \mathcal{X}_{\rm crs}} \psi_1(x)^* \psi_2(x)$$

where we use

$${\rm vol}_f = \varepsilon_T \varepsilon_X^{\rm d} \qquad {\rm vol}_c = (\varepsilon_T L_T)(\varepsilon_X L_X)^{\rm d}$$

to denote the volume of a single cell in $\mathcal{X}_{\rm fin}$, and $\mathcal{X}_{\rm crs}$, respectively. For the Bloch construction, it will also be useful to define the "single period" lattice

$$\mathcal{B} = \big(\varepsilon_T \mathbb{Z}/L_T\varepsilon_T\mathbb{Z}\big) \times \big(\varepsilon_X \mathbb{Z}^{\rm d}/L_X\varepsilon_X\mathbb{Z}^{\rm d}\big) \cong \mathcal{X}_{\rm fin}/\mathcal{X}_{\rm crs}$$

The lattices dual to $\mathcal{X}_{\rm fin}$, $\mathcal{X}_{\rm crs}$ and $\mathcal{B}$ are

$$\begin{aligned}
\hat{\mathcal{X}}_{\rm fin} &= \big(\tfrac{2\pi}{\varepsilon_T\mathcal{L}_T}\mathbb{Z}/\tfrac{2\pi}{\varepsilon_T}\mathbb{Z}\big) \times \big(\tfrac{2\pi}{\varepsilon_X\mathcal{L}_X}\mathbb{Z}^{\rm d}/\tfrac{2\pi}{\varepsilon_X}\mathbb{Z}^{\rm d}\big) \\
\hat{\mathcal{X}}_{\rm crs} &= \big(\tfrac{2\pi}{\varepsilon_T\mathcal{L}_T}\mathbb{Z}/\tfrac{2\pi}{L_T\varepsilon_T}\mathbb{Z}\big) \times \big(\tfrac{2\pi}{\varepsilon_X\mathcal{L}_X}\mathbb{Z}^{\rm d}/\tfrac{2\pi}{L_X\varepsilon_X}\mathbb{Z}^{\rm d}\big) \\
\hat{\mathcal{B}} &= \big(\tfrac{2\pi}{L_T\varepsilon_T}\mathbb{Z}/\tfrac{2\pi}{\varepsilon_T}\mathbb{Z}\big) \times \big(\tfrac{2\pi}{L_X\varepsilon_X}\mathbb{Z}^{\rm d}/\tfrac{2\pi}{\varepsilon_X}\mathbb{Z}^{\rm d}\big) \cong \hat{\mathcal{X}}_{\rm crs}/\hat{\mathcal{X}}_{\rm fin}
\end{aligned}$$

We denote by

$$\hat{\pi} : \hat{\mathcal{X}}_{\rm fin} \to \hat{\mathcal{X}}_{\rm crs}$$

the canonical projection from $\hat{\mathcal{X}}_{\rm fin}$ to $\hat{\mathcal{X}}_{\rm crs}$. It has kernel $\hat{\mathcal{B}}$. Observe that

$$p \cdot x = \hat{\pi}(p) \cdot x \bmod 2\pi \qquad \text{for all } x \in \mathcal{X}_{\rm crs},\ p \in \hat{\mathcal{X}}_{\rm fin}$$

The Fourier and inverse Fourier transforms are, for $\phi \in \mathcal{H}_f$, $\psi \in \mathcal{H}_c$, $\zeta \in L^2(\mathcal{B})$, $p \in \hat{\mathcal{X}}_{\rm fin}$, $k \in \hat{\mathcal{X}}_{\rm crs}$, $\ell \in \hat{\mathcal{B}}$, $u \in \mathcal{X}_{\rm fin}$, $x \in \mathcal{X}_{\rm crs}$ and $w \in \mathcal{B}$,

$$\hat{\phi}(p) = {\rm vol}_f \sum_{u \in \mathcal{X}_{\rm fin}} \phi(u) e^{-ip\cdot u} \qquad \phi(u) = \tfrac{\widehat{\rm vol}_f}{(2\pi)^{1+{\rm d}}} \sum_{p \in \hat{\mathcal{X}}_{\rm fin}} \hat{\phi}(p) e^{iu\cdot p}$$
$$\hat{\psi}(k) = {\rm vol}_c \sum_{x \in \mathcal{X}_{\rm crs}} \psi(x) e^{-ik\cdot x} \qquad \psi(x) = \tfrac{\widehat{\rm vol}_c}{(2\pi)^{1+{\rm d}}} \sum_{k \in \hat{\mathcal{X}}_{\rm crs}} \hat{\psi}(k) e^{ik\cdot x}$$
$$\hat{\zeta}(\ell) = {\rm vol}_f \sum_{w \in \mathcal{B}} \zeta(w) e^{-i\ell\cdot w} \qquad \zeta(w) = \tfrac{\widehat{\rm vol}_b}{(2\pi)^{1+{\rm d}}} \sum_{\ell \in \hat{\mathcal{B}}} \hat{\zeta}(\ell) e^{iw\cdot \ell}$$

where

$$\widehat{\rm vol}_f = \tfrac{(2\pi)^{1+{\rm d}}}{(\varepsilon_T\mathcal{L}_T)(\varepsilon_X\mathcal{L}_X)^{\rm d}} \qquad \widehat{\rm vol}_c = \tfrac{(2\pi)^{1+{\rm d}}}{(\varepsilon_T\mathcal{L}_T)(\varepsilon_X\mathcal{L}_X)^{\rm d}} \qquad \widehat{\rm vol}_b = \tfrac{(2\pi)^{1+{\rm d}}}{(\varepsilon_T L_T)(\varepsilon_X L_X)^{\rm d}}$$

denote the volume of a single cell in $\hat{\mathcal{X}}_{\text{fin}}$, $\hat{\mathcal{X}}_{\text{crs}}$ and $\hat{\mathcal{B}}$, respectively. Observe that

$$\begin{aligned} \frac{\text{vol}_f \widehat{\text{vol}}_f}{(2\pi)^{1+\text{d}}} &= \frac{1}{\mathcal{L}_T \mathcal{L}_X^{\text{d}}} = \frac{1}{|\mathcal{X}_{\text{fin}}|} = \frac{1}{|\hat{\mathcal{X}}_{\text{fin}}|} \\ \frac{\text{vol}_c \widehat{\text{vol}}_f}{(2\pi)^{1+\text{d}}} = \frac{\text{vol}_c \widehat{\text{vol}}_c}{(2\pi)^{1+\text{d}}} &= \frac{L_T L_X^{\text{d}}}{\mathcal{L}_T \mathcal{L}_X^{\text{d}}} = \frac{1}{|\mathcal{X}_{\text{crs}}|} = \frac{1}{|\hat{\mathcal{X}}_{\text{crs}}|} \end{aligned} \tag{2}$$

where $|\mathcal{X}_{\text{crs}}|$ denotes the number of points in $\mathcal{X}_{\text{crs}}$. By (2) and the fact that $\delta_{u,u'} = \frac{1}{|\hat{\mathcal{X}}_{\text{fin}}|} \sum_{p \in \hat{\mathcal{X}}_{\text{fin}}} e^{ip\cdot u} e^{-ip\cdot u'}$,

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle_f &= \text{vol}_f \sum_{u \in \mathcal{X}_{\text{fin}}} \phi_1(u)\phi_2(u) = \tfrac{\widehat{\text{vol}}_f}{(2\pi)^{1+\text{d}}} \sum_{p \in \hat{\mathcal{X}}_{\text{fin}}} \hat{\phi}_1(-p)\hat{\phi}_2(p) \\ \langle \psi_1, \psi_2 \rangle_c &= \text{vol}_c \sum_{x \in \mathcal{X}_{\text{crs}}} \psi_1(x)\psi_2(x) = \tfrac{\widehat{\text{vol}}_c}{(2\pi)^{1+\text{d}}} \sum_{k \in \hat{\mathcal{X}}_{\text{crs}}} \hat{\psi}_1(-k)\hat{\phi}_2(k) \end{aligned}$$

Internal Remark

$$\begin{aligned} \langle \phi_1, \phi_2 \rangle_f &= \text{vol}_f \sum_{u \in \mathcal{X}_{\text{fin}}} \phi_1(u)\phi_2(u) = \tfrac{\text{vol}_f}{|\hat{\mathcal{X}}_{\text{fin}}|} \sum_{\substack{u,u' \in \mathcal{X}_{\text{fin}} \\ p \in \hat{\mathcal{X}}_{\text{fin}}}} \phi_1(u) e^{ip\cdot u} e^{-ip\cdot u'} \phi_2(u') \\ &= \tfrac{\widehat{\text{vol}}_f}{(2\pi)^{1+\text{d}}} \sum_{p \in \hat{\mathcal{X}}_{\text{fin}}} \hat{\phi}_1(-p)\hat{\phi}_2(p) \\ \langle \psi_1, \psi_2 \rangle_c &= \text{vol}_c \sum_{x \in \mathcal{X}_{\text{crs}}} \psi_1(x)\psi_2(x) = \tfrac{\text{vol}_c}{|\hat{\mathcal{X}}_{\text{crs}}|} \sum_{\substack{x,x' \in \mathcal{X}_{\text{crs}} \\ k \in \hat{\mathcal{X}}_{\text{crs}}}} \psi_1(x) e^{ik\cdot x} e^{-ik\cdot x'} \psi_2(x') \\ &= \tfrac{\widehat{\text{vol}}_c}{(2\pi)^{1+\text{d}}} \sum_{k \in \hat{\mathcal{X}}_{\text{crs}}} \hat{\psi}_1(-k)\hat{\phi}_2(k) \end{aligned}$$

Let A be any operator on $\mathcal{H}_f$ that is translation invariant with respect to $\mathcal{X}_{\text{crs}}$. We call such an operator a "periodic operator". Denote by $A(u,u')$ its kernel, defined so that

$$(A\phi)(u) = \text{vol}_f \sum_{u' \in \mathcal{X}_{\text{fin}}} A(u,u')\phi(u')$$

By "translation invariant with respect to $\mathcal{X}_{\text{crs}}$", we mean that $A(u+x, u'+x) =$

$A(u,u')$ for all $u,u' \in \mathcal{X}_{\rm fin}$ and $x \in \mathcal{X}_{\rm crs}$. Set[4], for $p,p' \in \hat{\mathcal{X}}_{\rm fin}$,

$$\hat{A}(p,p') = \tfrac{{\rm vol}_f}{|\mathcal{X}_{\rm fin}|} \sum_{u,u' \in \mathcal{X}_{\rm fin}} e^{-ip\cdot u} A(u,u') e^{ip'\cdot u'} \tag{3}$$

and, for $u,u' \in \mathcal{X}_{\rm fin}$ and ${\rm k} \in \frac{2\pi}{\varepsilon_T \mathcal{L}_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X \mathcal{L}_X}\mathbb{Z}^{\rm d}$, the "universal cover" of $\hat{\mathcal{X}}_{\rm crs}$,

$$A_{\rm k}(u,u') = {\rm vol}_c \sum_{\substack{u'' \in \mathcal{X}_{\rm fin} \\ u''-u' \in \mathcal{X}_{\rm crs}}} e^{-i{\rm k}\cdot u} A(u,u'') e^{i{\rm k}\cdot u''} \tag{4}$$

For each fixed $u,u' \in \mathcal{X}_{\rm fin}$, ${\rm k} \mapsto A_{\rm k}(u,u')$ is not a function on the torus $\hat{\mathcal{X}}_{\rm crs}$ since, for ${\rm p} \in \frac{2\pi}{L_T \varepsilon_T}\mathbb{Z} \times \frac{2\pi}{L_X \varepsilon_X}\mathbb{Z}^{\rm d}$,

$$A_{\rm k+p}(u,u') = e^{-i{\rm p}(u-u')} A_{\rm k}(u,u')$$

This is why we defined $A_{\rm k}$ for ${\rm k} \in \frac{2\pi}{\varepsilon_T \mathcal{L}_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X \mathcal{L}_X}\mathbb{Z}^{\rm d}$, rather than ${\rm k} \in \hat{\mathcal{X}}_{\rm crs}$. On the other hand, $k \mapsto e^{ik(u-u')} A_k(u,u')$ is a well–defined function on $\hat{\mathcal{X}}_{\rm crs}$.

The following lemma is standard.

Lemma 1. *Let A be an operator on $\mathcal{H}_f$ that is translation invariant with respect to $\mathcal{X}_{\rm crs}$.*

(a) $A(u,u') = \frac{\widehat{\rm vol}_f}{(2\pi)^{1+{\rm d}}} \sum_{p,p' \in \hat{\mathcal{X}}_{\rm fin}} e^{ip\cdot u} \hat{A}(p,p') e^{-ip'\cdot u'}$

(b) $A(u,u') = \frac{\widehat{\rm vol}_c}{(2\pi)^{1+{\rm d}}} \sum_{\substack{[{\rm k}] \in \hat{\mathcal{X}}_{\rm crs} \\ \ell,\ell' \in \hat{\mathcal{B}}}} e^{i\ell\cdot u} \hat{A}({\rm k}+\ell, {\rm k}+\ell') e^{-i\ell'\cdot u'} e^{i{\rm k}\cdot(u-u')}$

Here $\sum_{[{\rm k}] \in \hat{\mathcal{X}}_{\rm crs}} f({\rm k})$ *means that one sums* k *over a subset of* $\frac{2\pi}{\varepsilon_T \mathcal{L}_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X \mathcal{L}_X}\mathbb{Z}^{\rm d}$ *that contains exactly one (arbitrary) representative from each equivalence class of* $\hat{\mathcal{X}}_{\rm crs}$. *Note that if* ${\rm k} \in \frac{2\pi}{\varepsilon_T \mathcal{L}_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X \mathcal{L}_X}\mathbb{Z}^{\rm d}$ *and* $\ell \in \hat{\mathcal{B}}$, *then* ${\rm k}+\ell \in \hat{\mathcal{X}}_{\rm fin}$.

(c) $\widehat{(A\phi)}(p) = \sum_{p' \in \hat{\mathcal{X}}_{\rm fin}} \hat{A}(p,p') \hat{\phi}(p')$ *for all* $\phi \in \mathcal{H}_f$.

(d) For each ${\rm k} \in \frac{2\pi}{\varepsilon_T \mathcal{L}_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X \mathcal{L}_X}\mathbb{Z}^{\rm d}$, $A_{\rm k}(u,u')$ *is periodic with respect to* $\mathcal{X}_{\rm crs}$ *in both* u *and* u' *and*

$$A(u,u') = \tfrac{\widehat{\rm vol}_c}{(2\pi)^{1+{\rm d}}} \sum_{k \in \hat{\mathcal{X}}_{\rm crs}} e^{ik\cdot u} A_k(u,u') e^{-ik\cdot u'}$$

[4]The "normal prefactor" for $\hat{A}$ would be ${\rm vol}_f^2$. We have chosen $\frac{{\rm vol}_f}{|\mathcal{X}_{\rm fin}|} = \frac{\widehat{\rm vol}_f}{(2\pi)^{1+{\rm d}}} {\rm vol}_f^2$ so as to replace approximate Dirac $(2\pi)^{1+{\rm d}}\delta(p-p')$'s with simple Kronecker $\delta_{p,p'}$'s in the translation invariant case.

(e) $A_{\mathrm{k}}(u,u') = \sum_{\ell,\ell' \in \hat{\mathcal{B}}} e^{i\ell\cdot u} \hat{A}(\mathrm{k}+\ell, \mathrm{k}+\ell') e^{-i\ell'\cdot u'}$

(f) Define the transpose of A *by* $A^\star(u,u') = A(u',u)$. *Then*

$$A^\star(u,u') = \tfrac{\widehat{\mathrm{vol}}_c}{(2\pi)^{1+\mathrm{d}}} \sum_{\substack{[\mathrm{k}]\in\hat{\mathcal{X}}_{\mathrm{crs}} \\ \ell,\ell'\in\hat{\mathcal{B}}}} e^{i\ell\cdot u} \hat{A}(-\mathrm{k}-\ell', -\mathrm{k}-\ell) e^{-i\ell'\cdot u'} e^{i\mathrm{k}\cdot(u-u')}$$

Internal Remark

Proof. (a) follows from

$$\delta_{u,u'} = \tfrac{1}{|\hat{\mathcal{X}}_{\mathrm{fin}}|} \sum_{p\in\hat{\mathcal{X}}_{\mathrm{fin}}} e^{ip\cdot u} e^{-ip\cdot u'} \qquad \text{for all } u,u' \in \mathcal{X}_{\mathrm{fin}} \tag{5}$$

and

$$\tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\mathrm{fin}}|} \tfrac{\widehat{\mathrm{vol}}_f}{(2\pi)^{1+\mathrm{d}}} |\hat{\mathcal{X}}_{\mathrm{fin}}|^2 = 1$$

(b) Since A is translation invariant with respect to $\mathcal{X}_{\mathrm{crs}}$, we have

$$A(x+u, x+u') = A(u,u')$$

for all $x \in \mathcal{X}_{\mathrm{crs}}$, and hence

$$\begin{aligned}
\hat{A}(p,p') &= \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\mathrm{fin}}|} \sum_{u,u'\in\mathcal{X}_{\mathrm{fin}}} e^{-ip\cdot u} A(u,u') e^{ip'\cdot u'} \\
&= \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\mathrm{fin}}|} \sum_{u,u'\in\mathcal{X}_{\mathrm{fin}}} e^{-ip\cdot u} A(u+x,u'+x) e^{ip'\cdot u'} \\
&= \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\mathrm{fin}}|} \sum_{u,u'\in\mathcal{X}_{\mathrm{fin}}} e^{-ip\cdot(u-x)} A(u,u') e^{ip'\cdot(u'-x)} \\
&= e^{i(p-p')\cdot x} \hat{A}(p,p')
\end{aligned}$$

for all $x \in \mathcal{X}_{\mathrm{crs}}$. This forces $\hat{A}(p,p')$ to vanish unless $p - p' \in \hat{\mathcal{B}}$, or equivalently, $\hat{\pi}(p) = \hat{\pi}(p')$, and the claim follows.

(c) By the definitions and (2),

$$\begin{aligned}\widehat{(A\phi)}(p) &= \mathrm{vol}_f^2 \sum_{u,u' \in \mathcal{X}_{\mathrm{fin}}} e^{-ip\cdot u} A(u,u')\phi(u') \\ &= \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\mathrm{fin}}|} \sum_{\substack{u,u' \in \mathcal{X}_{\mathrm{fin}} \\ p' \in \hat{\mathcal{X}}_{\mathrm{fin}}}} e^{-ip\cdot u} A(u,u') e^{iu'\cdot p'} \hat{\phi}(p') \\ &= \sum_{p' \in \hat{\mathcal{X}}_{\mathrm{fin}}} \hat{A}(p,p')\hat{\phi}(p')\end{aligned}$$

(d) The periodicity of $A_{\mathrm{k}}(u,u')$ in u' follows immediately from the definition and the observation that, if $x \in \mathcal{X}_{\mathrm{crs}}$, then $u''-u' \in \mathcal{X}_{\mathrm{crs}}$ if and only if $u''-(u'+x) \in \mathcal{X}_{\mathrm{crs}}$. Since

$$\begin{aligned}A_{\mathrm{k}}(u+x,u') &= \mathrm{vol}_c \sum_{\substack{u'' \in \mathcal{X}_{\mathrm{fin}} \\ u''-u' \in \mathcal{X}_{\mathrm{crs}}}} e^{-i\mathrm{k}\cdot(u+x)} A(u+x,u'') e^{i\mathrm{k}\cdot u''} \\ &= \mathrm{vol}_c \sum_{\substack{u'' \in \mathcal{X}_{\mathrm{fin}} \\ u''-u' \in \mathcal{X}_{\mathrm{crs}}}} e^{-i\mathrm{k}\cdot(u+x)} A(u+x,u''+x) e^{i\mathrm{k}\cdot(u''+x)} \\ &= \mathrm{vol}_c \sum_{\substack{u'' \in \mathcal{X}_{\mathrm{fin}} \\ u''-u' \in \mathcal{X}_{\mathrm{crs}}}} e^{-i\mathrm{k}\cdot u} A(u,u'') e^{i\mathrm{k}\cdot u''} \\ &= A_{\mathrm{k}}(u,u')\end{aligned}$$

for all $x \in \mathcal{X}_{\mathrm{crs}}$, $A_{\mathrm{k}}(u,u')$ is also periodic in u. For the representation of $A(u,u')$, we compute

$$\begin{aligned}\tfrac{\widehat{\mathrm{vol}}_c}{(2\pi)^{1+\mathrm{d}}} \sum_{\mathrm{k} \in \hat{\mathcal{X}}_{\mathrm{crs}}} e^{ik\cdot u} A_{\mathrm{k}}(u,u') e^{-ik\cdot u'} &= \tfrac{1}{|\hat{\mathcal{X}}_{\mathrm{crs}}|} \sum_{k \in \hat{\mathcal{X}}_{\mathrm{crs}}} e^{ik\cdot u} \sum_{\substack{u'' \in \mathcal{X}_{\mathrm{fin}} \\ u''-u' \in \mathcal{X}_{\mathrm{crs}}}} e^{-ik\cdot u} A(u,u'') e^{ik\cdot u''} e^{-ik\cdot u'} \\ &= \tfrac{1}{|\hat{\mathcal{X}}_{\mathrm{crs}}|} \sum_{\substack{u'' \in \mathcal{X}_{\mathrm{fin}} \\ u''-u' \in \mathcal{X}_{\mathrm{crs}}}} A(u,u'') \sum_{k \in \hat{\mathcal{X}}_{\mathrm{crs}}} e^{ik\cdot(u''-u')} \\ &= \sum_{\substack{u'' \in \mathcal{X}_{\mathrm{fin}} \\ u''-u' \in \mathcal{X}_{\mathrm{crs}}}} \delta_{u',u''} A(u,u'') \\ &= A(u,u')\end{aligned}$$

since $u''-u'$ is restricted to $\mathcal{X}_{\mathrm{crs}}$.

(e) By the definition of $A_{\mathrm{k}}(u,u')$ and part (c),

$$\begin{aligned}
A_{\mathrm{k}}(u,u') &= \mathrm{vol}_c \sum_{\substack{u''\in\mathcal{X}_{\mathrm{fin}}\\ u''-u'\in\mathcal{X}_{\mathrm{crs}}}} e^{-i\mathrm{k}\cdot(u-u'')}A(u,u'') \\
&= \tfrac{1}{|\mathcal{X}_{\mathrm{crs}}|}\sum_{\substack{u''\in\mathcal{X}_{\mathrm{fin}}\\ u''-u'\in\mathcal{X}_{\mathrm{crs}}}} e^{-i\mathrm{k}\cdot(u-u'')} \sum_{\substack{[\mathrm{k}']\in\hat{\mathcal{X}}_{\mathrm{crs}}\\ \ell,\ell'\in\hat{\mathcal{B}}}} e^{i\ell\cdot u}\hat{A}(\mathrm{k}'+\ell,\mathrm{k}'+\ell')e^{-i\ell'\cdot u''}e^{i\mathrm{k}'\cdot(u-u'')} \\
&= \tfrac{1}{|\mathcal{X}_{\mathrm{crs}}|}\sum_{\substack{[\mathrm{k}']\in\hat{\mathcal{X}}_{\mathrm{crs}}\\ \ell,\ell'\in\hat{\mathcal{B}}}} e^{i\ell\cdot u}\hat{A}(\mathrm{k}'+\ell,\mathrm{k}'+\ell') \sum_{x\in\mathcal{X}_{\mathrm{crs}}} e^{-i\ell'\cdot(u'+x)}e^{i(\mathrm{k}'-\mathrm{k})\cdot(u-u'-x)} \\
&= \sum_{\substack{[\mathrm{k}']\in\hat{\mathcal{X}}_{\mathrm{crs}}\\ \ell,\ell'\in\hat{\mathcal{B}}}} e^{i\ell\cdot u}\hat{A}(\mathrm{k}'+\ell,\mathrm{k}'+\ell')e^{-i\ell'\cdot u'}e^{i(\mathrm{k}'-\mathrm{k})\cdot(u-u')}\ \tfrac{1}{|\mathcal{X}_{\mathrm{crs}}|}\sum_{x\in\mathcal{X}_{\mathrm{crs}}} e^{-i(\mathrm{k}'-\mathrm{k})\cdot x} \\
&= \sum_{\substack{[\mathrm{k}']\in\hat{\mathcal{X}}_{\mathrm{crs}}\\ \ell,\ell'\in\hat{\mathcal{B}}}} e^{i\ell\cdot u}\hat{A}(\mathrm{k}'+\ell,\mathrm{k}'+\ell')e^{-i\ell'\cdot u'}e^{i(\mathrm{k}'-\mathrm{k})\cdot(u-u')}\delta_{[\mathrm{k}],[\mathrm{k}']} \\
&= \sum_{\ell,\ell'\in\hat{\mathcal{B}}} e^{i\ell\cdot u}\hat{A}(\mathrm{k}+\ell,\mathrm{k}+\ell')e^{-i\ell'\cdot u'}
\end{aligned}$$

If, in the second last line, a representative k' other than k was chosen for $[\mathrm{k}']=[\mathrm{k}]$, making the change of variables $\ell\to\ell-(\mathrm{k}'-\mathrm{k})$, $\ell'\to\ell'-(\mathrm{k}'-\mathrm{k})$ both makes k the representative and eliminates the factor $e^{i(\mathrm{k}'-\mathrm{k})\cdot(u-u')}$.

(f) Making the change of summation variables $\mathrm{k}\to-\mathrm{k}$, $\ell\to-\ell'$ and $\ell'\to-\ell$ gives

$$\begin{aligned}
A^\star(u,u') &= \tfrac{\widehat{\mathrm{vol}}_c}{(2\pi)^{1+\mathrm{d}}}\sum_{\substack{[\mathrm{k}]\in\hat{\mathcal{X}}_{\mathrm{crs}}\\ \ell,\ell'\in\hat{\mathcal{B}}}} e^{i\ell\cdot u'}\hat{A}(\mathrm{k}+\ell,\mathrm{k}+\ell')e^{-i\ell'\cdot u}e^{i\mathrm{k}\cdot(u'-u)} \\
&= \tfrac{\widehat{\mathrm{vol}}_c}{(2\pi)^{1+\mathrm{d}}}\sum_{\substack{[\mathrm{k}]\in\hat{\mathcal{X}}_{\mathrm{crs}}\\ \ell,\ell'\in\hat{\mathcal{B}}}} e^{i\ell\cdot u}\hat{A}(-\mathrm{k}-\ell',-\mathrm{k}-\ell)e^{-i\ell'\cdot u'}e^{i\mathrm{k}\cdot(u-u')}
\end{aligned}$$

as desired. □

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3 Periodized Operators

Define the (infinite) lattices

$$\mathcal{Z}_{\mathrm{fin}} = \varepsilon_T\mathbb{Z}\times\varepsilon_X\mathbb{Z}^{\mathrm{d}} \qquad \mathcal{Z}_{\mathrm{crs}} = L_T\varepsilon_T\mathbb{Z}\times L_X\varepsilon_X\mathbb{Z}^{\mathrm{d}}$$

Definition 2 (Periodization)**.** Suppose that $a(\mathrm{u},\mathrm{u}')$ is a function on $\mathcal{Z}_{\mathrm{fin}}\times\mathcal{Z}_{\mathrm{fin}}$ that

- is translation invariant with respect to $\mathcal{Z}_{\mathrm{crs}}$ in the sense that $a(\mathrm{u}+\mathrm{x},\mathrm{u}'+\mathrm{x}) = a(\mathrm{u},\mathrm{u}')$ for all $\mathrm{x}\in\mathcal{Z}_{\mathrm{crs}}$ and $\mathrm{u},\mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}$ and
- has finite L^1–L^∞ norm (i.e. $\sup\limits_{\mathrm{u}\in\mathcal{Z}_{\mathrm{fin}}}\sum\limits_{\mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}}\big|a(\mathrm{u},\mathrm{u}')\big|$ and $\sup\limits_{\mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}}\sum\limits_{\mathrm{u}\in\mathcal{Z}_{\mathrm{fin}}}\big|a(\mathrm{u},\mathrm{u}')\big|$ are both finite)

and that the operator A (on $\mathcal{H}_f$) acts by

$$(A\phi)([\mathrm{u}]) = \mathrm{vol}_f \sum_{\mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}} a(\mathrm{u},\mathrm{u}')\phi([\mathrm{u}']) \tag{6}$$

Here, for each $\mathrm{u}\in\mathcal{Z}_{\mathrm{fin}}$, the notation $[\mathrm{u}]$ means the equivalence class in $\mathcal{X}_{\mathrm{fin}}$ that contains u. Then we say that A is the periodization of a. It is "a with periodic boundary conditions on a box of size $\varepsilon_T\mathcal{L}_T\times\overbrace{\varepsilon_X\mathcal{L}_X\times\cdots\times\varepsilon_X\mathcal{L}_X}^{\mathrm{d\ factors}}$".

Remark 3.

(a) The right hand side of (6) is independent of the representative u chosen from $[\mathrm{u}]$ (by translation invariance with respect to $\varepsilon_T\mathcal{L}_T\mathbb{Z}\times\varepsilon_X\mathcal{L}_X\mathbb{Z}^{\mathrm{d}}\subset\mathcal{Z}_{\mathrm{crs}}$).

(b) The kernel of A is given by

$$A\big([\mathrm{u}],[\mathrm{u}']\big) = \sum_{\substack{\mathrm{u}''\in\mathcal{Z}_{\mathrm{fin}}\\ [\mathrm{u}'']=[\mathrm{u}']}} a(\mathrm{u},\mathrm{u}'') = \sum_{\mathrm{z}\in\varepsilon_T\mathcal{L}_T\mathbb{Z}\times\varepsilon_X\mathcal{L}_X\mathbb{Z}^{\mathrm{d}}} a(\mathrm{u},\mathrm{u}'+\mathrm{z})$$

The sum converges because a has finite L^1–L^∞ norm. This is the motivation for the name the "periodization of a".

(c) If A is the periodization of a and B is the periodization of b, then $C=AB$ is the periodization of

$$c(\mathrm{u},\mathrm{u}') = \mathrm{vol}_f \sum_{\mathrm{u}''\in\mathcal{Z}_{\mathrm{fin}}} a(\mathrm{u},\mathrm{u}'')b(\mathrm{u}'',\mathrm{u}')$$

Internal Remark

$$\begin{aligned}(AB\phi)([\mathrm{u}]) &= \mathrm{vol}_f \sum_{\mathrm{u}''\in\mathcal{Z}_{\mathrm{fin}}} a(\mathrm{u},\mathrm{u}'')(B\phi)([\mathrm{u}''])\\ &= \mathrm{vol}_f^2 \sum_{\mathrm{u}',\mathrm{u}''\in\mathcal{Z}_{\mathrm{fin}}} a(\mathrm{u},\mathrm{u}'')b(\mathrm{u}'',\mathrm{u}')\phi([\mathrm{u}'])\end{aligned}$$

Let $\hat{\mathcal{Z}}_{\rm crs} = \left(\mathbb{R}/\frac{2\pi}{\varepsilon_T L_T}\mathbb{Z}\right) \times \left(\mathbb{R}^{\rm d}/\frac{2\pi}{\varepsilon_X L_X}\mathbb{Z}^{\rm d}\right)$ be the dual space of $\mathcal{Z}_{\rm crs}$. Its universal cover is $\mathbb{R} \times \mathbb{R}^{\rm d}$. For each $\mathrm{k} \in \mathbb{R} \times \mathbb{R}^{\rm d}$, set, for $\mathrm{u}, \mathrm{u}' \in \mathcal{Z}_{\rm fin}$,

$$a_{\rm k}(\mathrm{u}, \mathrm{u}') = \mathrm{vol}_c \sum_{\substack{\mathrm{u}'' \in \mathcal{Z}_{\rm fin} \\ \mathrm{u}''-\mathrm{u}' \in \mathcal{Z}_{\rm crs}}} e^{-i\mathrm{k}\cdot\mathrm{u}} a(\mathrm{u}, \mathrm{u}'') e^{i\mathrm{k}\cdot\mathrm{u}''} \tag{7}$$

and, for $\ell, \ell' \in \hat{\mathcal{B}}$,

$$\begin{aligned} \hat{a}_{\rm k}(\ell, \ell') &= \tfrac{1}{|\mathcal{B}|^2} \sum_{[\mathrm{u}],[\mathrm{u}'] \in \mathcal{B}} e^{-i\ell\cdot\mathrm{u}} a_{\rm k}(\mathrm{u}, \mathrm{u}') e^{i\ell'\cdot\mathrm{u}'} \\ &= \tfrac{\mathrm{vol}_f}{|\mathcal{B}|} \sum_{\substack{[\mathrm{u}] \in \mathcal{B} \\ \mathrm{u}' \in \mathcal{Z}_{\rm fin}}} e^{-i\ell\cdot\mathrm{u}} a(\mathrm{u}, \mathrm{u}') e^{i\ell'\cdot\mathrm{u}'} e^{-i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \end{aligned} \tag{8}$$

(Recall that $\frac{1}{|\mathcal{B}|^2} = \frac{\mathrm{vol}_f}{\mathrm{vol}_c|\mathcal{B}|}$. We shall show in Lemma 5.a, below, that $a_{\rm k}(\mathrm{u}, \mathrm{u}')$ is periodic with respect to $\mathcal{Z}_{\rm crs}$ in both u and u′.) By the L^1–L^∞ hypothesis and the Lebesgue dominated convergence theorem, both $a_{\rm k}(\mathrm{u}, \mathrm{u}')$ and $a_{\rm k}(\ell, \ell')$ are continuous in k.

Remark 4. As was the case for $A_k(u, u')$, for each fized $\mathrm{u}, \mathrm{u}' \in \mathcal{Z}_{\rm fin}$, the map $\mathrm{k} \mapsto a_{\rm k}(\mathrm{u}, \mathrm{u}')$ is not a function on the torus $\hat{\mathcal{Z}}_{\rm crs}$ since, for $\mathrm{p} \in \frac{2\pi}{\varepsilon_T L_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X L_X}\mathbb{Z}^{\rm d}$,

$$a_{\rm k+p}(\mathrm{u}, \mathrm{u}') = e^{-i\mathrm{p}\cdot(\mathrm{u}-\mathrm{u}')} a_{\rm k}(\mathrm{u}, \mathrm{u}')$$

However

$$\mathrm{k} \in \hat{\mathcal{Z}}_{\rm crs} \mapsto e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} a_{\rm k}(\mathrm{u}, \mathrm{u}') = \mathrm{vol}_c \sum_{\mathrm{x} \in \mathcal{Z}_{\rm crs}} a(\mathrm{u}, \mathrm{u}' + \mathrm{x}) e^{i\mathrm{k}\cdot\mathrm{x}}$$

is a legitimate function on the torus $\hat{\mathcal{Z}}_{\rm crs}$ and is in fact the Fourier transform of the function

$$\mathrm{x} \in \mathcal{Z}_{\rm crs} \mapsto a(\mathrm{u}, \mathrm{u}' + \mathrm{x})$$

Correspondingly, for $\mathrm{p} \in \frac{2\pi}{\varepsilon_T L_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X L_X}\mathbb{Z}^{\rm d}$ and $\ell, \ell' \in \hat{\mathcal{B}}$

$$\hat{a}_{\rm k+p}(\ell, \ell') = \hat{a}_{\rm k}(\ell + \mathrm{p}, \ell' + \mathrm{p})$$

The following two lemmas are again standard.

Lemma 5. *Let $a(\mathrm{u}, \mathrm{u}') : \mathcal{Z}_{\rm fin} \times \mathcal{Z}_{\rm fin} \to \mathbb{C}$ obey the conditions of Definition 2, and, in particular, be translation invariant with respect to $\mathcal{Z}_{\rm crs}$.*

(a) For each $\mathrm{k} \in \mathbb{R} \times \mathbb{R}^{\mathrm{d}}$, $a_{\mathrm{k}}(\mathrm{u}, \mathrm{u}')$ *is periodic with respect to* $\mathcal{Z}_{\mathrm{crs}}$ *in both* u *and* u′ *and*

$$\begin{aligned} a(\mathrm{u}, \mathrm{u}') &= \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} a_{\mathrm{k}}(\mathrm{u}, \mathrm{u}') e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} \\ &= \sum_{\ell,\ell' \in \hat{\mathcal{B}}} \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} e^{i\ell\cdot\mathrm{u}} \hat{a}_{\mathrm{k}}(\ell, \ell') e^{-i\ell'\cdot\mathrm{u}'} e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} \end{aligned}$$

(b) If, in addition, $a(\mathrm{u}, \mathrm{u}') = \alpha(\mathrm{u} - \mathrm{u}')$ *is translation invariant with respect to* $\mathcal{Z}_{\mathrm{fin}}$*, then*

$$\hat{a}_{\mathrm{k}}(\ell, \ell') = \delta_{\ell',\ell}\ \hat{\alpha}(\mathrm{k} + \ell)$$

where $\hat{\alpha}(\mathrm{p}) = \mathrm{vol}_f \sum_{\mathrm{u} \in \mathcal{Z}_{\mathrm{fin}}} \alpha(\mathrm{u}) e^{-i\mathrm{p}\cdot\mathrm{u}}$.

(c) Let A *be the periodization of* a*. Then*

$$\begin{aligned} A_{\mathrm{k}}([\mathrm{u}], [\mathrm{u}']) &= a_{\mathrm{k}}(\mathrm{u}, \mathrm{u}') \\ \hat{A}(\mathrm{k} + \ell, \mathrm{k} + \ell') &= \hat{a}_{\mathrm{k}}(\ell, \ell') \end{aligned}$$

for all $\mathrm{k} \in \frac{2\pi}{\varepsilon_T \mathcal{L}_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X \mathcal{L}_X}\mathbb{Z}^{\mathrm{d}}$, $[\mathrm{u}], [\mathrm{u}'] \in \mathcal{X}_{\mathrm{fin}}$ *and* $\ell, \ell' \in \hat{\mathcal{B}}$.

Internal Remark

Proof. (a) The periodicity of $a_{\mathrm{k}}(\mathrm{u}, \mathrm{u}')$ in u′ follows immediately from the definition and the observation that, if $\mathrm{x} \in \mathcal{Z}_{\mathrm{crs}}$, then $\mathrm{u}'' - \mathrm{u}' \in \mathcal{Z}_{\mathrm{crs}}$ if and only if $\mathrm{u}'' - (\mathrm{u}' + \mathrm{x}) \in \mathcal{Z}_{\mathrm{crs}}$. Since

$$\begin{aligned} a_{\mathrm{k}}(\mathrm{u} + \mathrm{x}, \mathrm{u}') &= \mathrm{vol}_c \sum_{\substack{\mathrm{u}'' \in \mathcal{Z}_{\mathrm{fin}} \\ \mathrm{u}'' - \mathrm{u}' \in \mathcal{Z}_{\mathrm{crs}}}} e^{-i\mathrm{k}\cdot(\mathrm{u}+\mathrm{x})} a(\mathrm{u} + \mathrm{x}, \mathrm{u}'') e^{i\mathrm{k}\cdot\mathrm{u}''} \\ &= \mathrm{vol}_c \sum_{\substack{\mathrm{u}'' \in \mathcal{Z}_{\mathrm{fin}} \\ \mathrm{u}'' - \mathrm{u}' \in \mathcal{Z}_{\mathrm{crs}}}} e^{-i\mathrm{k}\cdot(\mathrm{u}+\mathrm{x})} a(\mathrm{u} + \mathrm{x}, \mathrm{u}'' + \mathrm{x}) e^{i\mathrm{k}\cdot(\mathrm{u}''+\mathrm{x})} \\ &= \mathrm{vol}_c \sum_{\substack{\mathrm{u}'' \in \mathcal{Z}_{\mathrm{fin}} \\ \mathrm{u}'' - \mathrm{u}' \in \mathcal{Z}_{\mathrm{crs}}}} e^{-i\mathrm{k}\cdot\mathrm{u}} a(\mathrm{u}, \mathrm{u}'') e^{i\mathrm{k}\cdot\mathrm{u}''} \\ &= a_{\mathrm{k}}(\mathrm{u}, \mathrm{u}') \end{aligned}$$

for all $\mathrm{x} \in \mathcal{Z}_{\mathrm{crs}}$, $a_{\mathrm{k}}(\mathrm{u}, \mathrm{u}')$ is also periodic in u. For the first representation of $a(\mathrm{u}, \mathrm{u}')$, we compute

$$\begin{aligned}
\int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} a_{\mathrm{k}}(\mathrm{u}, \mathrm{u}') e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} &= \mathrm{vol}_c \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} \sum_{\substack{\mathrm{u}'' \in \mathcal{Z}_{\mathrm{fin}} \\ \mathrm{u}''-\mathrm{u}' \in \mathcal{Z}_{\mathrm{crs}}}} e^{-i\mathrm{k}\cdot\mathrm{u}} a(\mathrm{u}, \mathrm{u}'') e^{i\mathrm{k}\cdot\mathrm{u}''} e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} \\
&= \mathrm{vol}_c \sum_{\substack{\mathrm{u}'' \in \mathcal{Z}_{\mathrm{fin}} \\ \mathrm{u}''-\mathrm{u}' \in \mathcal{Z}_{\mathrm{crs}}}} a(\mathrm{u}, \mathrm{u}'') \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} e^{i\mathrm{k}\cdot(\mathrm{u}''-\mathrm{u}')} \tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} \\
&= \sum_{\substack{\mathrm{u}'' \in \mathcal{Z}_{\mathrm{fin}} \\ \mathrm{u}''-\mathrm{u}' \in \mathcal{Z}_{\mathrm{crs}}}} \delta_{\mathrm{u}',\mathrm{u}''} a(\mathrm{u}, \mathrm{u}'') \\
&= a(\mathrm{u}, \mathrm{u}')
\end{aligned}$$

since $\mathrm{u}''-\mathrm{u}'$ is restricted to $\mathcal{Z}_{\mathrm{crs}}$. For the second representation of $a(\mathrm{u}, \mathrm{u}')$, we compute

$$\begin{aligned}
\sum_{\ell,\ell' \in \hat{\mathcal{B}}} \hat{a}_{\mathrm{k}}(\ell, \ell') e^{i\ell\cdot\mathrm{u}} e^{-i\ell'\cdot\mathrm{u}'} &= \sum_{\ell,\ell' \in \hat{\mathcal{B}}} \tfrac{\mathrm{vol}_f}{|\hat{\mathcal{B}}|} \sum_{\substack{[\tilde{\mathrm{u}}] \in \mathcal{B} \\ \tilde{\mathrm{u}}' \in \mathcal{Z}_{\mathrm{fin}}}} a(\tilde{\mathrm{u}}, \tilde{\mathrm{u}}') e^{-i\mathrm{k}\cdot(\tilde{\mathrm{u}}-\tilde{\mathrm{u}}')} e^{-i\ell\cdot\tilde{\mathrm{u}}} e^{i\ell'\cdot\tilde{\mathrm{u}}'} e^{i\ell\cdot\mathrm{u}} e^{-i\ell'\cdot\mathrm{u}'} \\
&= \mathrm{vol}_f |\hat{\mathcal{B}}| \sum_{\substack{[\tilde{\mathrm{u}}] \in \mathcal{B} \\ \tilde{\mathrm{u}}' \in \mathcal{Z}_{\mathrm{fin}}}} a(\tilde{\mathrm{u}}, \tilde{\mathrm{u}}') e^{-i\mathrm{k}\cdot(\tilde{\mathrm{u}}-\tilde{\mathrm{u}}')} \delta_{[\mathrm{u}],[\tilde{\mathrm{u}}]} \delta_{[\mathrm{u}'],[\tilde{\mathrm{u}}']} \\
&= \mathrm{vol}_c \sum_{[\tilde{\mathrm{u}}] \in \mathcal{B}} \sum_{\substack{\tilde{\mathrm{u}}' \in \mathcal{Z}_{\mathrm{fin}} \\ \tilde{\mathrm{u}}'-\mathrm{u}' \in \mathcal{Z}_{\mathrm{crs}}}} a(\tilde{\mathrm{u}}, \tilde{\mathrm{u}}') e^{-i\mathrm{k}\cdot(\tilde{\mathrm{u}}-\tilde{\mathrm{u}}')} \delta_{[\mathrm{u}],[\tilde{\mathrm{u}}]} \\
&= \sum_{[\tilde{\mathrm{u}}] \in \mathcal{B}} a_{\mathrm{k}}(\tilde{\mathrm{u}}, \mathrm{u}') \delta_{[\mathrm{u}],[\tilde{\mathrm{u}}]} \\
&= a_{\mathrm{k}}(\mathrm{u}, \mathrm{u}')
\end{aligned}$$

(b)

$$\begin{aligned}
\hat{a}_{\mathrm{k}}(\ell, \ell') &= \tfrac{\mathrm{vol}_f}{|\mathcal{B}|} \sum_{\substack{[\mathrm{u}] \in \mathcal{B} \\ \mathrm{u}' \in \mathcal{Z}_{\mathrm{fin}}}} \alpha(\mathrm{u}-\mathrm{u}') e^{-i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} e^{-i\ell\cdot\mathrm{u}} e^{i\ell'\cdot\mathrm{u}'} \\
&= \tfrac{\mathrm{vol}_f}{|\mathcal{B}|} \sum_{\substack{[\mathrm{u}] \in \mathcal{B} \\ \mathrm{u}' \in \mathcal{Z}_{\mathrm{fin}}}} \alpha(\mathrm{u}') e^{-i\mathrm{k}\cdot\mathrm{u}'} e^{-i\ell\cdot\mathrm{u}} e^{-i\ell'\cdot(\mathrm{u}'-\mathrm{u})} \\
&= \delta_{\ell',\ell}\ \mathrm{vol}_f \sum_{\mathrm{u}' \in \mathcal{Z}_{\mathrm{fin}}} \alpha(\mathrm{u}') e^{-i\mathrm{k}\cdot\mathrm{u}'} e^{-i\ell'\cdot\mathrm{u}'} \\
&= \delta_{\ell',\ell}\ \hat{\alpha}(\mathrm{k}+\ell)
\end{aligned}$$

(c) By definition, for any $\mathrm{k} \in \frac{2\pi}{\varepsilon_T \mathcal{L}_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X \mathcal{L}_X}\mathbb{Z}^{\mathrm{d}}$,

$$\begin{aligned}
A_{\mathrm{k}}([\mathrm{u}],[\mathrm{u}']) &= \mathrm{vol}_c \sum_{\substack{y'' \in \mathcal{X}_{\mathrm{fin}} \\ y''-[\mathrm{u}'] \in \mathcal{X}_{\mathrm{crs}}}} e^{-i\mathrm{k}\cdot\mathrm{u}} A([\mathrm{u}], y'') e^{i\mathrm{k}\cdot y''} \\
&= \mathrm{vol}_c \sum_{\substack{[\mathrm{u}''] \in \mathcal{X}_{\mathrm{fin}} \\ [\mathrm{u}'']-[\mathrm{u}'] \in \mathcal{X}_{\mathrm{crs}} \\ \mathrm{z} \in \varepsilon_T \mathcal{L}_T \mathbb{Z} \times \varepsilon_X \mathcal{L}_X \mathbb{Z}^{\mathrm{d}}}} e^{-i\mathrm{k}\cdot\mathrm{u}} a(\mathrm{u}, \mathrm{u}''+\mathrm{z}) e^{i\mathrm{k}\cdot(\mathrm{u}''+\mathrm{z})} \\
&= \mathrm{vol}_c \sum_{\mathrm{z}' \in \mathcal{Z}_{\mathrm{crs}}} e^{-i\mathrm{k}\cdot\mathrm{u}} a(\mathrm{u}, \mathrm{u}'+\mathrm{z}') e^{i\mathrm{k}\cdot(\mathrm{u}'+\mathrm{z}')} \\
&= a_k(\mathrm{u}, \mathrm{u}')
\end{aligned}$$

and, for any $\mathrm{k} \in \frac{2\pi}{\varepsilon_T \mathcal{L}_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X \mathcal{L}_X}\mathbb{Z}^{\mathrm{d}}$ and $\ell, \ell' \in \hat{\mathcal{B}}$,

$$\begin{aligned}
\hat{A}(\mathrm{k}+\ell, \mathrm{k}+\ell') &= \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\mathrm{fin}}|} \sum_{y,y' \in \mathcal{X}_{\mathrm{fin}}} e^{-i(\mathrm{k}+\ell)\cdot y} A(y, y') e^{i(\mathrm{k}+\ell')\cdot y'} \\
&= \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\mathrm{fin}}|} \sum_{\substack{[\mathrm{u}],[\mathrm{u}'] \in \mathcal{X}_{\mathrm{fin}} \\ \mathrm{z} \in \varepsilon_T \mathcal{L}_T \mathbb{Z} \times \varepsilon_X \mathcal{L}_X \mathbb{Z}^{\mathrm{d}}}} e^{-i(\mathrm{k}+\ell)\cdot\mathrm{u}} a(\mathrm{u}, \mathrm{u}'+\mathrm{z}) e^{i(\mathrm{k}+\ell')\cdot\mathrm{u}'} \\
&= \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\mathrm{fin}}|} \sum_{\substack{[\mathrm{u}] \in \mathcal{X}_{\mathrm{fin}} \\ \mathrm{u}' \in \mathcal{Z}_{\mathrm{fin}}}} e^{-i\ell\cdot\mathrm{u}} a(\mathrm{u}, \mathrm{u}') e^{i\ell'\cdot\mathrm{u}'} e^{-i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \\
&= \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\mathrm{fin}}|} \sum_{[\mathrm{u}] \in \mathcal{X}_{\mathrm{fin}}} b_{\ell,\ell'}(\mathrm{u})
\end{aligned}$$

where

$$b_{\ell,\ell'}(\mathrm{u}) = \sum_{\mathrm{u}' \in \mathcal{Z}_{\mathrm{fin}}} e^{-i\ell\cdot\mathrm{u}} a(\mathrm{u}, \mathrm{u}') e^{i\ell'\cdot\mathrm{u}'} e^{-i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')}$$

Since $b_{\ell,\ell'}(\mathrm{u}+\mathrm{x}) = b_{\ell,\ell'}(\mathrm{u})$ for all $\mathrm{u} \in \mathcal{Z}_{\mathrm{fin}}$ and $\mathrm{x} \in \mathcal{Z}_{\mathrm{crs}}$,

$$\begin{aligned}
\hat{A}(\mathrm{k}+\ell, \mathrm{k}+\ell') &= \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\mathrm{fin}}|} \tfrac{|\mathcal{X}_{\mathrm{fin}}|}{|\mathcal{B}|} \sum_{[\mathrm{u}] \in \mathcal{B}} b_{\ell,\ell'}(\mathrm{u}) \\
&= \tfrac{\mathrm{vol}_f}{|\mathcal{B}|} \sum_{\substack{[\mathrm{u}] \in \mathcal{B} \\ \mathrm{u}' \in \mathcal{Z}_{\mathrm{fin}}}} e^{-i\ell\cdot\mathrm{u}} a(\mathrm{u}, \mathrm{u}') e^{i\ell'\cdot\mathrm{u}'} e^{-i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \\
&= \hat{a}_{\mathrm{k}}(\ell, \ell')
\end{aligned}$$

□

Lemma 6.

(a) If $a(\mathrm{u},\mathrm{u}') = \frac{1}{\mathrm{vol}_f}\delta_{\mathrm{u},\mathrm{u}'}$ *is the kernel of the identity operator, then* $\hat{a}_{\mathrm{k}}(\ell,\ell') = \delta_{\ell,\ell'}$.

(b) Let $a(\mathrm{u},\mathrm{u}'), b(\mathrm{u},\mathrm{u}') : \mathcal{Z}_{\mathrm{fin}} \times \mathcal{Z}_{\mathrm{fin}} \to \mathbb{C}$ *both obey the conditions of Definition 2, and set*

$$c(\mathrm{u},\mathrm{u}') = \mathrm{vol}_f \sum_{\mathrm{u}''\in\mathcal{Z}_{\mathrm{fin}}} a(\mathrm{u},\mathrm{u}'')b(\mathrm{u}'',\mathrm{u}')$$

Then, for all $\mathrm{k} \in \frac{2\pi}{\varepsilon_T\mathcal{L}_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X\mathcal{L}_X}\mathbb{Z}^{\mathrm{d}}$ *and* $\ell,\ell' \in \hat{\mathcal{B}}$,

$$c_{\mathrm{k}}(\ell,\ell') = \sum_{\ell''\in\hat{\mathcal{B}}} a_{\mathrm{k}}(\ell,\ell'')b_{\mathrm{k}}(\ell'',\ell')$$

Internal Remark

Proof. (a) Just apply Lemma 5.b with $\alpha(\mathrm{u}) = \frac{1}{\mathrm{vol}_f}\delta_{\mathrm{u},0}$.

(b)

$$\begin{aligned}
c_{\mathrm{k}}(\ell,\ell') &= \tfrac{\mathrm{vol}_f}{|\mathcal{B}|} \sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}}} e^{-i\ell\cdot\mathrm{u}}c(\mathrm{u},\mathrm{u}')e^{i\ell'\cdot\mathrm{u}'}e^{-i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \\
&= \tfrac{\mathrm{vol}_f^2}{|\mathcal{B}|} \sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{u}',\mathrm{u}''\in\mathcal{Z}_{\mathrm{fin}}}} e^{-i\ell\cdot\mathrm{u}}a(\mathrm{u},\mathrm{u}'')b(\mathrm{u}'',\mathrm{u}')e^{i\ell'\cdot\mathrm{u}'}e^{-i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \\
&= \tfrac{\mathrm{vol}_f^2}{|\mathcal{B}|^2}\sum_{\ell''\in\hat{\mathcal{B}}} \sum_{\substack{[\mathrm{u}],[\tilde{\mathrm{u}}]\in\mathcal{B}\\ \mathrm{u}',\mathrm{u}''\in\mathcal{Z}_{\mathrm{fin}}}} e^{-i\ell\cdot\mathrm{u}}a(\mathrm{u},\mathrm{u}'')e^{i\ell''\cdot\mathrm{u}''}e^{-i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}'')}\ e^{-i\ell''\cdot\tilde{\mathrm{u}}}b(\tilde{\mathrm{u}},\mathrm{u}')e^{i\ell'\cdot\mathrm{u}'}e^{-i\mathrm{k}\cdot(\tilde{\mathrm{u}}-\mathrm{u}')} \\
&= \sum_{\ell''\in\hat{\mathcal{B}}} a_{\mathrm{k}}(\ell,\ell'')b_{\mathrm{k}}(\ell'',\ell')
\end{aligned}$$

In the transition from the second line to the third line, we used $\sum_{\ell''\in\hat{\mathcal{B}}} e^{i\ell''\cdot(\mathrm{u}''-\tilde{\mathrm{u}})} = |\mathcal{B}|\delta_{[\mathrm{u}''],[\tilde{\mathrm{u}}]}$. If the representative $\tilde{\mathrm{u}}$ chosen from $[\tilde{\mathrm{u}}]$ is not u'', make the change of variables $\mathrm{u} \to \mathrm{u}' + \tilde{\mathrm{u}} - \mathrm{u}''$. □

We now generalize the above discussion to include periodized operators from $L^2(\mathcal{X}_{\rm crs})$ to $L^2(\mathcal{X}_{\rm fin})$ and vice versa. If $b(\mathrm{u},\mathrm{x})$ and $c(\mathrm{x},\mathrm{u})$ are translation invariant with respect to $\mathcal{Z}_{\rm crs}$ (with x running over $\mathcal{Z}_{\rm crs}$ and with u running over $\mathcal{Z}_{\rm fin}$ as usual) and have finite L^1–L^∞ norm, we define, for $\mathrm{k}\in\mathbb{R}\times\mathbb{R}^{\rm d}$ and $\ell,\ell'\in\hat{\mathcal{B}}$,

$$\hat{b}_{\rm k}(\ell)=\mathrm{vol}_f\sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{x}\in\mathcal{Z}_{\rm crs}}}e^{-i(\mathrm{k}+\ell)\cdot\mathrm{u}}b(\mathrm{u},\mathrm{x})e^{i\mathrm{k}\cdot\mathrm{x}}\quad \hat{c}_{\rm k}(\ell')=\mathrm{vol}_f\sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{x}\in\mathcal{Z}_{\rm crs}}}e^{-i\mathrm{k}\cdot\mathrm{x}}c(\mathrm{x},\mathrm{u})e^{i(\mathrm{k}+\ell')\cdot\mathrm{u}} \tag{9}$$

For $\mathrm{p}\in\frac{2\pi}{\varepsilon_T L_T}\mathbb{Z}\times\frac{2\pi}{\varepsilon_X L_X}\mathbb{Z}^{\rm d}$ and $\ell,\ell'\in\hat{\mathcal{B}}$

$$\hat{b}_{\rm k+p}(\ell)=\hat{b}_{\rm k}(\ell+\mathrm{p})\qquad \hat{c}_{\rm k+p}(\ell')=\hat{c}_{\rm k}(\ell'+\mathrm{p})$$

The inverse transforms are

$$b(\mathrm{u},\mathrm{x})=\sum_{\ell\in\hat{\mathcal{B}}}\int_{\hat{\mathcal{Z}}_{\rm crs}}e^{i\ell\cdot\mathrm{u}}\hat{b}_{\rm k}(\ell)e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{x})}\tfrac{d^{1+\rm d}\mathrm{k}}{(2\pi)^{1+\rm d}}$$
$$c(\mathrm{x},\mathrm{u})=\sum_{\ell'\in\hat{\mathcal{B}}}\int_{\hat{\mathcal{Z}}_{\rm crs}}\hat{a}_{\rm k}(\ell,\ell')e^{-i\ell'\cdot\mathrm{u}}e^{i\mathrm{k}\cdot(\mathrm{x}-\mathrm{u})}\tfrac{d^{1+\rm d}\mathrm{k}}{(2\pi)^{1+\rm d}}$$

For $\psi\in L^2(\mathcal{X}_{\rm crs})$ and $\phi\in L^2(\mathcal{X}_{\rm fin})$

$$\begin{aligned}\widehat{b\psi}(k+\ell)&=\hat{b}_k(\ell)\hat{\psi}(k)\\ \widehat{c\phi}(k)&=\sum_{\ell'\in\hat{\mathcal{B}}}\hat{c}_k(\ell')\hat{\phi}(k+\ell')\end{aligned} \tag{10}$$

Internal Remark

To prove the inverse transform formula for $b(\mathrm{u},\mathrm{x})$ we first compute the sum

$$\begin{aligned}\sum_{\ell\in\hat{\mathcal{B}}}\hat{b}_{\rm k}(\ell)e^{i\ell\cdot\mathrm{u}}&=\sum_{\ell\in\hat{\mathcal{B}}}\mathrm{vol}_f\sum_{\substack{[\tilde{\mathrm{u}}]\in\mathcal{B}\\ \tilde{\mathrm{x}}\in\mathcal{Z}_{\rm crs}}}b(\tilde{\mathrm{u}},\tilde{\mathrm{x}})e^{-i\mathrm{k}\cdot(\tilde{\mathrm{u}}-\tilde{\mathrm{x}})}e^{-i\ell\cdot\tilde{\mathrm{u}}}e^{i\ell\cdot\mathrm{u}}\\ &=\mathrm{vol}_f|\hat{\mathcal{B}}|\sum_{\substack{[\tilde{\mathrm{u}}]\in\mathcal{B}\\ \tilde{\mathrm{x}}\in\mathcal{Z}_{\rm crs}}}b(\tilde{\mathrm{u}},\tilde{\mathrm{x}})e^{-i\mathrm{k}\cdot(\tilde{\mathrm{u}}-\tilde{\mathrm{x}})}\delta_{[\mathrm{u}],[\tilde{\mathrm{u}}]}\\ &=\mathrm{vol}_c\sum_{\substack{[\tilde{\mathrm{u}}]\in\mathcal{B}\\ \tilde{\mathrm{x}}\in\mathcal{Z}_{\rm crs}}}b(\tilde{\mathrm{u}},\tilde{\mathrm{x}})e^{-i\mathrm{k}\cdot(\tilde{\mathrm{u}}-\tilde{\mathrm{x}})}\delta_{[\mathrm{u}],[\tilde{\mathrm{u}}]}\\ &=\mathrm{vol}_c\sum_{\tilde{\mathrm{x}}\in\mathcal{Z}_{\rm crs}}b(\mathrm{u},\tilde{\mathrm{x}})e^{-i\mathrm{k}\cdot(\mathrm{u}-\tilde{\mathrm{x}})}\end{aligned}$$

Substituting this into the right hand side of the claimed inverse formula

$$\begin{aligned}\sum_{\ell\in\hat{\mathcal{B}}}\int_{\hat{\mathcal{Z}}_{\rm crs}} e^{i\ell\cdot \mathrm{u}}\hat{b}_{\mathrm{k}}(\ell)e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{x})}\tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} &= \mathrm{vol}_c\sum_{\tilde{\mathrm{x}}\in\mathcal{Z}_{\rm crs}}\int_{\hat{\mathcal{Z}}_{\rm crs}} b(\mathrm{u},\tilde{\mathrm{x}})e^{-i\mathrm{k}\cdot(\mathrm{u}-\tilde{\mathrm{x}})}e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{x})}\tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}}\\ &= \mathrm{vol}_c\sum_{\tilde{\mathrm{x}}\in\mathcal{Z}_{\rm crs}} b(\mathrm{u},\tilde{\mathrm{x}})\int_{\hat{\mathcal{Z}}_{\rm crs}} e^{i\mathrm{k}\cdot(\tilde{\mathrm{x}}-\mathrm{x})}\tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}}\\ &= \sum_{\tilde{\mathrm{x}}\in\mathcal{Z}_{\rm crs}}\delta_{\tilde{\mathrm{x}},\mathrm{x}}b(\mathrm{u},\tilde{\mathrm{x}})\\ &= b(\mathrm{u},\mathrm{x})\end{aligned}$$

For the action of b in momentum space

$$\begin{aligned}\widehat{b\psi}(p) &= \mathrm{vol}_f\sum_{u\in\mathcal{X}_{\rm fin}}(b\psi)(u)e^{-ip\cdot u}\\ &= \mathrm{vol}_c\mathrm{vol}_f\sum_{\substack{[\mathrm{u}]\in\mathcal{X}_{\rm fin}\\ \mathrm{x}\in\mathcal{Z}_{\rm crs}}} b(\mathrm{u},\mathrm{x})\psi([\mathrm{x}])e^{-ip\cdot\mathrm{u}}\\ &= \mathrm{vol}_c\mathrm{vol}_f\tfrac{\widehat{\mathrm{vol}_c}}{(2\pi)^{1+\mathrm{d}}}\sum_{\substack{[\mathrm{u}]\in\mathcal{X}_{\rm fin}\\ \mathrm{x}\in\mathcal{Z}_{\rm crs}}}\sum_{k\in\hat{\mathcal{X}}_{\rm crs}} e^{-ip\cdot\mathrm{u}}b(\mathrm{u},\mathrm{x})e^{ik\cdot\mathrm{x}}\hat{\psi}(k)\\ &= \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\rm crs}|}\sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ [\tilde{\mathrm{x}}]\in\mathcal{X}_{\rm crs}}}\sum_{\substack{k\in\hat{\mathcal{X}}_{\rm crs}\\ \mathrm{x}\in\mathcal{Z}_{\rm crs}}} e^{-ip\cdot(\mathrm{u}+\tilde{\mathrm{x}})}b(\mathrm{u}+\tilde{\mathrm{x}},\mathrm{x})e^{ik\cdot\mathrm{x}}\hat{\psi}(k)\\ &= \mathrm{vol}_f\sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{x}\in\mathcal{Z}_{\rm crs}}}\sum_{k\in\hat{\mathcal{X}}_{\rm crs}}\left[\tfrac{1}{|\mathcal{X}_{\rm crs}|}\sum_{[\tilde{\mathrm{x}}]\in\mathcal{X}_{\rm crs}} e^{i(k-p)\cdot\tilde{\mathrm{x}}}\right]e^{-ip\cdot\mathrm{u}}b(\mathrm{u},\mathrm{x})e^{ik\cdot\mathrm{x}}\hat{\psi}(k)\\ &= \mathrm{vol}_f\sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{x}\in\mathcal{Z}_{\rm crs}}} e^{-ip\cdot\mathrm{u}}b(\mathrm{u},\mathrm{x})e^{ik\cdot\mathrm{x}}\hat{\psi}(k)\big|_{k=\hat{\pi}(p)}\\ &= \hat{b}_k(\ell)\hat{\psi}(k)\big|_{\substack{k=\hat{\pi}(p)\\ \ell=p-\hat{\pi}(p)}}\end{aligned}$$

For the action of c in momentum space

$$\begin{aligned}\widehat{(c\phi)}(k) &= \mathrm{vol}_c\sum_{x\in\mathcal{X}_{\rm crs}}(c\phi)(x)e^{-ik\cdot x}\\ &= \mathrm{vol}_c\mathrm{vol}_f\sum_{\substack{x\in\mathcal{X}_{\rm crs}\\ \mathrm{u}\in\mathcal{Z}_{\rm fin}}} c(x,\mathrm{u})\phi([\mathrm{u}])e^{-ik\cdot x}\\ &= \mathrm{vol}_c\mathrm{vol}_f\tfrac{\widehat{\mathrm{vol}_f}}{(2\pi)^{1+\mathrm{d}}}\sum_{\substack{x\in\mathcal{X}_{\rm crs}\\ \mathrm{u}\in\mathcal{Z}_{\rm fin}}}\sum_{p\in\hat{\mathcal{X}}_{\rm fin}} c(x,\mathrm{u})\hat{\phi}(p)e^{i\mathrm{u}\cdot p}e^{-ik\cdot x}\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{vol}_c \mathrm{vol}_f \tfrac{\widehat{\mathrm{vol}}_f}{(2\pi)^{1+\mathrm{d}}} \sum_{\substack{\mathrm{x}\in\mathcal{Z}_{\mathrm{crs}}\\ [\mathrm{u}]\in\mathcal{X}_{\mathrm{fin}}}} \sum_{p\in\hat{\mathcal{X}}_{\mathrm{fin}}} c(\mathrm{x},\mathrm{u})\hat{\phi}(p)e^{i\mathrm{u}\cdot p}e^{-ik\cdot\mathrm{x}} \\
&= \mathrm{vol}_c \mathrm{vol}_f \tfrac{\widehat{\mathrm{vol}}_f}{(2\pi)^{1+\mathrm{d}}} \sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ [\tilde{\mathrm{x}}]\in\mathcal{X}_{\mathrm{crs}}}} \sum_{\substack{p\in\hat{\mathcal{X}}_{\mathrm{fin}}\\ \mathrm{x}\in\mathcal{Z}_{\mathrm{crs}}}} c(\mathrm{x},\mathrm{u}+\tilde{\mathrm{x}})\hat{\phi}(p)e^{i(\mathrm{u}+\tilde{\mathrm{x}})\cdot p}e^{-ik\cdot\mathrm{x}} \\
&= \mathrm{vol}_f \sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{x}\in\mathcal{Z}_{\mathrm{crs}}}} \sum_{p\in\hat{\mathcal{X}}_{\mathrm{fin}}} \left[\tfrac{1}{|\mathcal{X}_{\mathrm{crs}}|}\sum_{[\tilde{\mathrm{x}}]\in\mathcal{X}_{\mathrm{crs}}} e^{i\tilde{\mathrm{x}}\cdot(p-k)}\right] c(\mathrm{x},\mathrm{u})\hat{\phi}(p)e^{i\mathrm{u}\cdot p}e^{-ik\cdot\mathrm{x}} \\
&= \mathrm{vol}_f \sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{x}\in\mathcal{Z}_{\mathrm{crs}}}} \sum_{\substack{p\in\hat{\mathcal{X}}_{\mathrm{fin}}\\ \hat{\pi}(p)=k}} e^{-ik\cdot\mathrm{x}} c(\mathrm{x},\mathrm{u})e^{i\mathrm{u}\cdot p}\hat{\phi}(p) \\
&= \mathrm{vol}_f \sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{x}\in\mathcal{Z}_{\mathrm{crs}}}} \sum_{\ell'\in\hat{\mathcal{B}}} e^{-ik\cdot\mathrm{x}} c(\mathrm{x},\mathrm{u})e^{i\mathrm{u}\cdot(k+\ell')}\hat{\phi}(k+\ell') \\
&= \sum_{\ell'\in\hat{\mathcal{B}}} \hat{c}_k(\ell')\hat{\phi}(k+\ell')
\end{aligned}
$$

If $b^\star(\mathrm{x},\mathrm{u}) = b(\mathrm{u},\mathrm{x})$ and $c^\star(\mathrm{u},\mathrm{x}) = c(\mathrm{x},\mathrm{u})$ are the transposes of b and c, respectively, then

$$\hat{b}^\star_{\mathrm{k}}(\ell') = \hat{b}_{-\mathrm{k}}(-\ell') \qquad \hat{c}^\star_{\mathrm{k}}(\ell) = \hat{c}_{-\mathrm{k}}(-\ell) \tag{11}$$

4 Averaging Operators

In this subsection, we analyze "averaging operators" as examples of periodic operators. Fix a function $q : \mathcal{X}_{\mathrm{fin}} \to \mathbb{R}$ and define the "averaging operator" $Q : \mathcal{H}_f \to \mathcal{H}_c$ by

$$(Q\phi)(x) = \mathrm{vol}_f \sum_{u\in\mathcal{X}_{\mathrm{fin}}} q(x-u)\phi(u) \tag{12}$$

Lemma 7.

(a) The adjoint $Q^\star$ is given by

$$\big(Q^\star\psi\big)(u) = \mathrm{vol}_c \sum_{x\in\mathcal{X}_{\mathrm{crs}}} \psi(x)q(x-u)$$

(b) The composite operators $QQ^\star$ and $Q^\star Q$ are given by

$$(QQ^\star\psi)(x) = \mathrm{vol}_f\mathrm{vol}_c \sum_{\substack{u\in\mathcal{X}_{\mathrm{fin}}\\ x'\in\mathcal{X}_{\mathrm{crs}}}} q(x-u)q(x'-u)\psi(x')$$

$$(Q^\star Q\phi)(u) = \mathrm{vol}_f\mathrm{vol}_c \sum_{\substack{u'\in\mathcal{X}_{\mathrm{fin}}\\ x\in\mathcal{X}_{\mathrm{crs}}}} q(x-u)q(x-u')\phi(u')$$

Proof. trivial. □

Internal Remark
(a) follows immediately from

$$\mathrm{vol}_f \sum_{u\in\mathcal{X}_{\mathrm{fin}}} \overline{\big(Q^\star\psi\big)(u)}\phi(u) = \langle Q^\star\psi,\phi\rangle_f = \langle\psi,Q\phi\rangle_c$$
$$= \mathrm{vol}_f\mathrm{vol}_c \sum_{\substack{x\in\mathcal{X}_{\mathrm{crs}}\\ u\in\mathcal{X}_{\mathrm{fin}}}} \overline{\psi(x)}q(x-u)\phi(u)$$

(b) is trivial.

Example 8. Assume that L_T and L_X are odd and choose q to be $\frac{1}{\mathrm{vol}_c}$ times the characteristic function of the rectangle $\varepsilon_T\big[-\frac{L_T-1}{2},\frac{L_T-1}{2}\big]\times\varepsilon_X\big[-\frac{L_X-1}{2},\frac{L_X-1}{2}\big]^{\mathrm{d}}$ in $\mathcal{X}_{\mathrm{fin}}$. Observe that the number of points in this rectangle is exactly $L_TL_X^{\mathrm{d}}$. For $x\in\mathcal{X}_{\mathrm{crs}}$, denote by $\square_x$ the rectangle $x+\varepsilon_T\big[-\frac{L_T-1}{2},\frac{L_T-1}{2}\big]\times\varepsilon_X\big[-\frac{L_X-1}{2},\frac{L_X-1}{2}\big]^{\mathrm{d}}$ in $\mathcal{X}_{\mathrm{fin}}$. Also, for $u\in\mathcal{X}_{\mathrm{fin}}$, let $\xi(u)$ be the point of $\mathcal{X}_{\mathrm{crs}}$ closest to u. Then

$$(Q\phi)(x) = \tfrac{1}{L_TL_X^{\mathrm{d}}}\sum_{u\in\square_x}\phi(u) \qquad \big(Q^\star\psi\big)(u) = \psi\big(\xi(u)\big)$$

The composite operators are

$$(QQ^\star\psi)(x) = \tfrac{1}{L_TL_X^{\mathrm{d}}}\sum_{u\in\square_x}(Q^\star\psi)(u) = \tfrac{1}{L_TL_X^{\mathrm{d}}}\sum_{u\in\square_x}\psi(x) = \psi(x)$$
$$(Q^\star Q\phi)(u) = (Q\phi)\big(\xi(u)\big) = \tfrac{1}{L_TL_X^{\mathrm{d}}}\sum_{u'\in\square_{\xi(u)}}\phi(u')$$

Lemma 9. *Let $Q:\mathcal{H}_f\to\mathcal{H}_c$ be the averaging operator of (12), but with $q:\mathcal{Z}_{\mathrm{fin}}\to\mathbb{R}$ and $q(\mathrm{u})$ vanishing unless $|\mathrm{u}_0|<\frac{1}{2}\varepsilon_T\mathcal{L}_T$ and $|\mathrm{u}_\nu|<\frac{1}{2}\varepsilon_X\mathcal{L}_X$ for $\nu=1,2,3$.*

(a) For all $\phi \in \mathcal{H}_f$ and $\psi \in \mathcal{H}_c$,

$$\widehat{(Q\phi)}(k) = \sum_{\substack{p\in\hat{\mathcal{X}}_{\text{fin}}\\ \hat{\pi}(p)=k}} \hat{q}(p)\hat{\phi}(p) \qquad \widehat{(Q^\star\psi)}(p) = \overline{\hat{q}(p)}\,\hat{\psi}\big(\hat{\pi}(p)\big)$$

$$\widehat{(QQ^\star\psi)}(k) = \Big(\sum_{\substack{p\in\hat{\mathcal{X}}_{\text{fin}}\\ \hat{\pi}(p)=k}} \big|\hat{q}(p)\big|^2\Big)\hat{\psi}(k) \qquad \widehat{(Q^\star Q\phi)}(p) = \overline{\hat{q}(p)} \sum_{\substack{p'\in\hat{\mathcal{X}}_{\text{fin}}\\ \hat{\pi}(p')=\hat{\pi}(p)}} \hat{q}(p')\,\hat{\phi}(p')$$

(b) For $A = Q^\star Q$,

$$\hat{a}_{\mathrm{k}}(\ell,\ell') = \overline{\hat{q}(\mathrm{k}+\ell)}\,\hat{q}(\mathrm{k}+\ell')$$

Proof. (a) Using the definitions and (2),

$$\begin{aligned}
\widehat{(Q\phi)}(k) &= \mathrm{vol}_c \sum_{x\in\mathcal{X}_{\text{crs}}} (Q\phi)(x) e^{-ik\cdot x} && = \mathrm{vol}_f \mathrm{vol}_c \sum_{\substack{x\in\mathcal{X}_{\text{crs}}\\ u\in\mathcal{X}_{\text{fin}}}} e^{-ik\cdot x} q(x-u)\phi(u)\\
&= \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\text{crs}}|} \sum_{\substack{x\in\mathcal{X}_{\text{crs}}\\ u\in\mathcal{X}_{\text{fin}}\\ p\in\hat{\mathcal{X}}_{\text{fin}}}} e^{-ik\cdot x} e^{iu\cdot p} q(x-u)\hat{\phi}(p) && = \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\text{crs}}|} \sum_{\substack{x\in\mathcal{X}_{\text{crs}}\\ u\in\mathcal{X}_{\text{fin}}\\ p\in\hat{\mathcal{X}}_{\text{fin}}}} e^{-ik\cdot x} e^{i(x-u)\cdot p} q(u)\hat{\phi}(p)\\
&= \tfrac{1}{|\mathcal{X}_{\text{crs}}|} \sum_{\substack{x\in\mathcal{X}_{\text{crs}}\\ p\in\hat{\mathcal{X}}_{\text{fin}}}} e^{-i(k-p)\cdot x} \hat{q}(p)\hat{\phi}(p) && = \tfrac{1}{|\mathcal{X}_{\text{crs}}|} \sum_{\substack{x\in\mathcal{X}_{\text{crs}}\\ p\in\hat{\mathcal{X}}_{\text{fin}}}} e^{-i(k-\hat{\pi}(p))\cdot x} \hat{q}(p)\hat{\phi}(p)\\
&= \sum_{\substack{p\in\hat{\mathcal{X}}_{\text{fin}}\\ \hat{\pi}(p)=k}} \hat{q}(p)\hat{\phi}(p)
\end{aligned}$$

The computation for $\widehat{(Q^\star\psi)}(p)$ is similar.

Internal Remark

$$\begin{aligned}
\widehat{(Q^\star\psi)}(p) &= \mathrm{vol}_f \sum_{u\in\mathcal{X}_{\text{fin}}} (Q^\star\psi)(u) e^{-ip\cdot u} && = \mathrm{vol}_f \mathrm{vol}_c \sum_{\substack{u\in\mathcal{X}_{\text{fin}}\\ x\in\mathcal{X}_{\text{crs}}}} e^{-ip\cdot u}\psi(x)q(x-u)\\
&= \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\text{crs}}|} \sum_{\substack{x\in\mathcal{X}_{\text{crs}}\\ u\in\mathcal{X}_{\text{fin}}\\ k\in\hat{\mathcal{X}}_{\text{crs}}}} e^{ik\cdot x} e^{-ip\cdot u} q(x-u)\hat{\psi}(k) && = \tfrac{\mathrm{vol}_f}{|\mathcal{X}_{\text{crs}}|} \sum_{\substack{x\in\mathcal{X}_{\text{crs}}\\ u\in\mathcal{X}_{\text{fin}}\\ k\in\hat{\mathcal{X}}_{\text{crs}}}} e^{ik\cdot x} e^{-ip\cdot(x-u)} q(u)\hat{\psi}(k)\\
&= \tfrac{1}{|\mathcal{X}_{\text{crs}}|} \sum_{\substack{x\in\mathcal{X}_{\text{crs}}\\ k\in\hat{\mathcal{X}}_{\text{crs}}}} e^{i(k-p)\cdot x} \overline{\hat{q}(p)}\hat{\psi}(k) && = \tfrac{1}{|\mathcal{X}_{\text{crs}}|} \sum_{\substack{x\in\mathcal{X}_{\text{crs}}\\ k\in\hat{\mathcal{X}}_{\text{crs}}}} e^{i(k-\hat{\pi}(p))\cdot x} \overline{\hat{q}(p)}\hat{\psi}(k)\\
&= \overline{\hat{q}(p)}\,\hat{\psi}\big(\hat{\pi}(p)\big)
\end{aligned}$$

For the composite operators

$$\widehat{(QQ^\star\psi)}(k) = \sum_{\substack{p\in\hat{\mathcal{X}}_{\rm fin}\\ \hat\pi(p)=k}} \hat q(p)\widehat{(Q^\star\psi)}(p) = \sum_{\substack{p\in\hat{\mathcal{X}}_{\rm fin}\\ \hat\pi(p)=k}} \hat q(p)\overline{\hat q(p)}\,\hat\psi\big(\hat\pi(p)\big) = \sum_{\substack{p\in\hat{\mathcal{X}}_{\rm fin}\\ \hat\pi(p)=k}} \big|\hat q(p)\big|^2\,\hat\psi(k)$$

and similarly for $\widehat{(Q^\star Q\phi)}(p)$.

Internal Remark

$$\widehat{(Q^\star Q\phi)}(p) = \overline{\hat q(p)}\,\widehat{(Q\phi)}\big(\hat\pi(p)\big) = \sum_{\substack{p'\in\hat{\mathcal{X}}_{\rm fin}\\ \hat\pi(p')=\hat\pi(p)}} \overline{\hat q(p)}\hat q(p')\,\hat\phi(p')$$

(b) Since

$$a(\mathrm{u},\mathrm{u}') = \mathrm{vol}_c \sum_{\mathrm{x}\in\mathcal{Z}_{\rm crs}} q(\mathrm{x}-\mathrm{u})q(\mathrm{x}-\mathrm{u}')$$

we have

$$\begin{aligned}
\hat a_{\mathrm{k}}(\ell,\ell') &= \tfrac{\mathrm{vol}_f\mathrm{vol}_c}{|\mathcal{B}|} \sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{u}'\in\mathcal{Z}_{\rm fin}\\ \mathrm{x}\in\mathcal{Z}_{\rm crs}}} e^{-i(\mathrm{k}+\ell)\cdot(\mathrm{u}-\mathrm{x})} q(\mathrm{x}-\mathrm{u})q(\mathrm{x}-\mathrm{u}')e^{i(\mathrm{k}+\ell')\cdot(\mathrm{u}'-\mathrm{x})}\\
&= \mathrm{vol}_f^2 \sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{u}'\in\mathcal{Z}_{\rm fin}\\ \mathrm{x}\in\mathcal{Z}_{\rm crs}}} e^{-i(\mathrm{k}+\ell)\cdot(\mathrm{u}-\mathrm{x})} q(\mathrm{x}-\mathrm{u})q(\mathrm{u}')e^{-i(\mathrm{k}+\ell')\cdot\mathrm{u}'}\\
&= \mathrm{vol}_f^2 \sum_{\mathrm{u},\mathrm{u}'\in\mathcal{Z}_{\rm fin}} e^{i(\mathrm{k}+\ell)\cdot\mathrm{u}} q(\mathrm{u})q(\mathrm{u}')e^{-i(\mathrm{k}+\ell')\cdot\mathrm{u}'}\\
&= \overline{\hat q(\mathrm{k}+\ell)}\,\hat q(\mathrm{k}+\ell')
\end{aligned}$$

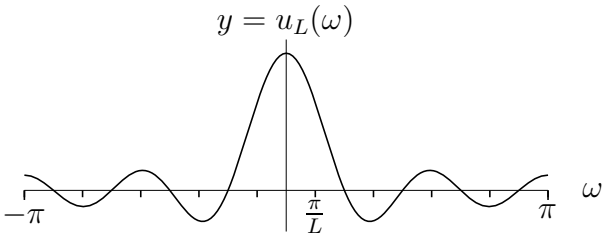
□

Example 10 (Example 8, continued)**.** In the notation of Example 8, the Fourier transform of q is

$$\hat q(p) = \tfrac{\mathrm{vol}_f}{\mathrm{vol}_c} \sum_{\substack{u\in\mathcal{X}_{\rm fin}\\ u\in\square_0}} e^{-ip\cdot u} = u_{L_T}(\varepsilon_T p_0)\prod_{\ell=1}^{\mathrm{d}} u_{L_X}(\varepsilon_X p_\ell)$$

where

$$u_L(\omega) = \tfrac{1}{L} \sum_{k=-\frac{L-1}{2}}^{\frac{L-1}{2}} e^{-i\omega k} = \begin{cases} \frac{1}{L}\frac{\sin\frac{L\omega}{2}}{\sin\frac{\omega}{2}} & \text{if } \omega \notin 2\pi\mathbb{Z} \\ 1 & \text{otherwise} \end{cases}$$



Remark 11. For the q of Examples 8 and 10, which is, up to a multiplicative constant, the characteristic function of a rectangle, the Fourier transform $\hat{q}(p)$ decays relatively slowly for large p. Choosing a smoother q increases the rate of decay of $\hat{q}(p)$. A convenient way to "smooth off" Q is to select an even $\mathfrak{q} \in \mathbb{N}$ and choose q to be the inverse Fourier transform of

$$\hat{q}(p) = u_{L_T}(\varepsilon_T p_0)^{\mathfrak{q}} \prod_{\ell=1}^{\mathrm{d}} u_{L_X}(\varepsilon_X p_\ell)^{\mathfrak{q}}$$

For example, when $\mathfrak{q} = 2$, q is the convolution of (a constant times) the characteristic function of a rectangle with itself and so is a "tent" function. In [6, 7], we use $\mathfrak{q} \geq 4$.

5 Analyticity of the Fourier Transform and L^1–L^∞ Norms

Define, for any $m \geq 0$ and $\mathrm{a} : \mathbb{X} \times \mathbb{X}' \to \mathbb{C}$, with $\mathbb{X}$ and $\mathbb{X}'$ being any of our lattices,

$$\|\mathrm{a}\|_m = \max\Big\{ \sup_{y\in\mathbb{X}} \mathrm{vol}_{X'} \sum_{y'\in\mathbb{X}'} e^{m|y-y'|} |\mathrm{a}(y,y')| \ , \ \sup_{y'\in\mathbb{X}'} \mathrm{vol}_X \sum_{y\in\mathbb{X}} e^{m|y-y'|} |\mathrm{a}(y,y')| \Big\}$$

Here vol_X and $\mathrm{vol}_{X'}$ is the volume of a single cell in X and X', respectively.

Lemma 12. *Let $a(\mathrm{u},\mathrm{u}') : \mathcal{Z}_{\mathrm{fin}} \times \mathcal{Z}_{\mathrm{fin}} \to \mathbb{C}$, $b(\mathrm{u},\mathrm{x}) : \mathcal{Z}_{\mathrm{fin}} \times \mathcal{Z}_{\mathrm{crs}} \to \mathbb{C}$ and $c(\mathrm{x},\mathrm{u}) : \mathcal{Z}_{\mathrm{crs}} \times \mathcal{Z}_{\mathrm{fin}} \to \mathbb{C}$ be translation invariant with respect to $\mathcal{Z}_{\mathrm{crs}}$ and have finite L^1–L^∞ norms. Let $0 < m'' < m' < m$.*

(a) If $\|a\|_m < \infty$, then, for each $\ell, \ell' \in \hat{\mathcal{B}}$, $\hat{a}_{\mathrm{k}}(\ell,\ell')$ is analytic in $|\mathrm{Im\,k}| < m$ and

$$\sup_{|\mathrm{Im\,k}|<m} \big|\hat{a}_{\mathrm{k}}(\ell,\ell')\big| \leq \|a\|_m$$

(b) If, for each $\ell, \ell' \in \hat{\mathcal{B}}$, $\hat{a}_{\mathrm{k}}(\ell, \ell')$ is analytic in $|\mathrm{Im}\,\mathrm{k}| < m$, then,

$$\sup_{\mathrm{u},\mathrm{u}' \in \mathcal{Z}_{\mathrm{fin}}} \big|a(\mathrm{u},\mathrm{u}')\big| e^{m'|\mathrm{u}-\mathrm{u}'|} \leq \tfrac{1}{\mathrm{vol}_c} \sup_{|\mathrm{Im}\,\mathrm{k}|=m'} \sum_{\ell,\ell' \in \hat{\mathcal{B}}} \big|\hat{a}_{\mathrm{k}}(\ell,\ell')\big| \;\leq \tfrac{|\mathcal{B}|}{\mathrm{vol}_f} \sup_{\substack{|\mathrm{Im}\,\mathrm{k}|=m' \\ \ell,\ell' \in \hat{\mathcal{B}}}} \big|\hat{a}_{\mathrm{k}}(\ell,\ell')\big|$$

$$\|A\|_{m''} \leq \|a\|_{m''} \leq \tfrac{C_{m'-m''}}{\mathrm{vol}_c} \sup_{|\mathrm{Im}\,\mathrm{k}|=m'} \sum_{\ell,\ell' \in \hat{\mathcal{B}}} \big|\hat{a}_{\mathrm{k}}(\ell,\ell')\big| \leq \tfrac{C_{m'-m''}|\mathcal{B}|}{\mathrm{vol}_f} \sup_{\substack{|\mathrm{Im}\,\mathrm{k}|=m' \\ \ell,\ell' \in \hat{\mathcal{B}}}} \big|\hat{a}_{\mathrm{k}}(\ell,\ell')\big|$$

where A is the periodization of a and $C_{m'-m''} = \mathrm{vol}_f \sum_{\mathrm{u} \in \mathcal{Z}_{\mathrm{fin}}} e^{-(m'-m'')|\mathrm{u}|}$.

(c) If, for each $\ell \in \hat{\mathcal{B}}$, $\hat{b}_{\mathrm{k}}(\ell)$ is analytic in $|\mathrm{Im}\,\mathrm{k}| < m$, then,

$$\sup_{\substack{\mathrm{u} \in \mathcal{Z}_{\mathrm{fin}} \\ \mathrm{x} \in \mathcal{Z}_{\mathrm{crs}}}} \big|b(\mathrm{u},\mathrm{x})\big| e^{m'|\mathrm{u}-\mathrm{x}|} \leq \tfrac{1}{\mathrm{vol}_c} \sup_{|\mathrm{Im}\,\mathrm{k}|=m'} \sum_{\ell \in \hat{\mathcal{B}}} \big|\hat{b}_{\mathrm{k}}(\ell)\big| \leq \tfrac{1}{\mathrm{vol}_f} \sup_{\substack{|\mathrm{Im}\,\mathrm{k}|=m' \\ \ell \in \hat{\mathcal{B}}}} \big|\hat{b}_{\mathrm{k}}(\ell)\big|$$

If, for each $\ell' \in \hat{\mathcal{B}}$, $\hat{c}_{\mathrm{k}}(\ell')$ is analytic in $|\mathrm{Im}\,\mathrm{k}| < m$, then,

$$\sup_{\substack{\mathrm{u} \in \mathcal{Z}_{\mathrm{fin}} \\ \mathrm{x} \in \mathcal{Z}_{\mathrm{crs}}}} \big|c(\mathrm{x},\mathrm{u})\big| e^{m'|\mathrm{x}-\mathrm{u}|} \leq \tfrac{1}{\mathrm{vol}_c} \sup_{|\mathrm{Im}\,\mathrm{k}|=m'} \sum_{\ell' \in \hat{\mathcal{B}}} \big|\hat{c}_{\mathrm{k}}(\ell')\big| \leq \tfrac{1}{\mathrm{vol}_f} \sup_{\substack{|\mathrm{Im}\,\mathrm{k}|=m' \\ \ell' \in \hat{\mathcal{B}}}} \big|\hat{c}_{\mathrm{k}}(\ell')\big|$$

Proof. (a) If $|\mathrm{Im}\,\mathrm{k}| < m$, then

$$\big|\hat{a}_{\mathrm{k}}(\ell,\ell')\big| \leq \tfrac{\mathrm{vol}_f}{|\mathcal{B}|} \sum_{\substack{[\mathrm{u}] \in \mathcal{B} \\ \mathrm{u}' \in \mathcal{Z}_{\mathrm{fin}}}} \big|a(\mathrm{u},\mathrm{u}')\big| e^{m|\mathrm{u}-\mathrm{u}'|} \leq \tfrac{1}{|\mathcal{B}|} \sum_{[\mathrm{u}] \in \mathcal{B}} \|a\|_m \leq \|a\|_m$$

Analyticity in k follows from the uniform convergence of the series on $|\mathrm{Im}\,\mathrm{k}| < m$.

(b) Fix any $\mathrm{u}, \mathrm{u}' \in \mathcal{Z}_{\mathrm{fin}}$. Set $\mathrm{q} = m' \frac{\mathrm{u}-\mathrm{u}'}{|\mathrm{u}-\mathrm{u}'|}$. Then

$$\begin{aligned} a(\mathrm{u},\mathrm{u}') e^{m'|\mathrm{u}-\mathrm{u}'|} &= \sum_{\ell,\ell' \in \hat{\mathcal{B}}} \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} \hat{a}_{\mathrm{k}}(\ell,\ell') e^{i(\mathrm{k}-i\mathrm{q})\cdot(\mathrm{u}-\mathrm{u}')} e^{i\ell\cdot\mathrm{u}} e^{-i\ell'\cdot\mathrm{u}'} \tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} \\ &= \sum_{\ell,\ell' \in \hat{\mathcal{B}}} \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} \hat{a}_{\mathrm{k}+i\mathrm{q}}(\ell,\ell') e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} e^{i\ell\cdot\mathrm{u}} e^{-i\ell'\cdot\mathrm{u}'} \tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} \end{aligned}$$

where we have applied Stokes' theorem, using analyticity in k and the fact that

$$e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \sum_{\ell,\ell' \in \hat{\mathcal{B}}} \hat{a}_{\mathrm{k}}(\ell,\ell') e^{i\ell\cdot\mathrm{u}} e^{-i\ell'\cdot\mathrm{u}'} = e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} a_{\mathrm{k}}(\mathrm{u},\mathrm{u}')$$

is periodic in the real part of k with respect to $\frac{2\pi}{\varepsilon_T L_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X L_X}\mathbb{Z}^{\mathrm{d}}$. Hence

$$\begin{aligned}\big|a(\mathrm{u},\mathrm{u}')\big|e^{m'|\mathrm{u}-\mathrm{u}'|} &\le \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} \sum_{\ell,\ell'\in\hat{\mathcal{B}}} \big|\hat{a}_{\mathrm{k}+i\mathrm{q}}(\ell,\ell')\big| \tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} \\ &\le \tfrac{1}{\mathrm{vol}_c} \sup_{\mathrm{k}\in\hat{\mathcal{Z}}_{\mathrm{crs}}} \sum_{\ell,\ell'\in\hat{\mathcal{B}}} \big|\hat{a}_{\mathrm{k}+i\mathrm{q}}(\ell,\ell')\big| \\ &\le \tfrac{|\mathcal{B}|}{\mathrm{vol}_f} \sup_{\substack{\mathrm{k}\in\hat{\mathcal{Z}}_{\mathrm{crs}} \\ \ell,\ell'\in\hat{\mathcal{B}}}} \big|\hat{a}_{\mathrm{k}+i\mathrm{q}}(\ell,\ell')\big|\end{aligned}$$

The second bound is obvious from

$$\begin{aligned}\mathrm{vol}_f \sum_{y'\in\mathcal{X}_{\mathrm{fin}}} \big|A\big([\mathrm{u}],y'\big)\big|e^{m''|[\mathrm{u}]-y'|} &= \mathrm{vol}_f \sum_{y'\in\mathcal{X}_{\mathrm{fin}}} \Big| \sum_{\substack{\mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}} \\ [\mathrm{u}']=y'}} a(\mathrm{u},\mathrm{u}')\Big|e^{m''|[\mathrm{u}]-y'|} \\ &\le \mathrm{vol}_f \sum_{\mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}} \big|a(\mathrm{u},\mathrm{u}')\big|e^{m''|\mathrm{u}-\mathrm{u}'|}\end{aligned}$$

(with the distance $|[\mathrm{u}]-y'|$ measured in $\mathcal{X}_{\mathrm{fin}}$ and the distance $|\mathrm{u}-\mathrm{u}'|$ measure in $\mathcal{Z}_{\mathrm{fin}}$) and the similar bound with the roles of u and u′ interchanged.

(c) The proof is much the same as that of part (b).

Internal Remark

If $\mathrm{q} = m'\frac{\mathrm{u}-\mathrm{x}}{|\mathrm{u}-\mathrm{x}|}$, then

$$\begin{aligned}b(\mathrm{u},\mathrm{x})e^{m'|\mathrm{u}-\mathrm{x}|} &= \sum_{\ell\in\hat{\mathcal{B}}} \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} e^{i\ell\cdot\mathrm{u}}\hat{b}_{\mathrm{k}}(\ell)e^{i(\mathrm{k}-i\mathrm{q})\cdot(\mathrm{u}-\mathrm{x})}\tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} \\ &= \sum_{\ell\in\hat{\mathcal{B}}} \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} e^{i\ell\cdot\mathrm{u}}\hat{b}_{\mathrm{k}+i\mathrm{q}}(\ell)e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{x})}\tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}}\end{aligned}$$

so that

$$\begin{aligned}|b(\mathrm{u},\mathrm{x})|e^{m'|\mathrm{u}-\mathrm{x}|} &\le \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} \sum_{\ell\in\hat{\mathcal{B}}} |\hat{b}_{\mathrm{k}+i\mathrm{q}}(\ell)|\tfrac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} \le \tfrac{1}{\mathrm{vol}_c} \sup_{\mathrm{k}\in\hat{\mathcal{Z}}_{\mathrm{crs}}} \sum_{\ell\in\hat{\mathcal{B}}} \big|\hat{b}_{\mathrm{k}+i\mathrm{q}}(\ell)\big| \\ &\le \tfrac{1}{\mathrm{vol}_f} \sup_{\substack{\mathrm{k}\in\hat{\mathcal{Z}}_{\mathrm{crs}} \\ \ell\in\hat{\mathcal{B}}}} \big|\hat{b}_{\mathrm{k}+i\mathrm{q}}(\ell)\big|\end{aligned}$$

□

Lemma 13. *Let $m > 0$. Let, for each $\ell, \ell' \in \hat{\mathcal{B}}$, $\hat{b}_{\mathrm{k}}(\ell, \ell')$ be analytic in $|\mathrm{Im\,k}| < m$. Assume that*

$$\hat{b}_{\mathrm{k+p}}(\ell, \ell') = \hat{b}_{\mathrm{k}}(\ell + \mathrm{p}, \ell' + \mathrm{p})$$

for all $p \in \frac{2\pi}{\varepsilon_T L_T}\mathbb{Z} \times \frac{2\pi}{\varepsilon_X L_X}\mathbb{Z}^{\mathrm{d}}$ and $\ell, \ell' \in \hat{\mathcal{B}}$. Set

$$a(\mathrm{u}, \mathrm{u}') = \sum_{\ell, \ell' \in \hat{\mathcal{B}}} \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} e^{i\ell\cdot\mathrm{u}} \hat{b}_{\mathrm{k}}(\ell, \ell') e^{-i\ell'\cdot\mathrm{u}'} e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \frac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}}$$

Then $a(\mathrm{u}, \mathrm{u}') : \mathcal{Z}_{\mathrm{fin}} \times \mathcal{Z}_{\mathrm{fin}} \to \mathbb{C}$ obeys the conditions of Definition 2 and

$$\hat{a}_{\mathrm{k}}(\ell, \ell') = \hat{b}_{\mathrm{k}}(\ell, \ell')$$

for all $\mathrm{k} \in \mathbb{R} \times \mathbb{R}^{\mathrm{d}}$ and $\ell, \ell' \in \hat{\mathcal{B}}$.

Proof. The proof is straightforward. □

Internal Remark

Proof. The $\hat{b}_{\mathrm{k+p}}(\ell, \ell') = \hat{b}_{\mathrm{k}}(\ell + \mathrm{p}, \ell' + \mathrm{p})$ hypothesis ensures that $a(\mathrm{u}, \mathrm{u}')$ is well–defined. The L^1–L^∞ condition of Definition 2 is satisfied by Lemma 12.b. For the translation invariance, let $\mathrm{z} \in \mathcal{Z}_{\mathrm{crs}}$ and $\mathrm{u}, \mathrm{u}' \in \mathcal{Z}_{\mathrm{fin}}$. Then $\ell \cdot \mathrm{z} \in 2\pi\mathbb{Z}$ for all $\ell \in \hat{\mathcal{B}}$ and

$$\begin{aligned} a(\mathrm{u}+\mathrm{z}, \mathrm{u}'+\mathrm{z}) &= \sum_{\ell, \ell' \in \hat{\mathcal{B}}} \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} e^{i\ell\cdot(\mathrm{u}+\mathrm{z})} \hat{b}_{\mathrm{k}}(\ell, \ell') e^{-i\ell'\cdot(\mathrm{u}'+\mathrm{z})} e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \frac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} \\ &= \sum_{\ell, \ell' \in \hat{\mathcal{B}}} \int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} e^{i\ell\cdot\mathrm{u}} \hat{b}_{\mathrm{k}}(\ell, \ell') e^{-i\ell'\cdot\mathrm{u}'} e^{i\mathrm{k}\cdot(\mathrm{u}-\mathrm{u}')} \frac{d^{1+\mathrm{d}}\mathrm{k}}{(2\pi)^{1+\mathrm{d}}} \\ &= a(\mathrm{u}, \mathrm{u}') \end{aligned}$$

Finally,

$$\begin{aligned}
\hat{a}_{\mathrm{k}}(\ell,\ell') &= \tfrac{\mathrm{vol}_f}{|\mathcal{B}|}\sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}}}\sum_{\tilde{\ell},\tilde{\ell}'\in\hat{\mathcal{B}}}\int_{\hat{\mathcal{Z}}_{\mathrm{crs}}} e^{i(\tilde{\ell}-\ell)\cdot\mathrm{u}}\hat{b}_{\tilde{\mathrm{k}}}(\tilde{\ell},\tilde{\ell}')e^{-i(\tilde{\ell}'-\ell')\cdot\mathrm{u}'}e^{i(\tilde{\mathrm{k}}-\mathrm{k})\cdot(\mathrm{u}-\mathrm{u}')}\tfrac{d^{1+\mathrm{d}}\tilde{\mathrm{k}}}{(2\pi)^{1+\mathrm{d}}}\\
&= \tfrac{\mathrm{vol}_f}{|\mathcal{B}|^2}\sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}}}\sum_{\tilde{\ell},\tilde{\ell}'\in\hat{\mathcal{B}}}\int_{\hat{\mathcal{Z}}_{\mathrm{fin}}} e^{i(\tilde{\ell}-\ell+\tilde{\mathrm{k}}-\mathrm{k})\cdot\mathrm{u}}\hat{b}_{\tilde{\mathrm{k}}}(\tilde{\ell},\tilde{\ell}')e^{-i(\tilde{\ell}'-\ell'+\tilde{\mathrm{k}}-\mathrm{k})\cdot\mathrm{u}'}\tfrac{d^{1+\mathrm{d}}\tilde{\mathrm{k}}}{(2\pi)^{1+\mathrm{d}}}\\
&= \tfrac{1}{|\mathcal{B}|^2}\sum_{[\mathrm{u}]\in\mathcal{B}}\sum_{\tilde{\ell},\tilde{\ell}'\in\hat{\mathcal{B}}} e^{i(\tilde{\ell}-\ell+\ell'-\tilde{\ell}')\cdot\mathrm{u}}\hat{b}_{\mathrm{k}+\ell'-\tilde{\ell}'}(\tilde{\ell},\tilde{\ell}')\\
&= \tfrac{1}{|\mathcal{B}|}\sum_{\tilde{\ell},\in\hat{\mathcal{B}}}\hat{b}_{\mathrm{k}-\tilde{\ell}+\ell}(\tilde{\ell},\tilde{\ell}-\ell+\ell') = \tfrac{1}{|\mathcal{B}|}\sum_{\tilde{\ell}\in\hat{\mathcal{B}}}\hat{b}_{\mathrm{k}}(\ell,\ell')\\
&= \hat{b}_{\mathrm{k}}(\ell,\ell')
\end{aligned}$$

Here $\hat{\mathcal{Z}}_{\mathrm{fin}} = \left(\mathbb{R}/\frac{2\pi}{\varepsilon_T}\mathbb{Z}\right)\times\left(\mathbb{R}^{\mathrm{d}}/\frac{2\pi}{\varepsilon_X}\mathbb{Z}^{\mathrm{d}}\right)$ and we have used, in the transition from the second to the third line, the Fourier series inversion theorem to give

$$\sum_{\mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}}\mathrm{vol}_f\int_{\hat{\mathcal{Z}}_{\mathrm{fin}}}\hat{f}(\tilde{\mathrm{k}})e^{-i(\tilde{\mathrm{k}}-\mathrm{k})\cdot\mathrm{u}'}\tfrac{d^{1+\mathrm{d}}\tilde{\mathrm{k}}}{(2\pi)^{1+\mathrm{d}}} = \hat{f}(\mathrm{k})$$

for all smooth functions $\hat{f}(\tilde{\mathrm{k}})$ that are periodic with respect to $\frac{2\pi}{\varepsilon_T}\mathbb{Z}\times\frac{2\pi}{\varepsilon_X}\mathbb{Z}^{\mathrm{d}}$. Note that $\hat{b}_{\tilde{\mathrm{k}}}(\ell,\ell')$ is indeed periodic in $\tilde{k}$ with respect to $\frac{2\pi}{\varepsilon_T}\mathbb{Z}\times\frac{2\pi}{\varepsilon_X}\mathbb{Z}^{\mathrm{d}}$, because it is periodic in ℓ and ℓ' with respect to $\frac{2\pi}{\varepsilon_T}\mathbb{Z}\times\frac{2\pi}{\varepsilon_X}\mathbb{Z}^{\mathrm{d}}$.

Internal Internal Remark

$$\sum_{\mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}}\mathrm{vol}_f\int_{\hat{\mathcal{Z}}_{\mathrm{fin}}}\hat{f}(\tilde{\mathrm{k}})e^{-i(\tilde{\mathrm{k}}-\mathrm{k})\cdot\mathrm{u}'}\tfrac{d^{1+\mathrm{d}}\tilde{\mathrm{k}}}{(2\pi)^{1+\mathrm{d}}} = \hat{f}(\mathrm{k})$$

is true for $\hat{f}(\tilde{\mathrm{k}}) = e^{i\tilde{k}\cdot\mathrm{u}}$, $\mathrm{u}\in\mathcal{Z}_{\mathrm{fin}}$. Then just take linear combinations and limits.

□

Internal Remark

Lemma 14. *Let A be an operator on $\mathcal{H}_f$ that is translation invariant with respect to $\mathcal{X}_{\rm crs}$. Let $A(y,y')$ be its kernel, and $\hat{A}(p,p')$ be defined by (3). Then*

$$\max_{y,y'\in\mathcal{X}_{\rm fin}} \big|A(y,y')\big| \le \tfrac{|\mathcal{B}|}{\mathrm{vol}_f} \max_{p,p'\in\hat{\mathcal{X}}_{\rm fin}} \big|\hat{A}(p,p')\big| = \tfrac{L_T L_X^{\rm d}}{\varepsilon_T \varepsilon_X^{\rm d}} \max_{p,p'\in\hat{\mathcal{X}}_{\rm fin}} \big|\hat{A}(p,p')\big|$$

Proof. By part (b) of Lemma 1,

$$\begin{aligned}
\max_{y,y'\in\mathcal{X}_{\rm fin}} \big|A(y,y')\big| &\le \tfrac{\widehat{\mathrm{vol}}_c}{(2\pi)^{1+{\rm d}}} \max_{p,p'\in\hat{\mathcal{X}}_{\rm fin}} \big|\hat{A}(p,p')\big|\ |\hat{\mathcal{X}}_{\rm crs}|\ |\hat{\mathcal{B}}|^2 \\
&\le \tfrac{1}{\varepsilon_T\varepsilon_X^{\rm d}\mathcal{L}_T\mathcal{L}_X^{\rm d}} \max_{p,p'\in\hat{\mathcal{X}}_{\rm fin}} \big|\hat{A}(p,p')\big| \tfrac{\mathcal{L}_T}{L_T}\big(\tfrac{\mathcal{L}_X}{L_X}\big)^{\rm d}\big(L_T L_X^{\rm d}\big)^2 \\
&= \tfrac{L_T L_X^{\rm d}}{\varepsilon_T\varepsilon_X^{\rm d}} \max_{p,p'\in\hat{\mathcal{X}}_{\rm fin}} \big|\hat{A}(p,p')\big|
\end{aligned}$$

□

6 Functions of Periodic Operators

Let C be a simple, closed, positively oriented, piecewise smooth curve in the complex plane and denote by $\mathcal{O}_C$ its interior. Denote by $\sigma(A)$ the spectrum of the bounded operator A and assume $\sigma(A)\subset\mathcal{O}_C$. Let $f(z)$ be analytic on the closure of $\mathcal{O}_C$. Then, by the Cauchy integral formula,

$$f(A) = \tfrac{1}{2\pi i}\oint_C \tfrac{f(\zeta)}{\zeta\mathbb{1}-A}\,d\zeta \tag{13}$$

and, for any $m\ge 0$,

$$\|f(A)\|_m \le \tfrac{1}{2\pi}|C| \sup_{\zeta\in C}|f(\zeta)| \sup_{\zeta\in C}\big\|(\zeta\mathbb{1}-A)^{-1}\big\|_m \tag{14}$$

Lemma 15. *Let*

- *$a(\mathrm{u},\mathrm{u}') : \mathcal{Z}_{\rm fin}\times\mathcal{Z}_{\rm fin}\to\mathbb{C}$ obey the conditions of Definition 2,*
- *C be a simple, closed, positively oriented, piecewise smooth curve in the complex plane with interior $\mathcal{O}_C$,*
- *$\mathcal{O}$ contain the closure of $\mathcal{O}_C$ and $f:\mathcal{O}\to\mathbb{C}$ be analytic, and*
- *$0<m''<m'<m$.*

Suppose that

- *for each* $\ell, \ell' \in \hat{\mathcal{B}}$, $\hat{a}_{\rm k}(\ell, \ell')$ *is analytic in* $|\mathrm{Im}\, {\rm k}| < m$.
- *for each* $\zeta \in \mathbb{C} \setminus \mathcal{O}_C$ *and each* ${\rm k}$ *with* $|\mathrm{Im}\, {\rm k}| < m$, *the matrix* $\big[\zeta \delta_{\ell,\ell'} - \hat{a}_{\rm k}(\ell, \ell')\big]_{\ell,\ell' \in \hat{\mathcal{B}}}$ *is invertible.*

Denote by A *the periodization of* a. *Then* $f(A)$, *defined by* (13), *exists and*

$$\begin{aligned} \|f(A)\|_{m''} &\leq \tfrac{C_{m'-m''}}{2\pi\ \mathrm{vol}_c} |C| \sup_{\zeta \in C} |f(\zeta)| \sup_{\substack{|\mathrm{Im}\, {\rm k}| = m' \\ \zeta \in C}} \sum_{\ell,\ell' \in \hat{\mathcal{B}}} \big|(\zeta \mathbb{1} - \hat{a}_{\rm k})^{-1}(\ell, \ell')\big| \\ &\leq \tfrac{C_{m'-m''|\mathcal{B}|}}{2\pi\ \mathrm{vol}_f} |C| \sup_{\zeta \in C} |f(\zeta)| \sup_{\substack{|\mathrm{Im}\, {\rm k}| = m' \\ \ell,\ell' \in \hat{\mathcal{B}} \\ \zeta \in C}} \big|(\zeta \mathbb{1} - \hat{a}_{\rm k})^{-1}(\ell, \ell')\big| \end{aligned}$$

Here $(\zeta \mathbb{1} - \hat{a}_{\rm k})^{-1}$ *refers to the inverse of the* $|\hat{\mathcal{B}}| \times |\hat{\mathcal{B}}|$ *matrix* $\big[\zeta \delta_{\ell,\ell'} - \hat{a}_{\rm k}(\ell, \ell')\big]_{\ell,\ell' \in \hat{\mathcal{B}}}$.

Proof. Each matrix element of $\zeta \mathbb{1} - \hat{a}_k$ is continuous on

$$\mathcal{D} = \big\{\ (\zeta, {\rm k}) \in \mathbb{C}^2 \ \big|\ \zeta \in \mathbb{C} \setminus \mathcal{O}_C,\ |\mathrm{Im}\, {\rm k}| < m\ \big\}$$

Furthermore $\det(\zeta \mathbb{1} - \hat{a}_k)$ does not vanish on $\mathcal{D}$. Hence every matrix element of $\big(\zeta \mathbb{1} - \hat{a}_k\big)^{-1}$ is also continuous on $\mathcal{D}$ and in particular is bounded on compact subsets of $\mathcal{D}$. Set, for each for each $\zeta \in \mathbb{C} \setminus \mathcal{O}_C$, and ${\rm u}, {\rm u}' \in \mathcal{Z}_{\rm fin}$,

$$r_\zeta({\rm u}, {\rm u}') = \sum_{\ell,\ell' \in \hat{\mathcal{B}}} \int_{\hat{\mathcal{Z}}_{\rm crs}} e^{i\ell \cdot {\rm u}} (\zeta \mathbb{1} - \hat{a}_{\rm k})^{-1}(\ell, \ell') e^{-i\ell' \cdot {\rm u}'} e^{i{\rm k} \cdot ({\rm u} - {\rm u}')} \tfrac{d^{1+{\rm d}} {\rm k}}{(2\pi)^{1+{\rm d}}}$$

By Lemma 13, $r_\zeta({\rm u}, {\rm u}')$ obeys the conditions of Definition 2 and

$$\hat{r}_{\zeta,{\rm k}}(\ell, \ell') = (\zeta \mathbb{1} - \hat{a}_{\rm k})^{-1}(\ell, \ell')$$

By Lemma 6, $r_\zeta = (\zeta \mathbb{1} - a)^{-1}$, as operators on $L^2(\mathcal{Z}_{\rm fin})$. By Remark 3.c, for each $\zeta \in \mathbb{C} \setminus \mathcal{O}_C$, the periodization of $r_\zeta({\rm u}, {\rm u}')$ is $\big(\zeta \mathbb{1} - A\big)^{-1}$. In particular, $\sigma(A) \subset \mathcal{O}_C$. By Lemma 12.b,

$$\begin{aligned} \|(\zeta \mathbb{1} - A)^{-1}\|_{m''} &\leq \tfrac{C_{m'-m''}}{\mathrm{vol}_c} \sup_{|\mathrm{Im}\, {\rm k}| = m'} \sum_{\ell,\ell' \in \hat{\mathcal{B}}} \big|(\zeta \mathbb{1} - \hat{a}_{\rm k})^{-1}(\ell, \ell')\big| \\ &\leq \tfrac{C_{m'-m''|\mathcal{B}|}}{\mathrm{vol}_f} \sup_{\substack{|\mathrm{Im}\, {\rm k}| = m' \\ \ell,\ell' \in \hat{\mathcal{B}}}} \big|(\zeta \mathbb{1} - \hat{a}_{\rm k})^{-1}(\ell, \ell')\big| \end{aligned}$$

Then, by (14),

$$\begin{aligned}\|f(A)\|_{m''} &\le \tfrac{1}{2\pi}|C| \sup_{\zeta\in C}|f(\zeta)| \sup_{\zeta\in C}\left\|(\zeta\mathbb{1}-A)^{-1}\right\|_{m''} \\ &\le \tfrac{C_{m'-m''}}{2\pi\ \mathrm{vol}_c}|C| \sup_{\zeta\in C}|f(\zeta)| \sup_{\substack{|\mathrm{Im\,k}|=m' \\ \zeta\in C}} \sum_{\ell,\ell'\in\hat{\mathcal{B}}} \left|(\zeta\mathbb{1}-\hat{a}_{\mathrm{k}})^{-1}(\ell,\ell')\right| \\ &\le \tfrac{C_{m'-m''}|\mathcal{B}|}{2\pi\ \mathrm{vol}_f}|C| \sup_{\zeta\in C}|f(\zeta)| \sup_{\substack{|\mathrm{Im\,k}|=m' \\ \ell,\ell'\in\hat{\mathcal{B}} \\ \zeta\in C}} \left|(\zeta\mathbb{1}-\hat{a}_{\mathrm{k}})^{-1}(\ell,\ell')\right|\end{aligned}$$

□

Internal Remark

Let D, $\mathcal{O}_C$, A and f be as in the framework specified just before (14). Let $g(z)$ be a second analytic function on the closure of $\mathcal{O}_C$. Also let C' be a second curve just inside C, also positively oriented, just as C, whose interior still contains all of the spectrum of A. Then combining

$$f(A) = \tfrac{1}{2\pi i}\oint_C \tfrac{f(\zeta)}{\zeta\mathbb{1}-A}\,d\zeta \qquad g(A) = \tfrac{1}{2\pi i}\oint_{C'} \tfrac{g(\xi)}{\xi\mathbb{1}-A}\,d\xi$$

we have

$$\begin{aligned}f(A)\,g(A) &= \tfrac{1}{(2\pi i)^2}\oint_{C'}\oint_C \tfrac{f(\zeta)}{\zeta\mathbb{1}-A}\tfrac{g(\xi)}{\xi\mathbb{1}-A}\,d\zeta\,d\xi \\ &= \tfrac{1}{(2\pi i)^2}\oint_{C'}\oint_C \tfrac{f(\zeta)g(\xi)}{\xi-\zeta}\left[\tfrac{1}{\zeta\mathbb{1}-A}-\tfrac{1}{\xi\mathbb{1}-A}\right]d\zeta\,d\xi \\ &= \tfrac{1}{(2\pi i)^2}\oint_C d\zeta\tfrac{f(\zeta)}{\zeta\mathbb{1}-A}\left[\oint_{C'}\tfrac{g(\xi)}{\xi-\zeta}\,d\xi\right] + \tfrac{1}{(2\pi i)^2}\oint_{C'} d\xi\tfrac{g(\xi)}{\xi\mathbb{1}-A}\left[\oint_C \tfrac{f(\zeta)}{\zeta-\xi}\,d\zeta\right]\end{aligned}$$

In the first term, the integral over C' vanishes, since ζ is always in the exterior of C'. In the second term, the integral over C is $2\pi i\,f(\xi)$, since ξ is always in the interior of C. So

$$f(A)\,g(A) = \tfrac{1}{2\pi i}\oint_{C'}\tfrac{f(\xi)g(\xi)}{\xi\mathbb{1}-A}\,d\xi = (fg)(A)$$

Note that this computation does not require that A be normal, or even diagonalizable.

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7 Scaling of Periodized Operators

Scaling plays an important role in the construction of [6, 7]. See, for example, [6, Definition 1.3 and §2] and [4, (1.3)]. For the current abstract setting, select scaling factors σ_T and σ_X and define the scaled lattices

$$\begin{aligned}
\mathcal{Z}_{\text{fin}}^{(s)} &= \tfrac{\varepsilon_T}{\sigma_T}\mathbb{Z} \times \tfrac{\varepsilon_X}{\sigma_X}\mathbb{Z}^{\text{d}} & \text{vol}_f^{(s)} &= \tfrac{\varepsilon_T\varepsilon_X^{\text{d}}}{\sigma_T\sigma_X^{\text{d}}} \\
\mathcal{Z}_{\text{crs}}^{(s)} &= L_T\tfrac{\varepsilon_T}{\sigma_T}\mathbb{Z} \times L_X\tfrac{\varepsilon_X}{\sigma_X}\mathbb{Z}^{\text{d}} & \text{vol}_c^{(s)} &= \tfrac{(L_T\varepsilon_T)(L_X\varepsilon_X)^{\text{d}}}{\sigma_T\sigma_X^{\text{d}}} \\
\hat{\mathcal{Z}}_{\text{crs}}^{(s)} &= \big(\mathbb{R}/\tfrac{2\pi\sigma_T}{\varepsilon_T L_T}\mathbb{Z}\big) \times \big(\mathbb{R}^{\text{d}}/\tfrac{2\pi\sigma_X}{\varepsilon_X L_X}\mathbb{Z}^{\text{d}}\big) \\
\mathcal{B}^{(s)} &= \big(\tfrac{\varepsilon_T}{\sigma_T}\mathbb{Z}/L_T\tfrac{\varepsilon_T}{\sigma_T}\mathbb{Z}\big) \times \big(\tfrac{\varepsilon_X}{\sigma_X}\mathbb{Z}^{\text{d}}/L_X\tfrac{\varepsilon_X}{\sigma_X}\mathbb{Z}^{\text{d}}\big) \cong \mathcal{X}_{\text{fin}}^{(s)}/\mathcal{X}_{\text{crs}}^{(s)} \\
\hat{\mathcal{B}}^{(s)} &= \big(\tfrac{2\pi\sigma_T}{L_T\varepsilon_T}\mathbb{Z}/\tfrac{2\pi\sigma_T}{\varepsilon_T}\mathbb{Z}\big) \times \big(\tfrac{2\pi\sigma_X}{L_X\varepsilon_X}\mathbb{Z}^{\text{d}}/\tfrac{2\pi\sigma_X}{\varepsilon_X}\mathbb{Z}^{\text{d}}\big)
\end{aligned}$$

The map $\mathbb{L}(\tau, \mathbf{x}) = (\sigma_T\tau, \sigma_X\mathbf{x})$ gives bijections

$$\mathbb{L} : \mathcal{Z}_{\text{fin}}^{(s)} \to \mathcal{Z}_{\text{fin}} \qquad \mathbb{L} : \mathcal{Z}_{\text{crs}}^{(s)} \to \mathcal{Z}_{\text{crs}} \qquad \mathbb{L} : \mathcal{B}^{(s)} \to \mathcal{B}$$

$\mathbb{L}$ induces linear bijections $\mathbb{L}_* : L^2\big(\mathcal{Z}_{\text{fin}}^{(s)}\big) \to L^2\big(\mathcal{Z}_{\text{fin}}\big)$ and $\mathbb{L}_* : L^2\big(\mathcal{Z}_{\text{crs}}^{(s)}\big) \to L^2\big(\mathcal{Z}_{\text{crs}}\big)$ by $\mathbb{L}_*(\alpha)(\mathbb{L}\text{u}) = \alpha(\text{u})$. Observe that

$$\langle \mathbb{L}_*\alpha, \mathbb{L}_*\beta\rangle_f = \sigma_T\sigma_X^{\text{d}}\,\langle\alpha,\beta\rangle_f^{(s)} \qquad \langle \mathbb{L}_*\alpha, \mathbb{L}_*\beta\rangle_c = \sigma_T\sigma_X^{\text{d}}\,\langle\alpha,\beta\rangle_c^{(s)}$$

Lemma 16. *Let $a : L^2\big(\mathcal{Z}_{\text{fin}}\big) \to L^2\big(\mathcal{Z}_{\text{fin}}\big)$ have kernel $a(\text{u}, \text{u}')$.*

(a) The kernel of $\mathbb{L}_*^{-1}a\,\mathbb{L}_*$ is

$$a^{(s)}(\text{v}, \text{v}') = \sigma_T\sigma_X^{\text{d}}\, a\big(\mathbb{L}\text{v}, \mathbb{L}\text{v}'\big)$$

(b) The Fourier transform of the kernel of $\mathbb{L}_*^{-1}a\mathbb{L}_*$ is

$$\hat{a}_{\text{k}}^{(s)}(\ell, \ell') = \hat{a}_{\mathbb{L}^{-1}\text{k}}(\mathbb{L}^{-1}\ell, \mathbb{L}^{-1}\ell') \qquad \textit{for } \text{k} \in \mathbb{R}\times\mathbb{R}^{\text{d}}, \quad \ell, \ell' \in \hat{\mathcal{B}}^{(s)}$$

(c) If $m \geq \max\big\{\frac{1}{\sigma_T}, \frac{1}{\sigma_X}\big\}m_s$, then $\|a^{(s)}\|_{m_s} \leq \|a\|_m$.

Proof. (a) For $\alpha \in L^2\big(\mathcal{Z}_{\text{fin}}^{(s)}\big)$ and $\text{v} \in \mathcal{Z}_{\text{fin}}^{(s)}$,

$$\begin{aligned}
(\mathbb{L}_*^{-1}a\mathbb{L}_*\,\alpha)(\text{v}) &= \text{vol}_f \sum_{\text{u}'\in\mathcal{Z}_{\text{fin}}} a(\mathbb{L}\text{v}, \text{u}')\,\big(\mathbb{L}_*\,\alpha\big)(\text{u}') \\
&= \text{vol}_f \sum_{\text{u}'\in\mathcal{Z}_{\text{fin}}} a(\mathbb{L}\text{v}, \text{u}')\,\alpha(\mathbb{L}_*^{-1}\text{u}') \\
&= \text{vol}_f^{(s)} \sum_{\text{v}'\in\mathcal{Z}_{\text{fin}}^{(s)}} \sigma_T\sigma_X^{\text{d}}\, a(\mathbb{L}\text{v}, \mathbb{L}\text{v}')\,\alpha(\text{v}')
\end{aligned}$$

(b) By (8) and part (a),

$$\begin{aligned}\hat{a}^{(s)}_{\mathrm{k}}(\ell,\ell') &= \tfrac{\mathrm{vol}_f}{|\hat{\mathcal{B}}|}\sum_{\substack{[\mathrm{v}]\in\mathcal{B}^{(s)}\\ \mathrm{v}'\in\mathcal{Z}^{(s)}_{\mathrm{fin}}}} e^{-i\ell\cdot\mathrm{v}}a\big(\mathbb{L}\mathrm{v},\mathbb{L}\mathrm{v}'\big)e^{i\ell'\cdot\mathrm{v}'}e^{-i\mathrm{k}\cdot(\mathrm{v}-\mathrm{v}')}\\ &= \tfrac{\mathrm{vol}_f}{|\hat{\mathcal{B}}|}\sum_{\substack{[\mathrm{u}]\in\mathcal{B}\\ \mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}}} e^{-i(\mathbb{L}^{-1}\ell)\cdot\mathrm{u}}a(\mathrm{u},\mathrm{u}')e^{i(\mathbb{L}^{-1}\ell')\cdot\mathrm{u}'}e^{-i(\mathbb{L}^{-1}\mathrm{k})\cdot(\mathrm{u}-\mathrm{u}')}\\ &= \hat{a}_{\mathbb{L}^{-1}\mathrm{k}}(\mathbb{L}^{-1}\ell,\mathbb{L}^{-1}\ell')\end{aligned}$$

(c) This part follows from the inequality

$$\begin{aligned}\sup_{\mathrm{v}\in\mathcal{Z}^{(s)}_{\mathrm{fin}}}\mathrm{vol}_f^{(s)}\sum_{\mathrm{v}'\in\mathcal{Z}^{(s)}_{\mathrm{fin}}} e^{m_s|\mathrm{v}-\mathrm{v}'|}|a^{(s)}(\mathrm{v},\mathrm{v}')| &= \sup_{\mathrm{v}\in\mathcal{Z}^{(s)}_{\mathrm{fin}}}\mathrm{vol}_f^{(s)}\sum_{\mathrm{v}'\in\mathcal{Z}^{(s)}_{\mathrm{fin}}} e^{m_s|\mathrm{v}-\mathrm{v}'|}\sigma_T\sigma_X^{\mathrm{d}}|a(\mathbb{L}\mathrm{v},\mathbb{L}\mathrm{v}')|\\ &= \sup_{\mathrm{u}\in\mathcal{Z}_{\mathrm{fin}}}\mathrm{vol}_f\sum_{\mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}} e^{m_s|\mathbb{L}^{-1}\mathrm{u}-\mathbb{L}^{-1}\mathrm{u}'|}|a(\mathrm{u},\mathrm{u}')|\\ &\le \sup_{\mathrm{u}\in\mathcal{Z}_{\mathrm{fin}}}\mathrm{vol}_f\sum_{\mathrm{u}'\in\mathcal{Z}_{\mathrm{fin}}} e^{m|\mathrm{u}-\mathrm{u}'|}|a(\mathrm{u},\mathrm{u}')|\end{aligned}$$

and the corresponding inequality with v summed over and v′ supped over. □

More generally,

Lemma 17. *Let* $b : L^2\big(\mathcal{Z}_{\mathrm{crs}}\big) \to L^2\big(\mathcal{Z}_{\mathrm{fin}}\big)$ *and* $c : L^2\big(\mathcal{Z}_{\mathrm{fin}}\big) \to L^2\big(\mathcal{Z}_{\mathrm{crs}}\big)$ *have kernels* $b(\mathrm{u},\mathrm{x})$ *and* $c(\mathrm{x},\mathrm{u})$ *respectively.*

(a) The kernels of $\mathbb{L}_*^{-1}b\,\mathbb{L}_*$ *and* $\mathbb{L}_*^{-1}c\,\mathbb{L}_*$ *are*

$$b^{(s)}(\mathrm{v},\mathrm{x}) = \sigma_T\sigma_X^{\mathrm{d}}\,b\big(\mathbb{L}\mathrm{v},\mathbb{L}\mathrm{x}\big) \qquad c^{(s)}(\mathrm{x},\mathrm{v}) = \sigma_T\sigma_X^{\mathrm{d}}\,c\big(\mathbb{L}\mathrm{x},\mathbb{L}\mathrm{v}\big)$$

(b) The Fourier transform of the kernels of $\mathbb{L}_*^{-1}b\,\mathbb{L}_*$ *and* $\mathbb{L}_*^{-1}c\,\mathbb{L}_*$ *are*

$$\hat{b}^{(s)}_{\mathrm{k}}(\ell) = \hat{b}_{\mathbb{L}^{-1}\mathrm{k}}(\mathbb{L}^{-1}\ell) \qquad \hat{c}^{(s)}_{\mathrm{k}}(\ell') = \hat{c}_{\mathbb{L}^{-1}\mathrm{k}}(\mathbb{L}^{-1}\ell') \qquad \textit{for } \mathrm{k}\in\mathbb{R}\times\mathbb{R}^{\mathrm{d}},\quad \ell,\ell'\in\hat{\mathcal{B}}^{(s)}$$

(c) If $m \ge \max\big\{\frac{1}{\sigma_T},\frac{1}{\sigma_X}\big\}m_s$, *then*

$$\|b^{(s)}\|_{m_s} \le \|b\|_m \qquad \|c^{(s)}\|_{m_s} \le \|c\|_m$$

References

[1] T. Balaban. The Ultraviolet Stability Bounds for Some Lattice σ–Models and Lattice Higgs–Kibble Models. In *Proc. of the International Conference on Mathematical Physics, Lausanne, 1979*, pages 237–240. Springer, 1980.

[2] T. Balaban. A low temperature expansion and "spin wave picture" for classical N-vector models. In *Constructive Physics (Palaiseau, 1994), Lecture Notes in Physics, 446*, pages 201–218. Springer, 1995.

[3] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. Complex Bosonic Many–body Models: Overview of the Small Field Parabolic Flow. Preprint, 2016.

[4] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. Operators for Parabolic Block Spin Transformations. Preprint, 2016.

[5] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. The Algebra of Block Spin Renormalization Group Transformations. Preprint, 2016.

[6] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. The Small Field Parabolic Flow for Bosonic Many–body Models: Part 1 — Main Results and Algebra. Preprint, 2016.

[7] T. Balaban, J. Feldman, H. Knörrer, and E. Trubowitz. The Small Field Parabolic Flow for Bosonic Many–body Models: Part 2 — Fluctuation Integral and Renormalization. Preprint, 2016.

[8] J. Dimock. The renormalization group according to Balaban – I. small fields. *Reviews in Mathematical Physics*, 25:1–64, 2013.

[9] K. Gawedzki and A. Kupiainen. A rigorous block spin approach to massless lattice theories. *Comm. Math. Phys.*, 77:31–64, 1980.

[10] L.P. Kadanoff. Scaling laws for Ising models near T_c. *Physics*, 2:263, 1966.

[11] P. Kuchment. *Floquet Theory for Partial Differential Equations*. Birkhäuser, 1993.

[12] K.G. Wilson. The renormalization group: critical phenomena and the Kondo problem. *Rev. Mod. Phys.*, 47:773, 1975.