The Small Field Parabolic Flow for Bosonic Many–body Models:
Part 2 — Fluctuation Integral and Renormalization

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Abstract

This paper is a contribution to a program to see symmetry breaking in a weakly interacting many Boson system on a three dimensional lattice at low temperature. It is part of an analysis of the "small field" approximation to the "parabolic flow" which exhibits the formation of a "Mexican hat" potential well. Here we complete the analysis of a renormalization group step, started in [7], by "evaluating" the fluctuation integral and renormalizing the chemical potential.

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Contents

Part of our program to construct and analyze an interacting many Boson system on a three dimensional lattice in the thermodynamic limit is the "small field parabolic flow" which exhibits the formation of a potential well in the effective interaction. For an overview of this part, see [3]. The starting point of this program is a representation of a "small field approximation" to the partition function which is written in the form of a functional integral $\int_{\mathcal{X}_0} e^{\mathcal{A}_0}$ over a $3+1$ dimensional unit lattice $\mathcal{X}_0$, with an action $\mathcal{A}_0$ of the form described in [7, §1.5]. This action is the outcome of the previous step in our program that had settled the temporal ultraviolet problem in imaginary time (see [2], [7, Appendix D]). For the "small field parabolic flow" we perform a number of approximate block spin transformations $\mathbb{T}_0^{(SF)}, \cdots, \mathbb{T}_n^{(SF)}$, each followed by a rescaling. Our main result [7, Theorem 1.17] is a representation of $\big((\mathbb{S}\mathbb{T}_n^{(SF)}) \circ \cdots \circ (\mathbb{S}\mathbb{T}_0^{(SF)})\big)\big(e^{\mathcal{A}_0}\big)$ for all integers n smaller than a given number n_p defined in [7, Definition 1.11.b]. The representation clearly shows the development of the potential well, see [7, (1.8)].

The proof of the main theorem consists of several steps, outlined in [3]. It is a combination of block spin transformation and complex stationary phase techniques. In [7] the algebraic aspects of these steps are presented in detail. The estimates needed to show that these algebraic steps are meaningful are presented in [8] and in this paper. [8] deals with the existence of, and estimates on, the background fields (introduced in [7, Definition 1.5]) on which the represention of the effective action in the main theorem is based. Here, we use these estimates as input to complete the inductive proof of [7, Theorem 1.17] which rewrites the representation $e^{\mathcal{C}_n}\mathcal{F}_n$ for $\big((\mathbb{S}\mathbb{T}_n^{(SF)}) \circ \cdots \circ (\mathbb{S}\mathbb{T}_0^{(SF)})\big)\big(e^{\mathcal{A}_0}\big)(\psi_*, \psi)$ given in [7, Corollary 4.3] in the form specified in [7, Theorem 1.17]. The main steps are

- "evaluation" of the fluctuation integral and
- renormalization of the chemical potential.

They are performed in the two main sections of this paper.

The symmetry breaking in the many Boson system is expected to happen only when the chemical potential is above a critical value. The renormalization of the chemical potential performed in this paper gives some insight into the leading term of the expansion of this critical chemical potential in powers of the coupling constant. This is presented in Appendix A. The more technical Appendixes B and C deal with the localization operation that we use during the course of renormalization, and with the effect of scaling on the norms we use.

We keep the terminology and notation of [7], which is summarized in [7, Appendix A].

5 One Block Spin Transformation — The Fluctuation Integral

In this section, we evaluate the fluctuation integral. Fix any $0 \le n \le n_p$. We assume that, if $n \ge 1$, the conclusions of [7, Theorem 1.17 and Remark 1.18] hold. In the case of $n = 0$, we use the data of [7, §1.5].

We start by introducing the main norm that will be used in this section. In this and the following section we abbreviate the weight factors of [7, Definition 1.11] by

$$\begin{aligned} &\kappa = \kappa(n) && \kappa' = \kappa'(n) \\ &\bar\kappa = \kappa(n+1) = L^\eta \kappa && \bar\kappa' = \kappa'(n+1) = L^{\eta'}\kappa' \quad \bar\kappa_{\mathfrak{l}} = \kappa_{\mathfrak{l}}(n+1) = 4r_n \end{aligned} \tag{5.1}$$

We also use the notation

$$\mathfrak{v}_n = \tfrac{\mathfrak{v}_0}{L^n} = 2\|\mathcal{V}_n^{(u)}\|_{2m}$$

Remark 5.1. By [7, Remark 1.18 and Corollary C.4],

$$\|\mathcal{V}_n\|_{2m} \le \mathfrak{v}_n$$
$$|\mu_n| \le 2L^{2n}(\mu_0 - \mu_*) + \mathfrak{v}_0^{1-\epsilon} \le 4\min\left\{\mathfrak{v}_0^{5\epsilon}\,,\ L^{2n}\mathfrak{v}_0^{\frac{8}{9}+\epsilon}\right\}$$

By Remark 5.1, [7, Definition 1.11, Remark 1.12 and Lemma C.1.b], we have, choosing $\mathfrak{v}_0$ small enough depending on ϵ and L,

$$\max\left\{\ L^2|\mu_n|\ ,\ \|V_n\|_{2m}(\bar\kappa + L^9\bar\kappa_{\mathfrak{l}})(\bar\kappa + \bar\kappa' + L^9\bar\kappa_{\mathfrak{l}})\ \right\} \le \tfrac{1}{2}\rho_{\rm bg} \tag{5.2}$$

with the $\rho_{\rm bg}$ of [8, Convention 1.2]. Denote by $\bar{\|}\tilde{\mathcal{G}}\bar{\|}$ the norm of the analytic function $\tilde{\mathcal{G}}(\tilde\psi_*, \tilde\psi, z_*, z)$ with mass $2m$ which associates the weight $\bar\kappa$ to the fields ψ_*, ψ, weight $\bar\kappa'$ to the fields $\psi_{\nu*}, \psi_\nu$, $\nu = 0, \cdots, 3$, and the weight $\bar\kappa_{\mathfrak{l}}$ to the fields z_*, z. Similarly we denote by $\bar{|\!|\!|}\tilde F\bar{|\!|\!|}$ the norm of the field map $\tilde F(\tilde\psi_*, \tilde\psi, z_*, z)$ with the same mass and field weights. See [7, Definition A.3].

In Lemma 5.5, below, we prove the bounds that will be needed for evaluation of the fluctuation integral. It uses

Definition 5.2 (Scaling Divergence Factor)**.** Set, for each constant $C \ge 1$ and each $\vec p = (p_u, p_0, p_{\rm sp})$,

$$\mathrm{sdf}(\vec p; C) = \left(\frac{C}{L^{3/2}}\frac{\bar\kappa}{\kappa}\right)^{p_u}\left(\frac{C}{L^{7/2}}\frac{\bar\kappa'}{\kappa'}\right)^{p_0}\left(\frac{C}{L^{5/2}}\frac{\bar\kappa'}{\kappa'}\right)^{p_{\rm sp}} = C^{p_u+p_0+p_{\rm sp}}L^{-\Delta(\vec p)}L^{\eta p_u + \eta' p_0 + \eta' p_{\rm sp}}$$

where

$$\Delta(\vec p) = \tfrac{3}{2}p_u + \tfrac{7}{2}p_0 + \tfrac{5}{2}p_{\rm sp}$$

Furthermore set

$$\mathrm{sdf}(C) = \sup_{\vec{p}\notin\mathfrak{D}_{\mathrm{rel}}} \mathrm{sdf}(\vec{p}; C)$$

Remark 5.3. Assuming that $L \ge \big(2C^8\big)^{1/\epsilon}$, we have $\mathrm{sdf}(C) \le \frac{1}{2L^5}$

Proof. When $\vec{p} = (p_u, p_0, p_{\mathrm{sp}})$ and $|\vec{p}| = p_u + p_0 + p_{\mathrm{sp}}$

$$\log_L \mathrm{sdf}(\vec{p}, C) = \log_L \mathrm{sdf}(\vec{p}, 1) + |\vec{p}| \log_L C$$

and

$$\begin{aligned}
\log_L \mathrm{sdf}(\vec{p}, 1) &= -(\tfrac{3}{2} - \eta)(p_u + p_0 + p_{\mathrm{sp}}) - \big(1 + (\eta - \eta')\big)(p_0 + p_{\mathrm{sp}}) - p_0 \\
&\le - \begin{cases} 8(\frac{3}{2} - \eta) & \text{if } |\vec{p}| \ge 8 \\ 10 - 5\eta - \eta' & \text{if } |\vec{p}| = 6 \text{ and } p_0 + p_{\mathrm{sp}} \ge 1 \\ 8 - 3\eta - \eta' & \text{if } |\vec{p}| = 4 \text{ and } p_0 \ge 1 \\ 8 - 2\eta - 2\eta' & \text{if } |\vec{p}| = 4 \text{ and } p_{\mathrm{sp}} \ge 2 \\ 7 - 2\eta' & \text{if } p_u = p_{\mathrm{sp}} = 0,\ p_0 = 2 \end{cases} \\
&\le -5 - \max\big\{\epsilon\,,\ \tfrac{1}{2}(|\vec{p}| - 8)\big\}
\end{aligned}$$

Consequently,

$$\begin{aligned}
\log_L \big(2L^5 \mathrm{sdf}(\vec{p}, C)\big) &\le \log_L(2C^8) + (|\vec{p}| - 8)\log_L C - \max\big\{\epsilon\,,\ \tfrac{1}{2}(|\vec{p}| - 8)\big\} \\
&\le \epsilon - \begin{cases} \epsilon & \text{if } |\vec{p}| \le 8 \\ \frac{1}{4}(|\vec{p}| - 8) & \text{if } |\vec{p}| \ge 10 \end{cases} \\
&\le 0
\end{aligned}$$

∎

Remark 5.4. This remark provides the motivation for our choice $\mathfrak{D}_{\mathrm{rel}}$ (in [7, Definition 1.16]) and $\mathfrak{D}$ (in [7, (1.18)]).

Let $\mathcal{M}$ be a monomial of type $\vec{p}$, as in [7, Definition 1.8]. By Lemma C.2.b,

$$\bar{\|}\mathbb{S}\mathcal{M}\bar{\|} \le L^5\, \mathrm{sdf}(\vec{p}; 1)\, \|\mathcal{M}\|^{(n)}$$

(If the mass were zero in both norms, this would be an equality.) So the "scale (n+1) norm" of the scaled monomial $\mathbb{S}\mathcal{M}$ is smaller than the "scale n norm" of the monomial $\mathcal{M}$ when $L^5\, \mathrm{sdf}(\vec{p}; 1) < 1$. This is the case if and only if $\vec{p} \in \mathfrak{D}_{\mathrm{rel}}$. In fact it was exactly this that determined our choice of $\mathfrak{D}_{\mathrm{rel}}$. Monomials of type $\vec{p}$ with

$\vec{p} \notin \mathfrak{D}_{\rm rel}$ are said to be "scaling–weight irrelevant". When such terms are generated during the course of renormalization group step number n, they are placed in the "high degree" part, $\mathcal{E}_n$ of the action.

Now let $\mathcal{M}$ be a monomial of type $\vec{p} \in \mathfrak{D}_{\rm rel}$. For some $\vec{p}$'s, the size of the kernel of $\mathcal{M}$ decreases, or at least does not increase, under scaling. (This does not contradict $\left\|\!\left|\mathbb{S}\mathcal{M}\right|\!\right\| > \|\mathcal{M}\|^{(n)}$ because the field weights in $\left\|\!\left|\mathbb{S}\mathcal{M}\right|\!\right\|$ are greater than the field weights in $\|\mathcal{M}\|^{(n)}$.) Indeed, by Lemma C.2.a,

$$\|\mathbb{S}\mathcal{M}\|_{2m} \le L^5\, L^{-\frac{3}{2}p_u - \frac{7}{2}p_0 - \frac{5}{2}p_{\rm sp}}\, \|\mathcal{M}\|_m$$

(Again, if the mass were zero in both norms, this would be an equality.) The only $\vec{p}$'s with $\frac{3}{2}p_u + \frac{7}{2}p_0 + \frac{5}{2}p_{\rm sp} < 5$, i.e. the only scaling relevent monomials, are those with $\vec{p} = (2,0,0), (1,0,1)$. Here is what we do with monomials $\mathcal{M}$ of type $\vec{p} \in \mathfrak{D}_{\rm rel}$ that are generated during the course of renormalization group step number n. See §6.

- If $\vec{p} = (6,0,0),\ (1,1,0),\ (0,1,1),\ (0,0,2)$, (i.e. if $\vec{p} \in \mathfrak{D}$) the monomial is placed in the "low degree" part, $\mathcal{R}_n$ of the action.
- If $\vec{p} = (4,0,0)$, the monomial is placed in the "main" part, A_n of the action, renormalizing $\mathcal{V}$.
- If $\vec{p} = (3,0,1)$, Lemma B.3.b is used to express $\mathcal{M}$ as a sum of monomials of type $\vec{p}\,'$ with $\vec{p}\,' = (2,1,1),\ (2,0,2)$, which are placed in the "high degree" part, $\mathcal{E}_n$ of the action.
- If $\vec{p} = (2,0,0)$, Lemma B.3.c is used to express $\mathcal{M}$ as a sum of a local degree two monomial, which is placed in the "main" part, A_n of the action, renormalizing the chemical potential μ, and a sum of monomials of type $\vec{p}\,'$ with $\vec{p}\,' = (1,1,0), (0,1,1),\ (0,0,2)$, which are placed in the "low degree" part, $\mathcal{R}_n$ of the action.
- If $\vec{p} = (1,0,1)$, Lemma B.3.a is used to express $\mathcal{M}$ as a sum of monomials of type $\vec{p}\,'$ with $\vec{p}\,' = (0,1,1),\ (0,0,2)$, which are placed in the "low degree" part, $\mathcal{R}_n$ of the action.

There is one other complication which we have supressed from these bullets. Monomials generated by the fluctuation integral are naturally functions of the fields $\psi_{(*)}$. But the "low degree" part $\mathcal{R}_n$ and the chemical potential and interaction parts of the "main" part A_n of the action are functions of the background field $\phi_{(*)n}(\psi_*, \psi, \mu_n, \mathcal{V}_n)$. Expressing the various functions above in terms of the "right" fields complicates the above procedure, but does not introduce any serious obstructions.

Lemma 5.5. *There is a constant $C_{\mathfrak{l}}$ that depends only on $\Gamma_{\rm op}$, $K_{\rm bg}$ and $\rho_{\rm bg}$ such that the following holds.*

(a) *Let* $\delta A_n^{(2)}$ *and* $\delta A_n^{(\ge 3)}$ *be the parts of* δA_n *that are of degree two and of degree at least three, respectively, in* $z_{(*)}$. *Then*

$$\bar{\|}\delta A_n^{(2)}\bar{\|} \le L^{42} C_{\mathfrak{l}} \left\{ \|V_n\|_{2m} (\bar{\kappa} + \bar{\kappa}_{\mathfrak{l}})^2 + |\mu_n| \right\} \bar{\kappa}_{\mathfrak{l}}^2$$
$$\bar{\|}\delta A_n^{(\ge 3)}\bar{\|} \le L^{42} C_{\mathfrak{l}} \, \|V_n\|_{2m} (\bar{\kappa} + \bar{\kappa}_{\mathfrak{l}}) \bar{\kappa}_{\mathfrak{l}}^3$$

(b) *Let* $\tilde{\mathcal{E}}_n$ *refer to the* $\tilde{\mathcal{E}}_n$ *of [7, Theorem 1.17] for* $n \ge 1$ *and the* $\mathcal{E}_0$ *of [7, §1.5] for* $n = 0$. *There are analytic functions* $\tilde{\mathcal{E}}_{n+1,1}(\tilde{\psi}_*, \tilde{\psi})$ *and* $\delta\tilde{\mathcal{E}}_n(\tilde{\psi}_*, \tilde{\psi}, z_*, z)$ *such that*

$$\mathcal{E}_{n+1,1}(\psi_*, \psi) = \tilde{\mathcal{E}}_{n+1,1}\big((\psi_*, \{\partial_\nu \psi_*\}) \, , \, (\psi, \{\partial_\nu \psi\}) \big)$$
$$\delta\mathcal{E}_n(\psi_*, \psi, z_*, z) = \delta\tilde{\mathcal{E}}_n\big((\psi_*, \{\partial_\nu \psi_*\}) \, , \, (\psi, \{\partial_\nu \psi\}) \, , \, z_* \, , \, z \big)$$

and

$$\bar{\|}\tilde{\mathcal{E}}_{n+1,1}\bar{\|} \le L^5 \, \mathrm{sdf}(C_{\mathfrak{l}}) \, \big\|\tilde{\mathcal{E}}_n\big\|^{(n)}$$
$$\bar{\|}\delta\tilde{\mathcal{E}}_n\bar{\|} \le L^{18} \, \tfrac{\bar{\kappa}_{\mathfrak{l}}}{\bar{\kappa}'} \, \mathrm{sdf}(C_{\mathfrak{l}}) \, \big\|\tilde{\mathcal{E}}_n\big\|^{(n)}$$

Furthermore, $\tilde{\mathcal{E}}_{n+1,1}$ *contains no scaling/weight relevant monomials.*

(c) *We have*

$$(\mathbb{S}\mathcal{R}_n)(\Phi_*, \Phi) = \sum_{\vec{p} \in \mathfrak{D}} (\mathbb{S}\tilde{\mathcal{R}}_n^{(\vec{p})})\big((\Phi_*, \{\partial_\nu \Phi_*\}) \, , \, (\Phi, \{\partial_\nu \Phi\}) \big)$$

and

$$\big\|\mathbb{S}\tilde{\mathcal{R}}_n^{(\vec{p})}\big\|_{2m} \le L^{5-\Delta(\vec{p})} \big\|\tilde{\mathcal{R}}_n^{(\vec{p})}\big\|_m$$

For each $\vec{p} = (p_u, p_0, p_{\mathrm{sp}}) \in \mathfrak{D}$, *there is an analytic function* $\delta\tilde{\mathcal{R}}_n^{(\vec{p})}(\tilde{\psi}_*, \tilde{\psi}, z_*, z)$ *such that*

$$\delta\mathcal{R}_n(\psi_*, \psi, z_*, z) = \sum_{\vec{p} \in \mathfrak{D}} \delta\tilde{\mathcal{R}}_n^{(\vec{p})}\big((\psi_*, \{\partial_\nu \psi_*\}) \, , \, (\psi, \{\partial_\nu \psi\}) \, , \, z_* \, , \, z \big)$$

and

$$\bar{\|}\delta\tilde{\mathcal{R}}_n^{(\vec{p})}\bar{\|} \le C_{\mathfrak{l}}^{\Delta(\vec{p})} \; L^{5-\Delta(\vec{p})} \; \tfrac{\bar{\kappa}^{\vec{p}}}{\sigma_n(\vec{p})} \; \|\tilde{\mathcal{R}}_n^{(\vec{p})}\|_m$$

where

$$\bar{\kappa}^{\vec{p}} = \bar{\kappa}^{p_u} \bar{\kappa}'^{p_0 + p_{\mathrm{sp}}}$$

and

$$\sigma_n(\vec{p}) = \frac{1}{L^{11}} \begin{cases} \frac{1}{L^2} \frac{\bar{\kappa}'}{\bar{\kappa}_{\mathfrak{l}}} & \text{if } p_0 + p_{\mathrm{sp}} \ne 0 \\ \frac{\bar{\kappa}}{\bar{\kappa}_{\mathfrak{l}}} & \text{if } p_0 = p_{\mathrm{sp}} = 0 \end{cases}$$

$\vec p$	$\bar\kappa^{\vec p}$	$\frac{1}{\sigma_n(\vec p)}$	$L^{5-\Delta(\vec p)}$	$L^{5-\Delta(\vec p)}\frac{\bar\kappa^{\vec p}}{\sigma_n(\vec p)}$
$(1,1,0)$	$\bar\kappa\bar\kappa'$	$L^{13}\frac{\bar\kappa_{\mathfrak l}}{\bar\kappa'}$	L^0	$(L^2\bar\kappa)(L^{11}\bar\kappa_{\mathfrak l})$
$(0,1,1)$	$\bar\kappa'^2$	$L^{13}\frac{\bar\kappa_{\mathfrak l}}{\bar\kappa'}$	L^{-1}	$(L\bar\kappa')(L^{11}\bar\kappa_{\mathfrak l})$
$(0,0,2)$	$\bar\kappa'^2$	$L^{13}\frac{\bar\kappa_{\mathfrak l}}{\bar\kappa'}$	L^0	$(L^2\bar\kappa')(L^{11}\bar\kappa_{\mathfrak l})$
$(6,0,0)$	$\bar\kappa^6$	$L^{11}\frac{\bar\kappa_{\mathfrak l}}{\bar\kappa}$	L^{-4}	$L^{-4}\bar\kappa^5(L^{11}\bar\kappa_{\mathfrak l})$

(5.3)

Proof. (a) We first consider the case $n\ge 1$. We apply [8, Proposition 3.1.b] with $\mathfrak m = 2m$, $\mathfrak k=\bar\kappa$, $\mathfrak k'=\bar\kappa'$ and $\mathfrak k_{\mathfrak l}=\bar\kappa_{\mathfrak l}$. The hypotheses of [8, Proposition 3.1] are fulfilled by (5.2) so that

$$\left|\!\left|\!\left|\delta\hat\phi^{(+)}_{(*)n+1}\right|\!\right|\!\right| \le L^{29}K_{\rm bg}\left\{\|V_n\|_{2m}(\bar\kappa+\bar\kappa_{\mathfrak l})^2+|\mu_n|\right\}\bar\kappa_{\mathfrak l}$$

The claim follows easily using [7, (4.8.a)].

We now consider $n=0$. The kernel $\frac12 V_0^{(s)}=\frac12 V_1^{(u)}$ of $\mathbb S\mathcal V_0$ is given in [7, Remark 2.2.h] and fulfills $\|V_0^{(s)}\|_{2m}\le\frac1L\|V_0\|_{2m}$ by Lemma C.2.a. Expanding the quartic

$$(\mathbb S\mathcal V_0)\big(\hat\psi_*+\delta\psi_*\,,\,\hat\psi+\delta\psi\big)\Big|_{\substack{\hat\psi_{(*)}=\hat\psi_{0(*)}(\psi_*,\psi,\mu_0)\\ \delta\psi_{(*)}=L^{3/2}\mathbb S D^{(0)(*)}\mathbb S^{-1}z_{(*)}}}$$

in powers of $z_{(*)}$, we get, by [8, Remark 5.2and Proposition 2.1.a] and [5, Proposition 3.2.a]

$$\begin{aligned}
\|\delta A_0^{(2)}\| &\le \tbinom42\|V_1\|_{2m}\big[\|S_1(L^2\mu_0)^{(*)}Q_1^*\mathfrak Q_1\|_{2m}+K_{\rm bg}\|V_1\|_{2m}\bar\kappa^2\big]^2\bar\kappa^2\big[L^{\frac32}\|\mathbb S D^{(0)}\mathbb S^{-1}\|_{2m}\bar\kappa_{\mathfrak l}\big]^2\\
&\qquad + |\mu_0|\,L^5\,\|\mathbb S C^{(0)}\mathbb S^{-1}\|_{2m}\bar\kappa_{\mathfrak l}^2\\
&\le L^{21}C_{\mathfrak l}\big(\|V_0\|_{2m}\bar\kappa^2+|\mu_0|\big)\bar\kappa_{\mathfrak l}^2\\
\|\delta A_0^{(\ge3)}\| &\le \tbinom43\|V_1\|_{2m}\big[\|S_1(L^2\mu_0)^{(*)}Q_1^*\mathfrak Q_1\|_{2m}+K_{\rm bg}\|V_1\|_{2m}\bar\kappa^2\big]\,\bar\kappa\,\big[L^{\frac32}\|\mathbb S D^{(0)}\mathbb S^{-1}\|_{2m}\bar\kappa_{\mathfrak l}\big]^3\\
&\qquad + \tbinom44\|V_1\|_{2m}\big[L^{\frac32}\|\mathbb S D^{(0)}\mathbb S^{-1}\|_{2m}\bar\kappa_{\mathfrak l}\big]^4\\
&\le L^{42}C_{\mathfrak l}\,\|V_0\|_{2m}(\bar\kappa+\bar\kappa_{\mathfrak l})\bar\kappa_{\mathfrak l}^3
\end{aligned}$$

(b) Set

$$\tilde{\mathcal E}_{n+1,1}(\tilde\psi_*,\tilde\psi)=(\mathbb S\tilde{\mathcal E}_n)\big(\,(\Psi_*,\{\Psi_{*\nu}\})\,,\,(\Psi,\{\Psi_\nu\})\,\big)\Big|_{\substack{\Psi_{(*)}=\hat\psi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal V_n)\\ \Psi_{(*)\nu}=\hat\psi_{(*)n,\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,\mu_n,\mathcal V_n)}}$$

and

$$\delta\tilde{\mathcal{E}}_n(\tilde\psi_*,\tilde\psi,z_*,z) = (\mathbb{S}\tilde{\mathcal{E}}_n)\big(\tilde\Psi_*,\tilde\Psi\big)\Big|_{\substack{\Psi_{(*)}=\hat\psi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)+L^{3/2}\mathbb{S}D^{(n)(*)}\mathbb{S}^{-1}z_{(*)}\\ \Psi_{(*)\nu}=\hat\psi_{(*)n,\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,\mu_n,\mathcal{V}_n)+L^{3/2}\mathbb{S}_\nu\partial_\nu D^{(n)(*)}\mathbb{S}^{-1}z_{(*)}}} - (\mathbb{S}\tilde{\mathcal{E}}_n)\big(\tilde\Psi_*,\tilde\Psi\big)\Big|_{\substack{\Psi_{(*)}=\hat\psi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)\\ \Psi_{(*)\nu}=\hat\psi_{(*)n,\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,\mu_n,\mathcal{V}_n)}}$$

with the $\tilde{\mathcal{E}}_n$ of [7, Theorem 1.17] and the $\hat\psi_{(*)n,\nu}$ of [8, Proposition 5.1]. By [7, Remark 2.2.b] the two equations of part (b) hold. That $\tilde{\mathcal{E}}_{n+1,1}$ contains no scaling/weight relevant monomials follows from the degree properties of $\hat\psi_{(*)n}$ and $\hat\psi_{(*)n,\nu}$ specified in [8, Proposition 5.1].

We set

$$\begin{aligned}
\lambda &= \big\{K_{\rm bg} + \tfrac{1}{L^9}\|\mathbb{S}D^{(n)}\mathbb{S}^{-1}\|_{2m}\big\}\ \bar\kappa \\
\lambda' &= \max_{0\le\nu\le3}\big\{K_{\rm bg} + \tfrac{1}{L^{11}}\|\partial_\nu\mathbb{S}D^{(n)}\mathbb{S}^{-1}\|_{2m}\big\}\bar\kappa' \\
\sigma &= \frac{1}{L^{13}}\frac{\bar\kappa'}{\bar\kappa_{\mathfrak{l}}}
\end{aligned}\tag{5.4}$$

As $\mathfrak{v}_0$ is being chosen sufficiently small, depending on L,

$$\sigma = \frac{1}{L^{11}}\min\left\{\frac{\bar\kappa}{\bar\kappa_{\mathfrak{l}}}\ ,\ \frac{\bar\kappa'}{L^2\bar\kappa_{\mathfrak{l}}}\right\} \ge 1 \tag{5.5}$$

by [7, Definition 1.11]. Denote by $\|\ \cdot\ \|_\lambda$ the (auxiliary) norm with mass $2m$ that assigns the weight factors λ to the fields $\Psi_{(*)}$ and λ' to the fields $\Psi_{\nu(*)}$. By [8, Proposition 5.1], with $\mathfrak{k}=\bar\kappa$, $\mathfrak{k}'=\bar\kappa'$ and $\mathfrak{k}_{\mathfrak{l}}=\bar\kappa_{\mathfrak{l}}$,

$$\begin{gathered}
\overline{|||}\hat\psi_{(*)n}\overline{|||} + \sigma\,\overline{|||}L^{3/2}\mathbb{S}D^{(n)(*)}\mathbb{S}^{-1}z_{(*)}\overline{|||} \le \lambda \\
\overline{|||}\hat\psi_{(*)n,\nu}\overline{|||} + \sigma\,\overline{|||}L^{3/2}\mathbb{S}_\nu\partial_\nu D^{(n)(*)}\mathbb{S}^{-1}z_{(*)}\overline{|||} \le \lambda',\ \ 0\le\nu\le3
\end{gathered}\tag{5.6}$$

so that, by [5, Proposition 3.2.a,b],

$$\overline{\|}\tilde{\mathcal{E}}_{n+1,1}\overline{\|} \le \|\mathbb{S}\tilde{\mathcal{E}}_n\|_\lambda \qquad \overline{\|}\delta\tilde{\mathcal{E}}_n\overline{\|} \le \tfrac{1}{\sigma}\|\mathbb{S}\tilde{\mathcal{E}}_n\|_\lambda$$

For each monomial $\mathcal{M}$ of type $\vec p$ in $\tilde{\mathcal{E}}_n$, Lemma C.2.b with $\mathfrak{m}=2m$, $\check{\mathfrak{m}}=m$, $\mathfrak{k}=\lambda$, $\mathfrak{k}'=\lambda'$, $\check{\mathfrak{k}}=\kappa$ and $\check{\mathfrak{k}}'=\kappa'$, gives

$$\big\|\mathbb{S}\mathcal{M}\big\|_\lambda \le L^5\,\mathrm{Sdf}(\mathcal{M})\,\|\mathcal{M}\|^{(n)}$$

with

$$\mathrm{Sdf}(\mathcal{M}) = \Big(\frac{1}{L^{3/2}}\frac{\lambda}{\kappa}\Big)^{p_u}\Big(\frac{1}{L^{7/2}}\frac{\lambda'}{\kappa'}\Big)^{p_0}\Big(\frac{1}{L^{5/2}}\frac{\lambda'}{\kappa'}\Big)^{p_{\rm sp}} \le \mathrm{sdf}(\vec p; C_{\mathfrak l})$$

provided

$$K_{\rm bg} + \tfrac{1}{L^9}\|\mathbb{S}D^{(n)}\mathbb{S}^{-1}\|_{2m} + \max_{0\le\nu\le 3}\tfrac{1}{L^{11}}\|\partial_\nu\mathbb{S}D^{(n)}\mathbb{S}^{-1}\|_{2m} \le K_{\rm bg} + 3e^{2m}\Gamma_{\rm op} \le C_{\mathfrak l}$$

So we have

$$\big\|\mathbb{S}\tilde{\mathcal{E}}_n\big\|_\lambda \le L^5\,\mathrm{sdf}(C_{\mathfrak l})\,\big\|\tilde{\mathcal{E}}_n\big\|^{(n)} \tag{5.7}$$

and the conclusion follows.

(c) The first equation holds by [7, Remark 2.2.b] and the bound on $\big\|\mathbb{S}\tilde{\mathcal{R}}_n^{(\vec p)}\big\|_{2m}$ is an immediate consequence of Lemma C.2.a.

Set

$$\begin{aligned}\delta\tilde{\mathcal{R}}_n^{(\vec p)}(\tilde\psi_*,\tilde\psi,z_*,z) = {}&(\mathbb{S}\tilde{\mathcal{R}}_n^{(\vec p)})\big(\tilde\Phi_*,\tilde\Phi\big)\Big|_{\substack{\Phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)+\delta\hat\phi_{(*)n+1}(\psi_*,\psi,z_*,z)\\ \Phi_{(*)\nu}=\phi_{n+1(*),\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\\ +\delta\hat\phi_{(*)n+1,\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,z_*,z)}}\\ &-(\mathbb{S}\tilde{\mathcal{R}}_n^{(\vec p)})\big(\tilde\Phi_*,\tilde\Phi\big)\Big|_{\substack{\Phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\\ \Phi_{(*)\nu}=\phi_{n+1(*),\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,L^2\mu_n,\mathbb{S}\mathcal{V}_n)}}\end{aligned}$$

where, by [8, Proposition 2.1],

$$\begin{aligned}\phi_{n+1(*),\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,L^2\mu_n,\mathbb{S}\mathcal{V}_n) = {}&B^{(\pm)}_{n+1,L^2\mu_n,\nu}\psi_{(*)\nu}\\ &+\phi^{(\ge 3)}_{(*)n+1,\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\end{aligned}$$

The properties of $\delta\hat\phi_{(*)n+1,\nu}$ are given in [8, Proposition 3.1.e]. By [7, Remark 2.2.b] we have $\delta\mathcal{R}_n = \sum_{\vec p\in\mathfrak{D}}\delta\tilde{\mathcal{R}}_n^{(\vec p)}$.

To bound $\big\vert\!\big\vert\!\big\vert\delta\tilde{\mathcal{R}}_n^{(\vec p)}\big\vert\!\big\vert\!\big\vert$ we proceed as we did in part (b), but setting

$$\begin{aligned}\lambda &= \big\{\big\|S_{n+1}(L^2\mu_n)^{(*)}Q^*_{n+1}\mathfrak{Q}_{n+1}\big\|_{2m} + K_{\rm bg}\Lambda_\phi + K_{\rm bg}\big\}\ \bar\kappa\\ \lambda' &= \max_{0\le\nu\le 3}\big\{\max_{\sigma\in\pm}\big\|B^{(\sigma)}_{n+1,L^2\mu_n,\nu}\big\|_{2m} + K_{\rm bg}\Lambda_\phi + K_{\rm bg}\big\}\bar\kappa'\end{aligned}$$

Denote by $\|\cdot\|_\lambda$ the (auxiliary) norm with mass $2m$ that assigns the weight factors λ to the fields $\Phi_{(*)}$ and λ' to the fields $\Phi_{\nu(*)}$. By [8, Propositions 2.1 and 3.1.a,e], with $\mathfrak{k}=\bar\kappa$, $\mathfrak{k}'=\bar\kappa'$ and $\mathfrak{k}_{\mathfrak l}=\bar\kappa_{\mathfrak l}$,

$$\begin{aligned}\big\vert\!\big\vert\!\big\vert\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\big\vert\!\big\vert\!\big\vert + \sigma_n(\vec p)\,\big\vert\!\big\vert\!\big\vert\delta\hat\phi_{(*)n+1}\big\vert\!\big\vert\!\big\vert &\le \lambda \quad \text{if } p_u\ne 0\\ \big\vert\!\big\vert\!\big\vert B^{(\pm)}_{n+1,L^2\mu_n,\nu}\psi_{(*)\nu} + \phi^{(\ge 3)}_{(*)n+1,\nu}\big\vert\!\big\vert\!\big\vert + \sigma_n(\vec p)\,\big\vert\!\big\vert\!\big\vert\delta\hat\phi_{(*)n+1,\nu}\big\vert\!\big\vert\!\big\vert &\le \lambda' \quad \text{if } p_0+p_{\rm sp}\ne 0\end{aligned}$$

As in (5.5), $\sigma_n(\vec{p}) \ge 1$, so that, by [5, Proposition 3.2.b],

$$\bar{\|}\delta\tilde{\mathcal{R}}_n^{(\vec{p})}\bar{\|} \le \tfrac{1}{\sigma_n(\vec{p})}\|\mathbb{S}\tilde{\mathcal{R}}_n^{(\vec{p})}\|_\lambda \le \tfrac{1}{\sigma_n(\vec{p})} L^{5-\Delta(\vec{p})}\|\tilde{\mathcal{R}}_n^{(\vec{p})}\|_m \, \lambda^{p_{*u}+p_u}\prod_{\nu=0}^{3}\lambda'^{p_{*\nu}+p_\nu}$$

□

Parts (b) and (c) of Lemma 5.5 provide bounds on the constituents $\mathcal{E}_{n+1,1}$ and $\mathbb{S}\mathcal{R}_n$ of the contribution, $\mathcal{C}_n$, from the critical field in [7, Corollary 4.3]. We now provide a bound on the fluctuation integral $\mathcal{F}_n$.

Proposition 5.6. *There is an analytic function $\tilde{\mathcal{E}}_{\mathfrak{l}}(\tilde{\psi}_*,\tilde{\psi})$ and a constant $\mathcal{Z}_n'$ such that*

$$\mathcal{F}_n(\psi_*,\psi) = \tfrac{1}{\mathcal{Z}_n'} e^{\tilde{\mathcal{E}}_{\mathfrak{l}}((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\}))} \qquad \textit{and} \qquad \bar{\|}\tilde{\mathcal{E}}_{\mathfrak{l}}\bar{\|} \le \mathfrak{e}_{\mathfrak{l}}(n)$$

Proof. By [7, 4.4] and Lemma 5.5 the fluctuation integral is

$$\mathcal{F}_n(\psi_*,\psi) = \int d\mu_{r_n}(z^*,z)\; e^{\tilde{\mathcal{D}}(\tilde{\psi}_*,\tilde{\psi},z_*,z)}\Big|_{\tilde{\psi}_{(*)}=(\psi_{(*)},\{\partial_\nu\psi_{(*)}\})}$$

where

$$\int d\mu_{r_n}(z^*,z) = \Big[\prod_{w\in\mathcal{X}_1^{(n)}}\int_{|z(w)|\le r_n} \tfrac{dz(w)^*\wedge dz(w)}{2\pi i} e^{-|z(w)|^2}\Big]$$

and

$$\begin{aligned}\tilde{\mathcal{D}}(\tilde{\psi}_*,\tilde{\psi},z_*,z) = -\delta A_n^{(2)}(\psi_*,\psi,z_*,z) - \delta A_n^{(\ge 3)}(\psi_*,\psi,z_*,z)\\ + \delta\tilde{\mathcal{E}}_n\big(\tilde{\psi}_*,\tilde{\psi},z_*,z\big) + \sum_{\vec{p}\in\mathfrak{D}}\delta\tilde{\mathcal{R}}_n^{(\vec{p})}\big(\tilde{\psi}_*,\tilde{\psi},z_*,z\big)\end{aligned}$$

By Remark 5.1 and [7, Definition 1.11, Lemma C.5.d and (C.1.b)],

$$\begin{aligned}2L^{42}C_{\mathfrak{l}}\,\{\|V_n\|_{2m}(\bar{\kappa}+\bar{\kappa}_{\mathfrak{l}})^2+|\mu_n|\}\bar{\kappa}_{\mathfrak{l}}^2 &\le 2L^{42}C_{\mathfrak{l}}\,\{4\tfrac{\mathfrak{v}_0}{L^n}\bar{\kappa}^2 + 2L^{2n}(\mu_0-\mu_*) + \mathfrak{v}_0^{1-\epsilon}\}\bar{\kappa}_{\mathfrak{l}}^2\\ &\le \tfrac{1}{16}\mathfrak{e}_{\mathfrak{l}}(n)\end{aligned} \tag{5.8.a}$$

By [7, Definition 1.11],

$$L^{18}\tfrac{\bar{\kappa}_{\mathfrak{l}}}{\bar{\kappa}'}\;\mathrm{sdf}(C_{\mathfrak{l}})\;\mathfrak{v}_0^{\epsilon} \le \tfrac{1}{16}\mathfrak{e}_{\mathfrak{l}}(n) \tag{5.8.b}$$

By [7, Lemma C.2.a] and (5.3),

$$L^5\big(\tfrac{C_{\mathfrak{l}}}{L}\big)^{\Delta(\vec p)}\ \tfrac{1}{\sigma_n(\vec p)}\ \bar\kappa^{\vec p}\,\mathfrak{r}_{\vec p}(n,C_{\mathcal R})\le \tfrac{1}{16}\mathfrak{e}_{\mathfrak{l}}(n)\qquad\text{for each } \vec p\in\mathfrak{D} \tag{5.8.c}$$

provided $\mathfrak{v}_0$ is small enough that the hypothesis $\epsilon|\log\mathfrak{v}_0|\ge 2\log(1+C_{\mathcal R})\,\Pi_0^\infty(C_{\mathcal R})$ of [7, Lemma C.2] is satisfied. By Lemma 5.5 and (5.8) we have $\big\|\tilde{\mathcal D}\big\|\le\frac12\mathfrak{e}_{\mathfrak{l}}(n)\le\frac1{32}$ by [7, (C.1.a)]. [1, Theorem 3.4] yields the existence of an analytic function $\tilde{\mathcal E}_{\mathfrak{l}}\big(\tilde\psi_*,\tilde\psi\big)$ such that

$$\frac{\int d\mu_{r_n}(z^*,z)\ e^{\tilde{\mathcal D}(\tilde\psi_*,\tilde\psi,z_*,z)}}{\int d\mu_{r_n}(z^*,z)\ e^{\tilde{\mathcal D}(0,0,z_*,z)}}=e^{\tilde{\mathcal E}_{\mathfrak{l}}(\tilde\psi_*,\tilde\psi)}$$

and $\big\|\tilde{\mathcal E}_{\mathfrak{l}}\big\|\le\mathfrak{e}_{\mathfrak{l}}(n)$. □

In order to renormalize the chemical potential we will need more detailed information about the monomial in $\tilde{\mathcal E}_{\mathfrak{l}}(\tilde\psi_*,\tilde\psi)$ that is of type $\psi_*\psi$. To extract that information, we need more detailed information about the part of

$$\tilde{\mathcal D}(\tilde\psi_*,\tilde\psi,z_*,z)=-\delta A_n(\psi_*,\psi,z_*,z)+\delta\tilde{\mathcal E}_n\big(\tilde\psi_*,\tilde\psi,z_*,z\big)+\sum_{\vec p\in\mathfrak{D}}\delta\tilde{\mathcal R}_n^{(\vec p)}\big(\tilde\psi_*,\tilde\psi,z_*,z\big)$$

that is of degree at most one in each of ψ_* and ψ and is of degree zero in $\psi_{(*)\nu}$. So we define, on the space of field maps $\tilde{\mathcal G}(\tilde\psi_*,\tilde\psi,z_*,z)$ with $\big\|\tilde{\mathcal G}\big\|<\infty$, the projections

- P_2^ψ which extracts the part which is of degree exactly one in each of ψ_* and ψ, of degree zero in the $\psi_{(*)\nu}$'s and of arbitrary degree in $z_{(*)}$ and
- P_1^ψ which extracts the part which is of degree exactly one in $\psi_{(*)}$, of degree zero in the $\psi_{(*)\nu}$'s and of arbitrary degree in $z_{(*)}$ and
- P_0^ψ which extracts the part which is of degree zero in $\psi_{(*)}$ and the $\psi_{(*)\nu}$'s and of arbitrary degree in $z_{(*)}$.

Lemma 5.7. *There is a constant Λ_1, depending only on L, $\Gamma_{\rm op}$, $K_{\rm bg}$ and $\rho_{\rm bg}$, such that the following holds.*

(a) If $n=0$, $\big\|P_0^\psi\delta A_0\big\|\le\Lambda_1\big(|\mu_0|+\mathfrak{v}_0\big)\bar\kappa_{\mathfrak{l}}^4$, $\big\|P_1^\psi\delta A_0\big\|\le\Lambda_1\mathfrak{v}_0\bar\kappa\bar\kappa_{\mathfrak{l}}^3$ and $P_2^\psi\,\delta A_0=-\mathcal M_0$ where

$$\begin{aligned}\mathcal M_0(\psi_*,\psi,z_*,z)=-\tfrac{2}{L^3}\int_{\mathcal X_0}dx_1\cdots dx_4\int_{\mathcal X_0}dx_1'dx_4'\int_{\mathcal X_0^{(1)}}&dx_2'dx_3'\ V_0(x_1,x_2,x_3,x_4)\\ &D^{(0)*}(x_1,x_1')\ z_*(\mathbb L^{-1}x_1')\\ &(S_1(L^2\mu_0)Q_1^*\mathfrak{Q}_1)(\mathbb L^{-1}x_2,x_2')\ \psi(x_2')\\ &(S_1(L^2\mu_0)^*Q_1^*\mathfrak{Q}_1)(\mathbb L^{-1}x_3,x_3')\ \psi_*(x_3')\\ &D^{(0)}(x_4,x_4')\ z(\mathbb L^{-1}x_4')\end{aligned}$$

If $n \ge 1$, $\left|\!\left|\!\left| P_0^{\psi} \delta A_n \right|\!\right|\!\right| \le \Lambda_1 \big(|\mu_n| + \mathfrak{v}_{n+1}\big) \bar\kappa_{\mathfrak{l}}^4$, $\left|\!\left|\!\left| P_1^{\psi} \delta A_n \right|\!\right|\!\right| \le \Lambda_1 \mathfrak{v}_{n+1} \bar\kappa \bar\kappa_{\mathfrak{l}}^3$ *and* $\left|\!\left|\!\left| P_2^{\psi}\, \delta A_n + \mathcal{M}_n \right|\!\right|\!\right| \le \Lambda_1 \mathfrak{v}_{n+1}^2 \bar\kappa^2 \bar\kappa_{\mathfrak{l}}^4$ *where*

$$\begin{aligned}\mathcal{M}_n = -\tfrac{2}{L^3} \int_{\mathcal{X}_n} du_1 \cdots du_4 \int_{\mathcal{X}_0^{(n+1)}} dx_2 dx_3 \int_{\mathcal{X}_0^{(n)}} dx_1 dx_4 \; & V_n(u_1,u_2,u_3,u_4) \\ & (D^{(n)} \mathfrak{Q}_n\, Q_n S_n(\mu_n))(x_1,u_1)\ z_*(\mathbb{L}^{-1}x_1) \\ & (S_{n+1}(L^2\mu_n) Q_{n+1}^* \mathfrak{Q}_{n+1})(\mathbb{L}^{-1}u_2, x_2)\ \psi(x_2) \\ & (S_{n+1}(L^2\mu_n)^* Q_{n+1}^* \mathfrak{Q}_{n+1})(\mathbb{L}^{-1}u_3, x_3)\ \psi_*(x_3) \\ & (S_n(\mu_n) Q_n^* \mathfrak{Q}_n\, D^{(n)})(u_4, x_4)\ z(\mathbb{L}^{-1}x_4)\end{aligned}$$

fulfills $\left|\!\left|\!\left| \mathcal{M}_n \right|\!\right|\!\right| \le \Lambda_1 \mathfrak{v}_n \bar\kappa^2 \bar\kappa_{\mathfrak{l}}^2$.

(b) $\left|\!\left|\!\left| \big(P_2^{\psi} + P_1^{\psi} + P_0^{\psi}\big) \delta \tilde{\mathcal{E}}_n \right|\!\right|\!\right| \le L^{31}\ \mathrm{sdf}(C_{\mathfrak{l}})\ \frac{\bar\kappa_{\mathfrak{l}}^2}{\bar\kappa'^2}\ \big\|\tilde{\mathcal{E}}_n\big\|^{(n)}$

(c) For each $\vec p \in \mathfrak{D}$,

$$\left|\!\left|\!\left| P_2^{\psi} \delta \tilde{\mathcal{R}}_n^{(\vec p)} \right|\!\right|\!\right| \le \Lambda_1\, \mathfrak{r}_{\vec p}(n, C_{\mathcal{R}}) \begin{cases} \bar\kappa^2 \bar\kappa_{\mathfrak{l}}^4 & if\, \vec p = (6,0,0) \\ \bar\kappa \bar\kappa_{\mathfrak{l}} & if\ \vec p = (1,1,0) \\ \bar\kappa_{\mathfrak{l}}^2 & if\ \vec p = (0,1,1), (0,0,2) \end{cases}$$

$$\left|\!\left|\!\left| P_1^{\psi} \delta \tilde{\mathcal{R}}_n^{(\vec p)} \right|\!\right|\!\right| \le \Lambda_1\, \mathfrak{r}_{\vec p}(n, C_{\mathcal{R}}) \begin{cases} \bar\kappa \bar\kappa_{\mathfrak{l}}^5 & if\ \vec p = (6,0,0) \\ \bar\kappa \bar\kappa_{\mathfrak{l}} & if\ \vec p = (1,1,0) \\ \bar\kappa_{\mathfrak{l}}^2 & otherwise \end{cases}$$

$$\left|\!\left|\!\left| P_0^{\psi} \delta \tilde{\mathcal{R}}_n^{(\vec p)} \right|\!\right|\!\right| \le \Lambda_1\, \mathfrak{r}_{\vec p}(n, C_{\mathcal{R}}) \begin{cases} \bar\kappa_{\mathfrak{l}}^6 & if\ \vec p = (6,0,0) \\ \bar\kappa_{\mathfrak{l}}^2 & otherwise \end{cases}$$

Proof. In this proof, when we say that a contribution has norm of order xyz, we mean that the $\left|\!\left|\!\left| \cdot \right|\!\right|\!\right|$ norm of the contribution is bounded by a constant, depending only on L, Γ_{op}, K_{bg} and ρ_{bg}, times xyz.

(a) We first consider the case $n = 0$. By [8, (4.8.b), Remark 5.2 and Proposition 2.1.a] and [7, Definition 2.1],

$$\begin{aligned}\delta A_0(\psi_*, \psi, z_*, z) = & \int_0^1 (1-t)\, \tfrac{d^2}{dt^2} (\mathbb{S}\mathcal{V}_0)\big(\hat\psi_* + t\delta\psi_*\,,\ \hat\psi + t\delta\psi\big)\, dt \Big|_{\substack{\hat\psi_{(*)} = \hat\psi_{0(*)}(\psi_*, \psi, \mu_0, \mathcal{V}_0) \\ \delta\psi_{(*)} = L^{3/2} \mathbb{S} D^{(0)(*)} \mathbb{S}^{-1} z_{(*)}}} \\ & + \mu_0 L^5 \big\langle z_*,\ \mathbb{S} C^{(0)} \mathbb{S}^{-1} z \big\rangle_1\end{aligned}$$

$$
\begin{aligned}
= \tfrac{1}{2}\cdot 2\cdot 2\int_0^1 (1-t)\,\tfrac{d^2}{dt^2}\int_{\mathcal{X}_0} dx_1\cdots dx_4 \int_{\mathcal{X}_0} dx'_1 dx'_4 \int_{\mathcal{X}_0^{(1)}} dx'_2 dx'_3\; & V_0(x_1,x_2,x_3,x_4)\\
& D^{(0)*}(x_1,x'_1)\; tz_*(\mathbb{L}^{-1}x'_1)\\
& L^{-3/2}(S_1(L^2\mu_0)Q_1^*\mathfrak{Q}_1)(\mathbb{L}^{-1}x_2,x'_2)\;\psi(x'_2)\\
& L^{-3/2}(S_1(L^2\mu_0)^*Q_1^*\mathfrak{Q}_1)(\mathbb{L}^{-1}x_3,x'_3)\,\psi_*(x'_3)\\
& D^{(0)}(x_4,x'_4)\; tz(\mathbb{L}^{-1}x'_4) + \text{h.o.}\\
= \tfrac{2}{L^3}\int_{\mathcal{X}_0} dx_1\cdots dx_4 \int_{\mathcal{X}_0} dx'_1 dx'_4 \int_{\mathcal{X}_0^{(1)}} dx'_2 dx'_3\; & V_0(x_1,x_2,x_3,x_4)\\
& D^{(0)*}(x_1,x'_1)\; z_*(\mathbb{L}^{-1}x'_1)\\
& (S_1(L^2\mu_0)Q_1^*\mathfrak{Q}_1)(\mathbb{L}^{-1}x_2,x'_2)\;\psi(x'_2)\\
& (S_1(L^2\mu_0)^*Q_1^*\mathfrak{Q}_1)(\mathbb{L}^{-1}x_3,x'_3)\;\psi_*(x'_3)\\
& D^{(0)}(x_4,x'_4)\; z(\mathbb{L}^{-1}x'_4) + \text{h.o.}
\end{aligned}
$$

with the contributions in h.o. being either

- independent of $\psi_{(*)}$ with norms of order $(|\mu_0| + \mathfrak{v}_0)\bar\kappa_{\mathfrak{l}}^4$ or
- of order precisely one in $\psi_{(*)}$ with norms of order $\mathfrak{v}_0\bar\kappa\bar\kappa_{\mathfrak{l}}^3$ or
- of order at least two in ψ or of order at least two in ψ_* or of order at least three in $\psi_{(*)}$.

We now consider the case $n \ge 1$. By [7, (4.8.a)],

$$
\begin{aligned}
\delta A_n(\psi_*,\psi,z_*,z) = & -L^{7/2}\int_0^1 dt\,\big\langle z_*, \mathbb{S}D^{(n)}\mathfrak{Q}_n\,Q_n\mathbb{S}^{-1}\,\delta\hat\phi_{n+1}^{(+)}\big(\psi_*,\psi;t\,z_*,t\,z\big)\big\rangle_1\\
& - L^{7/2}\int_0^1 dt\,\big\langle \mathbb{S}D^{(n)*}\mathfrak{Q}_n\,Q_n\mathbb{S}^{-1}\,\delta\hat\phi_{*n+1}^{(+)}\big(\psi_*,\psi;t\,z_*,t\,z\big),\ z\big\rangle_1
\end{aligned}
\tag{5.9}
$$

By [8, Proposition 3.1.d], using the notation of [7, Definition 3.1],

$$
\begin{aligned}
\delta\hat\phi_{(*)n+1}^{(+)}(\psi_*,\psi,z_*,z) = & L^{3/2}\mathbb{S}[S_n(\mu_n)^{(*)} - S_n^{(*)}]Q_n^*\mathfrak{Q}_nD^{(n)(*)}\mathbb{S}^{-1}z_{(*)}\\
& - L^{\frac{3}{2}}\mathbb{L}_*^{-1}S_n(\mu_n)^{(*)}\mathcal{V}'_{(*)}(\varphi_{(*)},\varphi_{(\bar *)},\varphi_{(*)})\Big|_{\substack{\varphi_*=\phi_*\\ \varphi=\phi}}^{\substack{\varphi_*=\phi_*+\delta\phi_*\\ \varphi=\phi+\delta\phi}} + \delta\hat\phi_{(*)}^{(\text{h.o.})}
\end{aligned}
$$

with the substitutions

$$
\begin{aligned}
\phi_{(*)} &= \mathbb{S}^{-1}S_{n+1}(L^2\mu_n)^{(*)}Q_{n+1}^*\mathfrak{Q}_{n+1}\,\psi_{(*)}\\
\delta\phi_{(*)} &= S_n(\mu_n)^{(*)}Q_n^*\mathfrak{Q}_n\,L^{3/2}D^{(n)(*)}\mathbb{S}^{-1}z_{(*)}
\end{aligned}
\tag{5.10}
$$

and with the contributions in $\delta\hat\phi^{(\rm h.o.)}_{(*)}$ being of order at least five in $(\psi_{(*)}, z_{(*)})$ and obeying

$$\big|\!\big|\!\big| P_j^\psi \delta\hat\phi^{(\rm h.o.)}_{(*)} \big|\!\big|\!\big| \le L^{47} K_{\rm bg}\, \mathfrak{v}_{n+1}^2 \bar\kappa^j \bar\kappa_{\mathfrak{l}}^{5-j} \qquad \text{for } j=0,1,2$$

Here $(\bar *)$ means the opposite of $(*)$ — i.e nothing when $(*)=*$ and $*$ when $(*)=$nothing.

So the two terms on the right hand side of (5.9) are (minus)

$$\begin{aligned} L^{7/2}\int_0^1 dt\, \big\langle z_*, \mathbb{S}D^{(n)}\mathfrak{Q}_n\, Q_n \mathbb{S}^{-1}\, \delta\hat\phi^{(+)}_{n+1}\big(\psi_*,\psi; t\,z_*, t\,z\big)\big\rangle_1 \\ = -2L^{7/2}\int_0^1 dt\, \big\langle z_*, \mathbb{S}D^{(n)}\mathfrak{Q}_n\, Q_n S_n(\mu_n)\mathcal{V}'(\phi,\phi_*,t\delta\phi)\big\rangle_1 + \text{h.o.} \\ = -L^{3/2}\big\langle \mathbb{S}^{-1}z_*, D^{(n)}\mathfrak{Q}_n\, Q_n S_n(\mu_n)\mathcal{V}'(\phi,\phi_*,\delta\phi)\big\rangle_0 + \text{h.o.} \\ = -L^{-3}\int_{\mathcal{X}_n} du_1\cdots du_4 \int_{\mathcal{X}_0^{(n+1)}} dx_2 dx_3 \int_{\mathcal{X}_0^{(n)}} dx_1 dx_4\ V_n(u_1,u_2,u_3,u_4) \quad (5.11) \\ (D^{(n)}\mathfrak{Q}_n\, Q_n S_n(\mu_n))(x_1,u_1)\ z_*(\mathbb{L}^{-1}x_1) \\ (S_{n+1}(L^2\mu_n)Q^*_{n+1}\mathfrak{Q}_{n+1})(\mathbb{L}^{-1}u_2,x_2)\ \psi(x_2) \\ (S_{n+1}(L^2\mu_n)^*Q^*_{n+1}\mathfrak{Q}_{n+1})(\mathbb{L}^{-1}u_3,x_3)\ \psi_*(x_3) \\ (S_n(\mu_n)Q_n^*\mathfrak{Q}_n\, D^{(n)})(u_4,x_4)\ z(\mathbb{L}^{-1}x_4) + \text{h.o.} \end{aligned}$$

and, similarly,

$$\begin{aligned} L^{7/2}\int_0^1 dt\, \big\langle \mathbb{S}D^{(n)*}\mathfrak{Q}_n\, Q_n \mathbb{S}^{-1}\, \delta\hat\phi^{(+)}_{*n+1}\big(\psi_*,\psi; t\,z_*, t\,z\big),\ z\big\rangle_1 \\ = -L^{-3}\int_{\mathcal{X}_n} du_1\cdots du_4 \int_{\mathcal{X}_0^{(n+1)}} dx_1 dx_2 \int_{\mathcal{X}_0^{(n)}} dx_3 dx_4\ V_n(u_1,u_2,u_3,u_4) \quad (5.12) \\ (D^{(n)*}\mathfrak{Q}_n\, Q_n S_n(\mu_n)^*)(x_4,u_4)\, z(\mathbb{L}^{-1}x_4) \\ (S_{n+1}(L^2\mu_n)^*Q^*_{n+1}\mathfrak{Q}_{n+1})(\mathbb{L}^{-1}u_1,x_1)\, \psi_*(x_1) \\ (S_{n+1}(L^2\mu_n)Q^*_{n+1}\mathfrak{Q}_{n+1})(\mathbb{L}^{-1}u_2,x_2)\, \psi(x_2) \quad (5.13) \\ (S_n(\mu_n)^*Q_n^*\mathfrak{Q}_n\, D^{(n)*})(u_3,x_3)\, z_*(\mathbb{L}^{-1}x_3) + \text{h.o.} \end{aligned}$$

with the contributions in h.o. being either

- independent of $\psi_{(*)}$ with norm of order $\big(|\mu_n| + \mathfrak{v}_{n+1}\big)\bar\kappa_{\mathfrak{l}}^4$ or
- of order precisely one in $\psi_{(*)}$ with norm of order $\mathfrak{v}_{n+1}\bar\kappa\bar\kappa_{\mathfrak{l}}^3$ or
- of order precisely two in $\psi_{(*)}$ with norm of order $\mathfrak{v}_{n+1}^2\bar\kappa^2\bar\kappa_{\mathfrak{l}}^4$ or

- of order at least two in ψ or of order at least two in ψ_* or of order at least three in $\psi_{(*)}$.

Finally, observe that the two integrals of the right hand sides of (5.11) and (5.12) are equal — just make the change of variables $x_1 \leftrightarrow x_3$, $u_1 \leftrightarrow u_3$ in one of them.

(b) The bound on $\delta\tilde{\mathcal{E}}_n$ given in Lemma 5.5.b is

$$\left\|\!\left|\delta\tilde{\mathcal{E}}_n\right|\!\right\| \le L^{18}\ \tfrac{\bar\kappa_{\mathfrak{l}}}{\bar\kappa'}\ \mathrm{sdf}(C_{\mathfrak{l}})\ \left\|\tilde{\mathcal{E}}_n\right\|^{(n)} \le L^{13}\ L^{-(\eta'-\epsilon/2)(n+1)}\mathfrak{v}_0^{1/3-\epsilon/2}$$

which implies that the kernel of the monomial of type $\psi_*\psi$ in $\delta\tilde{\mathcal{E}}_n$ has L^1–L^∞ norm bounded by

$$L^{13}\ L^{-(\eta'-\epsilon/2)(n+1)}\mathfrak{v}_0^{1/3-\epsilon/2}\ \tfrac{1}{\bar\kappa^2} = L^{13}\ L^{-(2\eta+\eta'-\epsilon/2)(n+1)}\mathfrak{v}_0^{1-5\epsilon/2}$$

This is not adequate for our present purposes. We want a power of $\mathfrak{v}_0$ that is strictly greater than one — the contributions from $\delta\tilde{\mathcal{E}}_n$ should be of higher order than the dominant contributions coming from δA_n, which are of order exactly one in the coupling constant. We can get that by exploiting the fact that we only care about the part of $\delta\tilde{\mathcal{E}}_n(\tilde\psi_*, \tilde\psi, z_*, z)$ that is of degree at most 2 in $\psi_{(*)}$ and of degree zero in the $\psi_{(*),\nu}$'s.

Recall that

$$\begin{aligned}\delta\tilde{\mathcal{E}}_n(\tilde\psi_*, \tilde\psi, z_*, z) = (\mathbb{S}\tilde{\mathcal{E}}_n)\big(\tilde\Psi_*, \tilde\Psi\big)\Big|_{\substack{\Psi_{(*)}=\hat\psi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)+L^{3/2}\mathbb{S}D^{(n)(*)}\mathbb{S}^{-1}z_{(*)}\\ \Psi_{(*)\nu}=\hat\psi_{(*)n,\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,\mu_n,\mathcal{V}_n)+L^{3/2}\mathbb{S}_\nu\partial_\nu D^{(n)(*)}\mathbb{S}^{-1}z_{(*)}}}\\ - (\mathbb{S}\tilde{\mathcal{E}}_n)\big(\tilde\Psi_*, \tilde\Psi\big)\Big|_{\substack{\Psi_{(*)}=\hat\psi_{(*)n}(\psi_*,\psi,\mu_n,\mathcal{V}_n)\\ \Psi_{(*)\nu}=\hat\psi_{(*)n,\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,\mu_n,\mathcal{V}_n)}}\end{aligned}$$

with the $\tilde{\mathcal{E}}_n$ of [7, Theorem 1.17] and the $\hat\psi_{(*)n,\nu}$ of [8, Proposition 5.1]. Let us denote by $\delta\tilde{\mathcal{E}}_n^{(\le 2)}(\tilde\Psi_*, \tilde\Psi, \delta\tilde\Psi_*, \delta\tilde\Psi_*)$ the part of

$$(\mathbb{S}\tilde{\mathcal{E}}_n)\big(\tilde\Psi_* + \delta\tilde\Psi_*, \tilde\Psi + \delta\tilde\Psi\big) - (\mathbb{S}\tilde{\mathcal{E}}_n)\big(\tilde\Psi_*, \tilde\Psi\big)$$

that is of degree at most two in $\Psi_{(*)}$ and of degree zero in $\Psi_{(*)\nu}$. Set $\delta\tilde{\mathcal{E}}_n^{(\le 2)}(\tilde\psi_*, \tilde\psi, z_*, z)$ to be $\delta\tilde{E}_n^{(\le 2)}(\tilde\Psi_*, \tilde\Psi, \delta\tilde\Psi_*, \delta\tilde\Psi_*)$ evaluated at

$$\begin{aligned}\Psi_{(*)} &= \hat\psi_{(*)n}(\psi_*, \psi, \mu_n, \mathcal{V}_n)\\ \Psi_{(*)\nu} &= \hat\psi_{(*)n,\nu}(\psi_*, \psi, \psi_{*\nu}, \psi_\nu, \mu_n, \mathcal{V}_n)\\ \delta\Psi_{(*)} &= L^{3/2}\mathbb{S}D^{(n)(*)}\mathbb{S}^{-1}z_{(*)}\\ \delta\Psi_{(*)\nu} &= L^{3/2}\mathbb{S}_\nu\partial_\nu D^{(n)(*)}\mathbb{S}^{-1}z_{(*)}\end{aligned}$$

Since $\hat{\psi}_{(*)n}$ is of degree at least one in $\psi_{(*)}$ and $\hat{\psi}_{(*)n,\nu}$ is of degree at least one in $\psi_{(*)\nu}$, every monomial in $\delta\tilde{\mathcal{E}}_n - \delta\tilde{\mathcal{E}}_n^{(\le 2)}$ is either of degree at least one in the $\psi_{(*)\nu}$'s or of degree at least three in $\psi_{(*)}$. So

$$\big(P_2^\psi + P_1^\psi + P_0^\psi\big)\delta\tilde{\mathcal{E}}_n = \big(P_2^\psi + P_1^\psi + P_0^\psi\big)\delta\tilde{\mathcal{E}}_n^{(\le 2)}$$

Note further that, since every monomial in $\tilde{\mathcal{E}}_n$ that is of degree at least one in $\psi_{(*)}$ is actually of degree at least 4 in $(\psi_{(*)}, \psi_{(*)\nu})$, and every monomial in $\tilde{\mathcal{E}}_n$ is of degree at least 2, we have that every monomial in $\delta\tilde{\mathcal{E}}_n^{(\le 2)}$ is of degree at least 2 in $z_{(*)}$.

We define λ, λ' and σ by (5.4). As in the proof of Lemma 5.5.b, denote by $\|\,\cdot\,\|_\lambda$ the (auxiliary) norm with mass $2m$ that assigns the weight factors λ to the fields $\Psi_{(*)}$ and λ' to the fields $\Psi_{\nu(*)}$. Then, by [5, Lemma 3.1.a and Proposition 3.2.a], (5.5), (5.6) and (5.7),

$$\bar{\|}\delta\tilde{\mathcal{E}}_n^{(\le 2)}\bar{\|} \le \tfrac{1}{\sigma^2}\|\mathbb{S}\tilde{\mathcal{E}}_n\|_\lambda \le \tfrac{1}{\sigma^2}L^5\,\mathrm{sdf}(C_{\mathfrak{l}})\,\big\|\tilde{\mathcal{E}}_n\big\|^{(n)} \le L^{31}\ \mathrm{sdf}(C_{\mathfrak{l}})\ \tfrac{\bar{\kappa}_{\mathfrak{l}}^2}{\bar{\kappa}'^2}\ \big\|\tilde{\mathcal{E}}_n\big\|^{(n)}$$

(c) Recall from Lemma 5.5.c that

$$\begin{aligned}\delta\tilde{\mathcal{R}}_n^{(\vec{p})}(\tilde{\psi}_*, \tilde{\psi}, z_*, z) = {} & (\mathbb{S}\tilde{\mathcal{R}}_n^{(\vec{p})})\big(\tilde{\Phi}_*, \tilde{\Phi}\big)\Big|_{\substack{\Phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)+\delta\hat{\phi}_{(*)n+1}(\psi_*,\psi,z_*,z)\\ \Phi_{(*)\nu}=\phi_{n+1(*),\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\\ +\delta\hat{\phi}_{(*)n+1,\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,z_*,z)}} \\ & - (\mathbb{S}\tilde{\mathcal{R}}_n^{(\vec{p})})\big(\tilde{\Phi}_*, \tilde{\Phi}\big)\Big|_{\substack{\Phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\\ \Phi_{(*)\nu}=\phi_{n+1(*),\nu}(\psi_*,\psi,\psi_{*\nu},\psi_\nu,L^2\mu_n,\mathbb{S}\mathcal{V}_n)}}\end{aligned}$$

The claims follow from the observations that

- $\phi_{(*)n+1}$ is of degree at least one in $\psi_{(*)}$ and $\phi_{(*)n+1,\nu}$ is of degree one in $\psi_{(*),\nu}$, and,
- by [8, Propositions 2.1.a and 3.1.a,e],

$$\begin{aligned}\bar{|||}\phi_{(*)n+1}(\psi_*, \psi, L^2\mu_n, \mathbb{S}\mathcal{V}_n)\bar{|||} &\le \big\{\big\|S_{n+1}^{(*)}Q_{n+1}^*\mathfrak{Q}_{n+1}\big\|_{2m} + K_{\mathrm{bg}}\rho_{\mathrm{bg}}\big\}\ \bar{\kappa} \\ \bar{|||}\delta\hat{\phi}_{(*)n+1}\bar{|||} &\le L^{11}K_{\mathrm{bg}}\,\bar{\kappa}_{\mathfrak{l}} \\ \bar{|||}\delta\hat{\phi}_{(*)n+1,\nu}\bar{|||} &\le L_\nu L^{11}K_{\mathrm{bg}}\,\bar{\kappa}_{\mathfrak{l}}\end{aligned}$$

  and,
- by Lemma C.2.a,

$$\|\mathbb{S}\tilde{\mathcal{R}}_n^{(\vec{p})}\|_{2m} \le L^{5-\Delta(\vec{p})}\|\tilde{\mathcal{R}}_n^{(\vec{p})}\|_m \le L^{5-\Delta(\vec{p})}\mathfrak{r}_{\vec{p}}(n, C_{\mathcal{R}})$$

□

Proposition 5.8. *There is a constant Λ_2, depending only on L, $\Gamma_{\rm op}$, $K_{\rm bg}$ and $\rho_{\rm bg}$ such that the following holds.*

(a) *Denote by $P_{\psi_*\psi}$ the projection, on the space of analytic functions $\tilde{\mathcal{G}}(\tilde\psi_*,\tilde\psi)$, which extracts the part which is of degree exactly one in each of ψ_* and ψ, of degree zero in the $\psi_{(*)\nu}$'s. Also set, for $n\ge 0$,*

$$M_n'(\psi_*,\psi)=\frac{\int d\mu_{r_n}(z^*,z)\ \mathcal{M}_n(\psi_*,\psi,z^*,z)}{\int d\mu_{r_n}(z^*,z)}$$

Then

$$\|P_{\psi_*\psi}\tilde{\mathcal{E}}_{\mathfrak{l}}-M_n'(\psi_*,\psi)\|_{2m}\le \Lambda_2\,\mathfrak{v}_0\left(\mathfrak{v}_0^{\frac{1}{3}-5\epsilon}+|\mu_n|\right)\bar\kappa_{\mathfrak{l}}^6$$

(b) *Set*

$$\begin{aligned}M_0(\psi_*,\psi)=-\tfrac{2}{L^3}\int_{\mathcal{X}_0}dx_1\cdots dx_4\int_{\mathcal{X}_0^{(1)}}&dx_2'dx_3'\ V_0(x_1,x_2,x_3,x_4)\\&(S_1(L^2\mu_0)Q_1^*\mathfrak{Q}_1)(\mathbb{L}^{-1}x_2,x_2')\ \psi(x_2')\\&(S_1(L^2\mu_0)^*Q_1^*\mathfrak{Q}_1)(\mathbb{L}^{-1}x_3,x_3')\ \psi_*(x_3')\\&C^{(0)}(x_4,x_1)\end{aligned}$$

and, for $n\ge 1$,

$$\begin{aligned}M_n(\psi_*,\psi)=-\tfrac{2}{L^3}\int_{\mathcal{X}_n}du_1\cdots du_4\int_{\mathcal{X}_0^{(n+1)}}&dx_2dx_3\ V_n(u_1,u_2,u_3,u_4)\\&(S_n(\mu_n)Q_n^*\mathfrak{Q}_n\,C^{(n)}\mathfrak{Q}_n\,Q_nS_n(\mu_n))(u_4,u_1)\\&(S_{n+1}(L^2\mu_n)Q_{n+1}^*\mathfrak{Q}_{n+1})(\mathbb{L}^{-1}u_2,x_2)\ \psi(x_2)\\&(S_{n+1}(L^2\mu_n)^*Q_{n+1}^*\mathfrak{Q}_{n+1})(\mathbb{L}^{-1}u_3,x_3)\ \psi_*(x_3)\end{aligned}$$

Then

$$\|M_n'(\psi_*,\psi)-M_n(\psi_*,\psi)\|_{2m}\le\Lambda_2\,\mathfrak{v}_n\,r_n^2e^{-r_n^2}$$

Proof. (a) Set

$$\tilde{\mathcal{D}}_2(\tilde\psi_*,\tilde\psi,z_*,z)=\left(P_2^\psi+P_1^\psi+P_0^\psi\right)\Big\{-\delta A_n+\delta\tilde{\mathcal{E}}_n+\sum_{\vec p\in\mathfrak{D}}\delta\tilde{\mathcal{R}}_n^{(\vec p)}\Big\}$$

We are to bound

$$P_{\psi_*\psi}\tilde{\mathcal{E}}_{\mathfrak{l}} - M'_n(\psi_*,\psi) = P_{\psi_*\psi}\ln\left[\frac{\int d\mu_{r_n}(z^*,z)\ e^{\tilde{\mathcal{D}}(\tilde\psi_*,\tilde\psi,z_*,z)}}{\int d\mu_{r_n}(z^*,z)\ e^{\tilde{\mathcal{D}}(0,0,z_*,z)}}\right] - M'_n(\psi_*,\psi)$$
$$= P_{\psi_*\psi}\ln\left[\frac{\int d\mu_{r_n}(z^*,z)\ e^{\tilde{\mathcal{D}}_2(\tilde\psi_*,\tilde\psi,z_*,z)}}{\int d\mu_{r_n}(z^*,z)\ e^{\tilde{\mathcal{D}}_2(0,0,z_*,z)}}\right] - M'_n(\psi_*,\psi)$$

We use [1, Corollary 3.5] with $n=1$, $d\mu$ being the normalized measure

$$d\mu_{r_n}\Big/\int d\mu_{r_n}(z^*,z)$$

with $f=\tilde{\mathcal{D}}_2$, and with the norm $\|\ \cdot\ \|_w$ of mass $2m$ which assigns a weight w (that we shall choose shortly) to ψ_*,ψ and the weight $\bar\kappa_{\mathfrak{l}}$ to the fields z_*, z. We have, by Lemma 5.7 and [7, Theorem 1.17, Definition 1.11 and Lemma C.2.b],

$$\begin{aligned}
\big\|P_2^\psi\tilde{\mathcal{D}}_2-\mathcal{M}_n\big\|_{w\,\frac{1}{w^2}} &\le \Lambda_1\mathfrak{v}_{n+1}^2\bar\kappa_{\mathfrak{l}}^4 + L^{31}\ \tfrac{\bar\kappa_{\mathfrak{l}}^2}{\bar\kappa^2\bar\kappa'^2}\ \ \mathrm{sdf}(C_{\mathfrak{l}})\ \big\|\tilde{\mathcal{E}}_n\big\|^{(n)}\\
&\qquad + \Lambda_1\sum_{\vec p\in\mathfrak{D}}\mathfrak{r}_{\vec p}(n,C_{\mathcal{R}})\begin{cases}\bar\kappa_{\mathfrak{l}}/\bar\kappa & \text{if } \vec p=(1,1,0)\\ \bar\kappa_{\mathfrak{l}}^2/\bar\kappa^2 & \text{if } \vec p=(0,1,1),\ (0,0,2)\\ \bar\kappa_{\mathfrak{l}}^4 & \text{if } \vec p=(6,0,0)\end{cases}\\
&\le \Lambda_1\mathfrak{v}_n^2\bar\kappa_{\mathfrak{l}}^4 + L^{-2(\eta+\eta')(n+1)}\mathfrak{v}_0^{4/3-5\epsilon}\mathfrak{v}_0^{\epsilon}\bar\kappa_{\mathfrak{l}}^2 + \mathfrak{v}_0^{\frac{\epsilon}{2}}\bar\kappa_{\mathfrak{l}}^2\min\left\{\mathfrak{v}_0^{\frac{4}{3}-7\epsilon},\tfrac{\mathfrak{v}_0}{L^n}\right\}\\
&\le \bar\kappa_{\mathfrak{l}}^2\min\left\{\mathfrak{v}_0^{\frac{4}{3}-7\epsilon},\mathfrak{v}_n\right\}
\end{aligned}\tag{5.14}$$

and

$$\big\|P_2^\psi\tilde{\mathcal{D}}_2\big\|_{w\,\frac{1}{w^2}} \le \big\|P_2^\psi\tilde{\mathcal{D}}_2-\mathcal{M}_n\big\|_{w\,\frac{1}{w^2}} + \lVert\!|\mathcal{M}_n|\!\rVert_{\frac{1}{\bar\kappa^2}} \le \Lambda_2'\mathfrak{v}_n\bar\kappa_{\mathfrak{l}}^2$$

Also

$$\begin{aligned}
\big\|P_1^\psi\tilde{\mathcal{D}}_2\big\|_{w\,\frac{1}{w}} &\le \Lambda_1\mathfrak{v}_n\bar\kappa_{\mathfrak{l}}^3 + L^{31}\ \tfrac{\bar\kappa_{\mathfrak{l}}^2}{\bar\kappa\bar\kappa'^2}\ \ \mathrm{sdf}(C_{\mathfrak{l}})\ \big\|\tilde{\mathcal{E}}_n\big\|^{(n)}\\
&\qquad + \Lambda_1\sum_{\vec p\in\mathfrak{D}}\mathfrak{r}_{\vec p}(n,C_{\mathcal{R}})\begin{cases}\bar\kappa_{\mathfrak{l}} & \text{if } \vec p=(1,1,0)\\ \bar\kappa_{\mathfrak{l}}^2/\bar\kappa & \text{if } \vec p=(0,1,1),\ (0,0,2)\\ \bar\kappa_{\mathfrak{l}}^5 & \text{if } \vec p=(6,0,0)\end{cases}\\
&\le \Lambda_1\tfrac{\mathfrak{v}_0}{L^n}\bar\kappa_{\mathfrak{l}}^3 + L^{31}\mathrm{sdf}(C_{\mathfrak{l}})\ L^{-(\eta+2\eta')n}\mathfrak{v}_0^{1-2\epsilon}\ \bar\kappa_{\mathfrak{l}}^2 + \mathfrak{v}_0^{1-\frac{9}{2}\epsilon}\bar\kappa_{\mathfrak{l}}^3\\
&\le \mathfrak{v}_0^{1-5\epsilon}\bar\kappa_{\mathfrak{l}}^3
\end{aligned}$$

if $\mathfrak{v}_0$ is small enough, by [7, Definition 1.11, Lemma C.1.d], in the case (1,1,0) and [7, Lemma C.2.b] in the other cases. Furthermore

$$\begin{aligned}\big\|P_0^\psi\tilde{\mathcal{D}}_2\big\|_w &\le \Lambda_1\big(|\mu_n|+\mathfrak{v}_n\big)\bar\kappa_\mathfrak{l}^4 + L^{31}\ \tfrac{\bar\kappa_\mathfrak{l}^2}{\bar\kappa'^2}\ \mathrm{sdf}(C_\mathfrak{l})\ \big\|\tilde{\mathcal{E}}_n\big\|^{(n)}\\ &\qquad + \Lambda_1\sum_{\vec p\in\mathfrak{D}}\mathfrak{r}_{\vec p}(n,C_\mathcal{R})\begin{cases}\bar\kappa_\mathfrak{l}^6 & \text{if } \vec p=(6,0,0)\\ \bar\kappa_\mathfrak{l}^2 & \text{otherwise}\end{cases}\\ &\le \Lambda_1\big(|\mu_n|+\mathfrak{v}_n\big)\bar\kappa_\mathfrak{l}^4 + L^{31}\mathrm{sdf}(C_\mathfrak{l})\ L^{-2\eta' n}\mathfrak{v}_0^{2/3-\epsilon}\ \bar\kappa_\mathfrak{l}^2 + \mathfrak{v}_0^{\frac{2}{3}}\bar\kappa_\mathfrak{l}^4\\ &\le \Lambda_2'\big(|\mu_n|+\mathfrak{v}_0^{\frac{2}{3}-\epsilon}\big)\bar\kappa_\mathfrak{l}^4\end{aligned}$$

by [7, Definition 1.11 and Lemma C.1.d]. We now set

$$w=\bar\kappa_\mathfrak{l}\sqrt{\frac{|\mu_n|+\mathfrak{v}_0^{\frac{2}{3}-\epsilon}}{\mathfrak{v}_0}}$$

so that

$$\begin{aligned}\big\|\tilde{\mathcal{D}}_2\big\|_w &\le \Lambda_2'\Big\{L^{-n}\big(|\mu_n|+\mathfrak{v}_0^{\frac{2}{3}-\epsilon}\big)+\mathfrak{v}_0^{1-5\epsilon}\sqrt{\frac{|\mu_n|+\mathfrak{v}_0^{\frac{2}{3}-\epsilon}}{\mathfrak{v}_0}}+\big(|\mu_n|+\mathfrak{v}_0^{\frac{2}{3}-\epsilon}\big)\Big\}\bar\kappa_\mathfrak{l}^4\\ &\le 3\Lambda_2'\big(|\mu_n|+\mathfrak{v}_0^{\frac{2}{3}-\epsilon}\big)\bar\kappa_\mathfrak{l}^4\end{aligned}$$

and, by [1, Corollary 3.5] with $d\mu(z^*,z)$ being the normalized $d\mu_{r_n}(z^*,z)$,

$$\Big\|P_{\psi_*\psi}\Big(\tilde{\mathcal{E}}_\mathfrak{l}-\int d\mu\ \tilde{\mathcal{D}}_2(\psi_*,\psi,z^*,z)\Big)\Big\|_w \le 40^2\big\|\tilde{\mathcal{D}}_2\big\|_w^2 \le \big(120\Lambda_2'\big)^2\big(|\mu_n|+\mathfrak{v}_0^{\frac{2}{3}-\epsilon}\big)^2\bar\kappa_\mathfrak{l}^8$$

Hence

$$\begin{aligned}\Big\|P_{\psi_*\psi}\Big(\tilde{\mathcal{E}}_\mathfrak{l}-\int d\mu\ \tilde{\mathcal{D}}_2(\psi_*,\psi,z^*,z)\Big)\Big\|_{2m} &\le \tfrac{1}{w^2}\Big\|P_{\psi_*\psi}\Big(\tilde{\mathcal{E}}_\mathfrak{l}-\int d\mu\ \tilde{\mathcal{D}}_2(\psi_*,\psi,z^*,z)\Big)\Big\|_w\\ &\le \big(120\Lambda_2'\big)^2\mathfrak{v}_0\big(|\mu_n|+\mathfrak{v}_0^{\frac{2}{3}-\epsilon}\big)\bar\kappa_\mathfrak{l}^6\end{aligned}$$

It now suffices to observe that, by (5.14),

$$\begin{aligned}&\Big\|P_{\psi_*\psi}\Big(M_n'(\psi_*,\psi)-\int d\mu\ \tilde{\mathcal{D}}_2(\psi_*,\psi,z^*,z)\Big)\Big\|_{2m}\\ &\qquad\le \tfrac{1}{w^2}\big\|P_2^\psi\big(\mathcal{M}_n(\psi_*,\psi,z_*,z)-\tilde{\mathcal{D}}_2(\psi_*,\psi,z_*,z)\big)\big\|_w\\ &\qquad\le \mathfrak{v}_0^{\frac{4}{3}-7\epsilon}\bar\kappa_\mathfrak{l}^2\end{aligned}$$

(b) This follows from the observations that, for $r_n \geq 1$,

$$\left| \frac{\int d\mu_{r_n}(z^*, z)\ z^*(\mathbb{L}^{-1}x_1) z(\mathbb{L}^{-1}x_2)}{\int d\mu_{r_n}(z^*, z)} \ - \ \delta_{x_1, x_2} \right| \leq 2r_n^2 e^{-r_n^2}$$

and, recalling that $D^{(0)*}$ is the transpose of $D^{(0)}$,

$$\begin{aligned} \int_{\mathcal{X}_0} dx'\ D^{(0)*}(x_1, x)\, D^{(0)}(x_4, x) &= C^{(0)}(x_4, x_1) \\ \int_{\mathcal{X}_0^{(n)}} dx'\ D^{(n)}(x_1, x)\, D^{(n)}(x, x_2) &= C^{(n)}(x_1, x_2) \end{aligned}$$

$\square$

6 One Block Spin Transformation — Renormalization and Conclusion of the Induction

Lemma 5.5 and Proposition 5.6 provide, for each $0 \le n \le n_p$, an integral free representation for the $e^{\mathcal{C}_n}\mathcal{F}_n = \tilde{N}_{\mathbb{T}}^{(n)} \mathcal{Z}_n \Big((\mathbb{S}\mathbb{T}_n^{(SF)}) \circ \cdots \circ (\mathbb{S}\mathbb{T}_0^{(SF)}) \Big) \big(e^{\mathcal{A}_0} \big)(\psi_*, \psi)$ of [7, Corollary 4.3]. It is

$$\begin{aligned} &\log e^{\mathcal{C}_n}\mathcal{F}_n + \log \mathcal{Z}_n' \\ &= \Big[-A_{n+1}(\psi_*, \psi, \phi_*, \phi, L^2\mu_n, \mathbb{S}\mathcal{V}_n) + (\mathbb{S}\mathcal{R}_n)(\phi_*, \phi) \Big]_{\phi_{(*)} = \phi_{(*)n+1}(\psi_*, \psi, L^2\mu_n, \mathbb{S}\mathcal{V}_n)} \\ &\qquad + \Big[\tilde{\mathcal{E}}_{n+1,1}(\tilde{\psi}_*, \tilde{\psi}) + \tilde{\mathcal{E}}_{\mathfrak{l}}(\tilde{\psi}_*, \tilde{\psi}) \Big]_{\tilde{\psi}_{(*)} = (\psi_{(*)}, \{\partial_\nu \psi_{(*)}\})} \end{aligned} \tag{6.1}$$

To convert this representation into the form specified in [7, Theorem 1.17 and Remark 1.18] (with n replaced by $n+1$) we shall

- move the scaling/weight relevant part of $\tilde{\mathcal{E}}_{\mathfrak{l}}$ into $\mathcal{R}_{n+1}$ and
- renormalize the chemical potential and the interaction.

We again fix any $0 \le n \le n_p$ and assume that, if $n \ge 1$, the conclusions of [7, Theorem 1.17 and Remark 1.18] hold. In the case of $n = 0$, we use the data of [7, §1.5]. In this section, we will show that these assumptions imply [7, Theorem 1.17 and Remark 1.18] with n replaced by $n+1$, thus concluding the induction step.

We shall construct a chemical potential μ_{n+1}, an interaction $\mathcal{V}_{n+1}$ and a polynomial $\mathcal{R}_{n+1}$ that fulfil the conclusions of [7, Theorem 1.17 and Remark 1.18], and an analytic function $\tilde{\mathcal{E}}_{n+1,2}(\tilde{\psi}_*, \tilde{\psi})$ whose power series expansion does not contain scaling/weight relevant monomials such that

$$\begin{aligned} &\Big[-A_{n+1}(\psi_*, \psi, \phi_*, \phi, L^2\mu_n, \mathbb{S}\mathcal{V}_n) + (\mathbb{S}\mathcal{R}_n)(\phi_*, \phi) \Big]_{\phi_{(*)} = \phi_{(*)n+1}(\psi_*, \psi, L^2\mu_n, \mathbb{S}\mathcal{V}_n)} \\ &\qquad + \tilde{\mathcal{E}}_{\mathfrak{l}}(\tilde{\psi}_*, \tilde{\psi}) \Big|_{\tilde{\psi}_{(*)} = (\psi_{(*)}, \{\partial_\nu \psi_{(*)}\})} \\ &= \Big[-A_{n+1}(\psi_*, \psi, \phi_*, \phi, \mu_{n+1}, \mathcal{V}_{n+1}) + \mathcal{R}_{n+1}(\phi_*, \phi) \Big]_{\phi_{(*)} = \phi_{(*)n+1}(\psi_*, \psi, \mu_{n+1}, \mathcal{V}_{n+1})} \\ &\qquad + \tilde{\mathcal{E}}_{n+1,2}(\tilde{\psi}_*, \tilde{\psi}) \Big|_{\tilde{\psi}_{(*)} = (\psi_{(*)}, \{\partial_\nu \psi_{(*)}\})} \end{aligned} \tag{6.2}$$

With $\tilde{\mathcal{E}}_{n+1} = \tilde{\mathcal{E}}_{n+1,1} + \tilde{\mathcal{E}}_{n+1,2}$ we will get the desired representation for

$$\Big((\mathbb{S}\mathbb{T}_n^{(SF)}) \circ \cdots \circ (\mathbb{S}\mathbb{T}_0^{(SF)}) \Big) \Big(e^{\mathcal{A}_0} \Big)(\psi_*, \psi)$$

The first term on the left hand side of (6.2) is written in terms of the background fields $\phi_{(*)n+1}(\psi_*, \psi, L^2\mu_n, \mathbb{S}\mathcal{V}_n)$, and the second term in terms of the fields $\psi_{(*)}$ themselves. For the proof of (6.2) we will reshuffle this arrangement — i.e. write background fields in terms of $\psi_{(*)}$ fields and conversely. To take care of the special degree properties of the relevant monomials, the chemical potential and the interaction, we have to keep track of the degrees of the monomials arising in the conversion process. The background fields are defined in terms of the $\psi_{(*)}$ fields (see [7, Proposition 1.14]), so conversion from ϕ fields to ψ is in principle "easy". The converse is taken care of in Lemma 6.3.

We set up and solve (in Lemma 6.2) the equation for $\delta\mu_n = \mu_{n+1} - L^2\mu_n$. Then, we derive and solve the equation for $\delta\mathcal{V}_n = \mathcal{V}_{n+1} - \mathbb{S}\mathcal{V}_n$. See Lemma 6.4. The polynomial $\mathcal{R}_{n+1}$ and the function $\tilde{\mathcal{E}}_{n+1,2}$ of (6.2) are constructed in Lemma 6.6.

We again use the abbreviations (5.1) and the norms $\bar{\|} \cdot \bar{\|}$ (for analytic functions) and $\bar{|||} \cdot \bar{|||}$ (for field maps) with mass $2m$ and weight factors $\bar{\kappa}$, $\bar{\kappa}'$ defined at the beginning of §5. For the output of the renormalization procedure, we use the norm $\| \cdot \|^{(n+1)}$ with mass m which associates the same weight $\bar{\kappa} = \kappa(n+1)$ to the fields ψ_*, ψ , and the same weight $\bar{\kappa}' = \kappa'(n+1)$ to the fields $\psi_{\nu*}, \psi_\nu$, $\nu = 0, \cdots, 3$. We abbreviate $\underline{\|} \cdot \underline{\|} = \| \cdot \|^{(n+1)}$ and use $\underline{|||} \cdot \underline{|||}$ to denote the corresponding norm for field maps. Recall also, from [7, Definition 1.11], that $\| \cdot \|_{2m}$ is the norm with mass $2m$ which associates the weight 1 to all fields.

To keep track of relevant monomials, we use

Definition 6.1. For a vector $\vec{p} = (p_u, p_0, p_{\rm sp})$ of nonnegative integers denote by $\mathfrak{P}_{\vec{p}}$, respectively $\mathfrak{R}_{\vec{p}}$, the space of $\mathfrak{S}$ invariant, particle number preserving, polynomials in the fields $\tilde{\psi}_*$, $\tilde{\psi}$, respectively $\tilde{\phi}_*$, $\tilde{\phi}$, of type $\vec{p}$, as in [7, Definition 1.8]. For any analytic function $\tilde{\mathcal{F}}(\tilde{\psi}_*, \tilde{\psi})$, denote by $\mathcal{M}_{\vec{p}}(\tilde{\mathcal{F}})$ the part of $\mathcal{F}$ that is in $\mathfrak{P}_{\vec{p}}$. Let

- $\mathfrak{P}_{\mathfrak{D}}$, respectively $\mathfrak{R}_{\mathfrak{D}}$, denote the space of $\mathfrak{S}$ invariant, particle number preserving, polynomials in the fields $\tilde{\psi}_*$,$\tilde{\psi}$, respectively $\tilde{\phi}_*$,$\tilde{\phi}$, that contain only monomials of type $\vec{p} \in \mathfrak{D}$ as in [7, (1.18)].
- $\mathfrak{P}_{\rm rel}$ denote the space of $\mathfrak{S}$ invariant, particle number preserving, polynomials in the fields $\tilde{\psi}_*$, $\tilde{\psi}$ that contain only monomials of type $\vec{p} \in \mathfrak{D}_{\rm rel}$ as in [7, Definition 1.16].
- $\mathfrak{P}_{\rm irr}$ denote the space of $\mathfrak{S}$ invariant, particle number preserving, analytic functions of the fields $\tilde{\psi}_*$, $\tilde{\psi}$, that contain only scaling/weight irrelevant monomials, i.e. of type $\vec{p} \notin \mathfrak{D}_{\rm rel}$.

$\mathfrak{P}_{\mathfrak{D}}$ and $\mathfrak{P}_{\rm rel}$ are direct sums of $\mathfrak{P}_{\vec{p}}$'s with $\vec{p}$ running over the vectors specified in [7, (1.18) and Definition 1.16], respectively. By construction, $\mathfrak{P}_{\mathfrak{D}}$ is a subspace of $\mathfrak{P}_{\rm rel}$.

We also use the projections $\mathcal{L}_4$, $\mathcal{L}_{\mathfrak{D}}$ and $\mathcal{I}$ and mass extraction operator ℓ of Proposition B.4 and Definition B.5.

By Corollary B.6 and Proposition 5.6,

$$\begin{aligned}\tilde{\mathcal{E}}_{\mathfrak{l}}\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\big) = \ell(\tilde{\mathcal{E}}_{\mathfrak{l}})\int dx\ \psi_*(x)\psi(x) + \mathcal{L}_4(\tilde{\mathcal{E}}_{\mathfrak{l}})(\psi_*,\psi)\\ + \mathcal{L}_{\mathfrak{D}}(\tilde{\mathcal{E}}_{\mathfrak{l}})\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\big)\\ + \mathcal{I}(\tilde{\mathcal{E}}_{\mathfrak{l}})\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\big)\\ |\ell(\tilde{\mathcal{E}}_{\mathfrak{l}})| \le \tfrac{1}{\bar\kappa^2}\mathfrak{e}_{\mathfrak{l}}(n)\\ \bar{\|}\mathcal{L}_4(\tilde{\mathcal{E}}_{\mathfrak{l}})\bar{\|}\,,\ \underline{\|}\mathcal{L}_{\mathfrak{D}}(\tilde{\mathcal{E}}_{\mathfrak{l}})\underline{\|}\,,\ \underline{\|}\mathcal{I}(\tilde{\mathcal{E}}_{\mathfrak{l}})\underline{\|}\ \le\ \Big[1+18c_{\rm loc}\tfrac{\bar\kappa'}{\bar\kappa}\Big]^2\,\mathfrak{e}_{\mathfrak{l}}(n)\end{aligned} \tag{6.3}$$

Set

$$\begin{aligned}&A^{\rm var}(\psi_*,\psi,\delta\mu,\delta\mathcal{V})\\ &\quad= A_{n+1}(\psi_*,\psi,\phi_*,\phi,L^2\mu_n+\delta\mu,\mathbb{S}\mathcal{V}_n)\Big|_{\phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n+\delta\mu,\mathbb{S}\mathcal{V}_n+\delta\mathcal{V})}\\ &\qquad\qquad - A_{n+1}(\psi_*,\psi,\phi_*,\phi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\Big|_{\phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)}\end{aligned} \tag{6.4}$$

Note that the last argument of the first A_{n+1} on the right hand side is $\mathbb{S}\mathcal{V}_n$, rather than $\mathbb{S}\mathcal{V}_n+\delta\mathcal{V}$. See the definition of $\tilde{A}^{\rm var}$ before (6.15).

Lemma 6.2 (Renormalization of the Chemical Potential)**.** *There is a unique $\delta\mu_n$ in $\big[-\rho_{\rm bg},\rho_{\rm bg}\big]$ such that for all $\delta\mathcal{V}\in\mathfrak{P}_{(4,0,0)}$*

$$\ell\big(A^{\rm var}(\,\cdot\,,\,\cdot\,,\delta\mu_n,\delta\mathcal{V})\big)+\ell(\tilde{\mathcal{E}}_{\mathfrak{l}})=0$$

Furthermore $\delta\mu_n$ has the same sign as $\ell(\tilde{\mathcal{E}}_{\mathfrak{l}})$ and

$$\tfrac14|\ell(\tilde{\mathcal{E}}_{\mathfrak{l}})|\le|\delta\mu_n|\le\tfrac94|\ell(\tilde{\mathcal{E}}_{\mathfrak{l}})|\le\tfrac{3}{\bar\kappa^2}\mathfrak{e}_{\mathfrak{l}}(n)$$

Proof. The part of $\phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n+\delta\mu,\mathbb{S}\mathcal{V}_n+\delta\mathcal{V})$ that is linear in ψ_*,ψ is $B_{(*)}\psi_{(*)}+\delta\mu\,\Delta B_{(*)}(\delta\mu)\,\psi_{(*)}$ with

$$\begin{aligned}B_{(*)}&=S_{n+1}(L^2\mu_n)^{(*)}Q_{n+1}^*\mathfrak{Q}_{n+1} &\qquad \Delta B_{(*)}(\delta\mu)&=\tilde{S}(\delta\mu)^{(*)}\,B_{(*)}\\ \tilde{S}(\delta\mu)^{(*)}&=S_{n+1}^{(*)}\big[1-(L^2\mu_n+\delta\mu)S_{n+1}^{(*)}\big]^{-1}\end{aligned} \tag{6.5}$$

See [8, Propositions 2.1 and 4.1]. In particular it is independent of $\delta\mathcal{V}$. Similarly, the part of $\phi_D = D_{n+1}\phi_{(*)n+1}(\psi_*, \psi, L^2\mu_n + \delta\mu, \mathbb{S}\mathcal{V}_n + \delta\mathcal{V})$ that is linear in ψ is $B_D\,\psi + \delta\mu\,\Delta B_D\,\psi$ where

$$B_D = B^{(-)}_{n+1,L^2\mu_n,D} \qquad \Delta B_D = B_{n+1,L^2\mu_n,D}$$

Therefore, the part of $A^{\rm var}$ that is quadratic in the fields ψ_*, ψ is

$$\begin{aligned} A_2^{\rm var}(\psi_*, \psi, \delta\mu) = \Big[&\big\langle \psi_* - Q_{n+1}\phi_*, \mathfrak{Q}_{n+1}\big(\psi - Q_{n+1}\phi\big)\big\rangle_0 \\ &+ \langle \phi_*,\, \phi_D\rangle_{n+1} - L^2\mu_n\,\langle \phi_*,\, \phi\rangle_{n+1}\Big]^{\phi_{(*)}=(B_{(*)}+\delta\mu\Delta B_{(*)})\psi_{(*)},\ \phi_D=(B_D+\delta\mu\Delta B_D)\psi}_{\phi_{(*)}=B_{(*)}\psi_{(*)},\ \phi_D=B_D\psi} \\ &- \delta\mu\,\langle \phi_*, \phi\rangle_{n+1}\Big|_{\phi_{(*)}=(B_{(*)}+\delta\mu\Delta B_{(*)})\psi_{(*)}} \end{aligned} \tag{6.6}$$

In particular

$$\big\|\!\big| A_2^{\rm var}\big\|\!\big| \le c_A\big[1 + L^2|\mu_n|\big]\,|\delta\mu|\bar\kappa^2 \tag{6.7}$$

Denote by 1 and $1_{\rm fin}$ the functions on $\mathcal{X}_0^{(n+1)}$ and $\mathcal{X}_{n+1}$, respectively, which always take the value 1. By Remark B.7,

$$\begin{aligned} B_{(*)}1 &= \tfrac{a_{n+1}}{a_{n+1}-L^2\mu_n}1_{\rm fin} & \Delta B_{(*)}1 &= \tfrac{a_{n+1}}{[a_{n+1}-L^2\mu_n-\delta\mu][a_{n+1}-L^2\mu_n]}1_{\rm fin} \\ (B_{(*)} + \delta\mu\,\Delta B_{(*)})1 &= \tfrac{a_{n+1}}{a_{n+1}-L^2\mu_n-\delta\mu}1_{\rm fin} \\ B_D 1 &= \Delta B_D 1 = 0 \end{aligned}$$

where

$$a_{n+1} = a\big(1 + \sum_{j=1}^{n} \tfrac{1}{L^{2j}}\big)^{-1} \tag{6.8}$$

Therefore, by Corollary B.6.b.

$$\begin{aligned} \ell\big(A^{\rm var}(\,\cdot\,,\,\cdot\,,\delta\mu,\delta\mathcal{V})\big) &= \ell\big(A_2^{\rm var}(\,\cdot\,,\,\cdot\,,\delta\mu)\big) = \tfrac{1}{\langle 1,1\rangle_0}A_2^{\rm var}(1\,,\,1\,,\delta\mu) \\ &= a_{n+1}\Big[\big(\tfrac{L^2\mu_n+\delta\mu}{a_{n+1}-L^2\mu_n-\delta\mu}\big)^2 - \big(\tfrac{L^2\mu_n}{a_{n+1}-L^2\mu_n}\big)^2\Big] - L^2\mu_n\Big[\big(\tfrac{a_{n+1}}{a_{n+1}-L^2\mu_n-\delta\mu}\big)^2 - \big(\tfrac{a_{n+1}}{a_{n+1}-L^2\mu_n}\big)^2\Big] \\ &\qquad - \delta\mu\Big[\tfrac{a_{n+1}}{a_{n+1}-L^2\mu_n-\delta\mu}\Big]^2 \\ &= -\tfrac{a_{n+1}(L^2\mu_n+\delta\mu)}{a_{n+1}-L^2\mu_n-\delta\mu} + \tfrac{a_{n+1}\,L^2\mu_n}{a_{n+1}-L^2\mu_n} \\ &= -\tfrac{a_{n+1}^2}{a_{n+1}-L^2\mu_n-\delta\mu} + \tfrac{a_{n+1}^2}{a_{n+1}-L^2\mu_n} \end{aligned} \tag{6.9}$$

This function vanishes when $\delta\mu = 0$ and has first derivative, with respect to $\delta\mu$, given by

$$\tfrac{\partial}{\partial\delta\mu}\ell\big(A_2^{\rm var}(\,\cdot\,,\,\cdot\,,\delta\mu)\big) = -\frac{a_{n+1}^2}{[a_{n+1}-(L^2\mu_n+\delta\mu)]^2} = -\frac{1}{\big[1-\frac{L^2\mu_n+\delta\mu}{a_{n+1}}\big]^2}$$

For $|\delta\mu| \le \frac{1}{8}$, this derivative is between $-\frac{4}{9}$ and -4. So, as $\delta\mu$ runs from $-\frac{1}{8}$ to $+\frac{1}{8}$, $\ell\big(A^{\rm var}(\,\cdot\,,\,\cdot\,,\delta\mu,\delta\mathcal{V})\big)$ decreases monotonically over an interval that contains $[-\frac{1}{18},\frac{1}{18}]$. As $\ell(\tilde{\mathcal{E}}_{\mathfrak{l}})$ is a constant, independent of $\delta\mu$, the claims follow by (6.3). $\square$

Set $\mu_{n+1} = L^2\mu_n + \delta\mu_n$.

Lemma 6.3 (ψ to ϕ conversion)**.** *There exists a constant c_Ω, depending only on $\Gamma_{\rm op}$ and $K_{\rm bg}$, and there are maps*

$$\Omega : \mathfrak{P}_{\mathfrak{D}} \to \mathfrak{R}_{\mathfrak{D}} \qquad \Omega_4 : \mathfrak{P}_{(4,0,0)} \to \mathfrak{R}_{(4,0,0)} \qquad \Omega_6 : \mathfrak{P}_{(4,0,0)} \times \mathfrak{R}_{(4,0,0)} \to \mathfrak{R}_{(6,0,0)}$$
$$\Omega_{\rm irr} : (\mathfrak{P}_{\mathfrak{D}} \oplus \mathfrak{P}_{(4,0,0)}) \times \mathfrak{R}_{(4,0,0)} \to \mathfrak{P}_{\rm irr}$$

with Ω, Ω_4 and Ω_6 being linear and with $\Omega_{\rm irr}$ being linear in the first variable, such that the following holds for all $\delta\mathcal{V} \in \mathfrak{R}_{(4,0,0)}$.

- *For all $\mathcal{P} \in \mathfrak{P}_{\mathfrak{D}}$,*

$$\mathcal{P}\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\big) = \Omega(\mathcal{P})(\tilde\phi_*,\tilde\phi)\Big|_{\substack{\phi_{(*)}=\phi_{(*),n+1}(\psi_*,\psi,\mu_{n+1},\mathbb{S}\mathcal{V}_n+\delta\mathcal{V})\\ \phi_{(*),\nu}=\partial_\nu\phi_{(*),n+1}(\psi_*,\psi,\mu_{n+1},\mathbb{S}\mathcal{V}_n+\delta\mathcal{V})}} + \Omega_{\rm irr}(\mathcal{P},\delta\mathcal{V})\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\big)$$

  and for all $\mathcal{P} \in \mathfrak{P}_{(4,0,0)}$,

$$\mathcal{P}(\psi_*,\psi) = \Big[\Omega_4(\mathcal{P})(\phi_*,\phi) + \Omega_6(\mathcal{P},\delta\mathcal{V})(\phi_*,\phi)\Big]_{\phi_{(*)}=\phi_{(*),n+1}(\psi_*,\psi,\mu_{n+1},\mathbb{S}\mathcal{V}_n+\delta\mathcal{V})} + \Omega_{\rm irr}(\mathcal{P},\delta\mathcal{V})\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\big)$$

- *If $\mathcal{P} \in \mathfrak{P}_{\vec p}$, for some $\vec p \in \mathfrak{D}$ then $\Omega(\mathcal{P}) \in \mathfrak{R}_{\vec p}$ and*

$$\|\Omega(\mathcal{P})\|_m \le \tfrac{c_\Omega}{\bar\kappa^{\vec p}}\underline{\|\mathcal{P}\|} \qquad \|\Omega(\mathcal{P})\|_{2m} \le \tfrac{c_\Omega}{\bar\kappa^{\vec p}}\overline{\|\mathcal{P}\|}$$

- *If $\mathcal{P} \in \mathfrak{P}_{(4,0,0)}$, then*

$$\|\Omega_4(\mathcal{P})\|_{2m} \le \tfrac{c_\Omega}{\bar\kappa^4}\overline{\|\mathcal{P}\|} \qquad \|\Omega_6(\mathcal{P},\mathcal{V})\|_{2m} \le \tfrac{c_\Omega}{\bar\kappa^4}\|\mathbb{S}\mathcal{V}_n+\delta\mathcal{V}\|_{2m}\,\overline{\|\mathcal{P}\|}$$

- *For all* $\mathcal{P} \in \mathfrak{P}_{\mathfrak{D}} \cup \mathfrak{P}_{(4,0,0)}$,

$$\underline{\|}\Omega_{\rm irr}(\mathcal{P},\delta\mathcal{V})\underline{\|} \le c_\Omega \|\mathbb{S}\mathcal{V}_n + \delta\mathcal{V}\|_m\,\bar\kappa^2\ \underline{\|}\mathcal{P}\underline{\|}$$
$$\bar{\|}\Omega_{\rm irr}(\mathcal{P},\delta\mathcal{V})\bar{\|} \le c_\Omega \|\mathbb{S}\mathcal{V}_n + \delta\mathcal{V}\|_{2m}\,\bar\kappa^2\ \bar{\|}\mathcal{P}\bar{\|}$$

Proof. Let $B_{(*)} = S^{(*)}_{n+1,\mu_{n+1}} Q^*_{n+1}\mathfrak{Q}_{n+1}$ be the operators of [8, Proposition 2.1.a][1] and $B^{(\pm)}_{n,\mu_{n+1},\nu}$ be the operators of [8, Proposition 2.1.b]. By [4, Lemma 5.7] the operators B^*B, $B^*_*B_*$ and $\big(B^{(\pm)}_{n+1,\mu_{n+1},\nu}\big)^* B^{(\pm)}_{n+1,\mu_{n+1},\nu}$ all have bounded inverses. Consequently the operators

$$R_{(*)} = \big[B^*_{(*)}B_{(*)}\big]^{-1}B^*_{(*)}$$
$$R^{(\pm)}_\nu = \big[\big(B^{(\pm)}_{n+1,\mu_{n+1},\nu}\big)^* B^{(\pm)}_{n+1,\mu_{n+1},\nu}\big]^{-1}\big(B^{(\pm)}_{n+1,\mu_{n+1},\nu}\big)^*$$

are left inverses of $B_{(*)}$ and $B^{(\pm)}_{n+1,\mu_{n+1},\nu}$, respectively. All have uniformly bounded $\|\ \cdot\ \|_{2m}$ norms.

For $\mathcal{P} \in \mathfrak{P}_{\vec p}$ and $\vec p \in \mathfrak{D} \cup \{(4,0,0)\}$, set

$$\Omega'(\mathcal{P})(\tilde\phi_*,\tilde\phi) = \mathcal{P}\big((R_*\phi_*,\{R^{(+)}_\nu\phi_{*\nu}\})\,,\,(R\phi,\{R^{(-)}_\nu\phi_\nu\})\big)$$

Then $\|\Omega'(\mathcal{P})\|_m \le \frac{c_\Omega}{\bar\kappa^{\vec p}}\underline{\|}\mathcal{P}\underline{\|}$ and $\|\Omega'(\mathcal{P})\|_{2m} \le \frac{c_\Omega}{\bar\kappa^{\vec p}}\bar{\|}\mathcal{P}\bar{\|}$.

- If $\vec p \in \mathfrak{D}$ and $\mathcal{P} \in \mathfrak{P}_{\vec p}$, we set $\Omega(\mathcal{P}) = \Omega'(\mathcal{P})$. In this case, by [8, Proposition 2.1.a,b] and [5, Corollary 3.3],

$$\Omega(\mathcal{P})(\tilde\phi_*,\tilde\phi)\Big|_{\substack{\phi_{(*)}=\phi_{(*),n+1}(\psi_*,\psi,\mu_{n+1},\mathbb{S}\mathcal{V}_n+\delta\mathcal{V})\\ \phi_{(*),\nu}=\partial_\nu\phi_{(*),n+1}(\psi_*,\psi,\mu_{n+1},\mathbb{S}\mathcal{V}_n+\delta\mathcal{V})}}$$
$$= \mathcal{P}\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\big) - \Omega_{\rm irr}(\mathcal{P},\delta\mathcal{V})\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\big)$$

  with an $\Omega_{\rm irr}(\mathcal{P},\delta\mathcal{V}) \in \mathfrak{P}_{\rm irr}$ satisfying $\underline{\|}\Omega_{\rm irr}(\mathcal{P})\underline{\|} \le c_\Omega\|\mathbb{S}\mathcal{V}_n+\delta\mathcal{V}\|_m\,\bar\kappa^2\ \underline{\|}\mathcal{P}\underline{\|}$ and $\bar{\|}\Omega_{\rm irr}(\mathcal{P})\bar{\|} \le c_\Omega\|\mathbb{S}\mathcal{V}_n+\delta\mathcal{V}\|_{2m}\,\bar\kappa^2\ \bar{\|}\mathcal{P}\bar{\|}$.
- If $\mathcal{P} \in \mathfrak{P}_{(4,0,0)}$, then there are $\mathcal{P}_6 \in \mathfrak{P}_{(6,0,0)}$ and $\Omega'_{\rm irr}(\mathcal{P},\delta\mathcal{V}) \in \mathfrak{P}_{\rm irr}$, fulfilling

$$\underline{\|}\Omega'_{\rm irr}(\mathcal{P},\delta\mathcal{V})\underline{\|} \le c_\Omega\|\mathbb{S}\mathcal{V}_n+\delta\mathcal{V}\|_m\,\bar\kappa^2\ \underline{\|}\mathcal{P}\underline{\|}$$
$$\bar{\|}\mathcal{P}_6\bar{\|}\,,\ \bar{\|}\Omega'_{\rm irr}(\mathcal{P},\delta\mathcal{V})\bar{\|} \le c_\Omega\|\mathbb{S}\mathcal{V}_n+\delta\mathcal{V}\|_{2m}\,\bar\kappa^2\ \bar{\|}\mathcal{P}\bar{\|}$$

[1] The hypothesis of this Proposition is fulfilled by (5.2).

such that

$$\Omega'(\mathcal{P})(\tilde{\phi}_*,\tilde{\phi})\Big|_{\phi_{(*)}=\phi_{(*),n+1}(\psi_*,\psi,\mu_{n+1},\mathbb{S}\mathcal{V}_n+\delta\mathcal{V})} = \mathcal{P}(\psi_*,\psi) - \mathcal{P}_6(\psi_*,\psi) - \Omega'_{\rm irr}(\mathcal{P},\delta\mathcal{V})(\psi_*,\psi)$$

Set

$$\Omega_4(\mathcal{P}) = \Omega'(\mathcal{P}) \quad \Omega_6(\mathcal{P},\delta\mathcal{V}) = \Omega(\mathcal{P}_6) \quad \Omega_{\rm irr}(\mathcal{P},\delta\mathcal{V}) = \Omega'_{\rm irr}(\mathcal{P},\delta\mathcal{V}) + \Omega_{\rm irr}(\mathcal{P}_6,\delta\mathcal{V})$$

□

Lemma 6.4 (Renormalization of the Interaction)**.**

(a) There exists a constant $c_{\delta\mathcal{V}}$, depending only on $\Gamma_{\rm op}$, $K_{\rm bg}$, $\rho_{\rm bg}$ and m, and a unique $\delta\mathcal{V}_n \in \mathfrak{P}_{(4,0,0)}$ such that

$$\delta\mathcal{V}_n(\phi_*,\phi) + \Omega_4\Big(\mathcal{L}_4\big(A^{\rm var}(\,\cdot\,,\,\cdot\,,\delta\mu_n,\delta\mathcal{V}_n) + \tilde{\mathcal{E}}_{\mathfrak{l}}\big)\Big) = 0$$

It fulfills the estimate $\|\delta\mathcal{V}_n\|_{2m} \le c_{\delta\mathcal{V}}\frac{\mathfrak{e}_{\mathfrak{l}}(n)}{\bar{\kappa}^4}$.

(b) Set

$$\mathcal{V}_{n+1} = \mathbb{S}\mathcal{V}_n + \delta\mathcal{V}_n \qquad \text{and} \qquad C_{\delta\mathcal{V}} = c_{\delta\mathcal{V}}$$

Then

$$\big\|\mathcal{V}_{n+1} - \mathcal{V}^{(u)}_{n+1}\big\|_{2m} \le \tfrac{C_{\delta\mathcal{V}}}{L^{n+1}} \sum_{\ell=1}^{n+1} \tfrac{L^\ell}{\kappa(\ell)^4}\mathfrak{e}_{\mathfrak{l}}(\ell-1)$$

Part (b) provides our choice for the $C_{\delta\mathcal{V}}$ of [7, Remark 1.18]. By [7, Corollary C.4],

$$\big\|\mathcal{V}_{n+1}\big\|_{2m} \le \mathfrak{v}_{n+1}$$

Proof. (a) By [8, Propositions 2.1 and 4.1 and (4.3)]

$$\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n) = \Phi^{(1)}_{(*)} + \Phi^{(3)}_{(*)} + \Phi^{(\ge 5)}_{(*)}$$

$$\phi_{(*)n+1}(\psi_*,\psi,\mu_{n+1},\mathbb{S}\mathcal{V}_n+\delta\mathcal{V}) - \phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n) = \Delta\Phi^{(1)}_{(*)} + \Delta\Phi^{(3)}_{(*)} + \Delta\Phi^{(\ge 5)}_{(*)} \tag{6.10}$$

where

$$\Phi^{(1)}_{(*)} = B_{(*)}\,\psi_* \qquad\qquad \Delta\Phi^{(1)}_{(*)} = \delta\mu_n\,\Delta B_{(*)}(\delta\mu_n)\,\psi_{(*)} = \delta\mu_n\tilde{S}(\delta\mu_n)^{(*)}\Phi^{(1)}_{(*)}$$

with

- $B_{(*)}$, $\Delta B_{(*)}(\delta\mu_n)$ and $\tilde{S}(\delta\mu_n)$ as in (6.5),
- $\Phi^{(3)}_{(*)}$ is the part of $\phi_{(*)n+1}(\psi_*, \psi, L^2\mu_n, \mathbb{S}\mathcal{V}_n)$ that is of degree exactly three in the fields ψ_*, ψ,
- $\Delta\Phi^{(3)}_{(*)} = \varphi^{(c)}_{(*)} + \varphi^{(l)}_{(*)}(\delta\mathcal{V})$ with, using the notation of[7, Definition 3.1],

$$\varphi^{(c)}_* = \delta\mu_n\, \tilde{S}(\delta\mu_n)^*\, \Phi^{(3)}_* - \tilde{S}(\delta\mu_n)^*\, (\mathbb{S}\mathcal{V}_n)'_*(\Phi_*, \Phi, \Phi_*)\Big|^{\Phi_{(*)}=[\mathbb{1}+\delta\mu_n \tilde{S}(\delta\mu_n)^{(*)}]\Phi^{(1)}_{(*)}}_{\Phi_{(*)}=\Phi^{(1)}_{(*)}}$$

$$\varphi^{(c)} = \delta\mu_n\, \tilde{S}(\delta\mu_n)\, \Phi^{(3)} - \tilde{S}(\delta\mu_n)\, (\mathbb{S}\mathcal{V}_n)'(\Phi, \Phi_*, \Phi)\Big|^{\Phi_{(*)}=[\mathbb{1}+\delta\mu_n \tilde{S}(\delta\mu_n)^{(*)}]\Phi^{(1)}_{(*)}}_{\Phi_{(*)}=\Phi^{(1)}_{(*)}}$$

$$\varphi^{(l)}_*(\delta\mathcal{V}) = -\tilde{S}(\delta\mu_n)^*\, \delta\mathcal{V}'_*\big(\Phi^{(1)}_* + \Delta\Phi^{(1)}_*, \Phi^{(1)} + \Delta\Phi^{(1)}, \Phi^{(1)}_* + \Delta\Phi^{(1)}_*\big)$$
$$\varphi^{(l)}(\delta\mathcal{V}) = -\tilde{S}(\delta\mu_n)\, \delta\mathcal{V}'\big(\Phi^{(1)} + \Delta\Phi^{(1)}, \Phi^{(1)}_* + \Delta\Phi^{(1)}_*, \Phi^{(1)} + \Delta\Phi^{(1)}\big)$$

- and $\Phi^{(\geq 5)}_{(*)}$, $\Delta\Phi^{(\geq 5)}_{(*)}$ being of degree at least five in the fields ψ_*, ψ.

Observe that $\Phi^{(1)}_{(*)}$, $\Phi^{(3)}_{(*)}$, $\Phi^{(\geq 5)}_{(*)}$, $\Delta\Phi^{(1)}_{(*)}$ and $\varphi^{(c)}_{(*)}$ are independent of $\delta\mathcal{V}$ and that $\varphi^{(l)}_{(*)}(\delta\mathcal{V})$ is linear in $\delta\mathcal{V}$. By [8, Propositions 2.1 and 4.1] and [4], there is a constant K_Φ, depending only on $\Gamma_{\rm op}$ and $K_{\rm bg}$, such that

$$\begin{aligned} \bar{|||}\Phi^{(1)}_{(*)}\bar{|||}\,,\ \bar{|||}D^{(*)}_{n+1}\Phi^{(1)}_{(*)}\bar{|||} &\leq K_\Phi\bar{\kappa} \qquad \bar{|||}\Phi^{(3)}_{(*)}\bar{|||} \leq K_\Phi \tfrac{\mathfrak{v}_n}{L}\bar{\kappa}^3 \\ \bar{|||}\Delta\Phi^{(1)}_{(*)}\bar{|||}\,,\ \bar{|||}D^{(*)}_{n+1}\Delta\Phi^{(1)}_{(*)}\bar{|||} &\leq K_\Phi|\delta\mu_n|\bar{\kappa} \end{aligned} \tag{6.11}$$

and

$$\bar{|||}\varphi^{(c)}_{(*)}\bar{|||} \leq K_\Phi\, |\delta\mu_n| \tfrac{\mathfrak{v}_n}{L}\, \bar{\kappa}^3 \qquad\qquad \bar{|||}\varphi^{(l)}_{(*)}(\delta\mathcal{V})\bar{|||} \leq K_\Phi\, \|\delta V\|_{2m}\ \bar{\kappa}^3 \tag{6.12}$$

By inspection

$$
\begin{aligned}
&\mathcal{L}_4\big(A^{\rm var}(\psi_*,\psi,\delta\mu_n,\delta\mathcal{V})\big)\\
&\quad = -\big\langle \big(\psi_* - Q_{n+1}\Phi_*^{(1)} - Q_{n+1}\Delta\Phi_*^{(1)}\big)\ ,\ \mathfrak{Q}_{n+1}Q_{n+1}\Delta\Phi^{(3)}\big\rangle_0\\
&\qquad\qquad - \big\langle \Delta\Phi_*^{(3)}\ ,\ Q_{n+1}^*\mathfrak{Q}_{n+1}\big(\psi - Q_{n+1}\Phi^{(1)} - Q_{n+1}\Delta\Phi^{(1)}\big)\big\rangle_{n+1}\\
&\qquad\qquad + \big\langle Q_{n+1}\Delta\Phi_*^{(1)}\ ,\ \mathfrak{Q}_{n+1}Q_{n+1}\Phi^{(3)}\big\rangle_0 + \big\langle \Phi_*^{(3)}\ ,\ Q_{n+1}^*\mathfrak{Q}_{n+1}Q_{n+1}\Delta\Phi^{(1)}\big\rangle_{n+1}\\
&\qquad + \mathbb{S}\mathcal{V}_n(\phi_*,\phi)\Big|_{\phi_{(*)}=\Phi_{(*)}^{(1)}}^{\phi_{(*)}=\Phi_{(*)}^{(1)}+\Delta\Phi_{(*)}^{(1)}}\\
&\qquad - L^2\mu_n\Big(\big\langle \Phi_*^{(3)}+\Delta\Phi_*^{(3)}\ ,\ \Phi^{(1)}+\Delta\Phi^{(1)}\big\rangle_{n+1} - \big\langle \Phi_*^{(3)}\ ,\ \Phi^{(1)}\big\rangle_{n+1}\\
&\qquad\qquad + \big\langle \Phi_*^{(1)}+\Delta\Phi_*^{(1)}\ ,\ \Phi^{(3)}+\Delta\Phi^{(3)}\big\rangle_{n+1} - \big\langle \Phi_*^{(1)}\ ,\ \Phi^{(3)}\big\rangle_{n+1}\Big)\\
&\qquad - \delta\mu_n\Big(\big\langle \Phi_*^{(3)}+\Delta\Phi_*^{(3)}\ ,\ \Phi^{(1)}+\Delta\Phi^{(1)}\big\rangle_{n+1} + \big\langle \Phi_*^{(1)}+\Delta\Phi_*^{(1)}\ ,\ \Phi^{(3)}+\Delta\Phi^{(3)}\big\rangle_{n+1}\Big)\\
&\qquad + \big\langle \Delta\Phi_*^{(3)}\ ,\ D_{n+1}\Phi^{(1)}\big\rangle_{n+1} + \big\langle \Phi_*^{(3)}+\Delta\Phi_*^{(3)}\ ,\ D_{n+1}\Delta\Phi^{(1)}\big\rangle_{n+1}\\
&\qquad\qquad + \big\langle D_{n+1}^*\Phi_*^{(1)}\ ,\ \Delta\Phi^{(3)}\big\rangle_{n+1} + \big\langle \Delta\Phi_*^{(1)}\ ,\ D_{n+1}(\Phi^{(3)}+\Delta\Phi^{(3)})\big\rangle_{n+1}
\end{aligned}
$$

By [8, Proposition 2.1.c],

$$
\begin{aligned}
D_{n+1}^{(*)}\Phi_{(*)}^{(1)} &= Q_{n+1}^*\mathfrak{Q}_{n+1}\psi_{(*)} - (Q_{n+1}^*\mathfrak{Q}_{n+1}Q_{n+1} - L^2\mu_n)B_{(*)}\psi_*\\
&= Q_{n+1}^*\mathfrak{Q}_{n+1}\big(\psi_{(*)} - Q_{n+1}\Phi_{(*)}^{(1)}\big) + L^2\mu_n\Phi_{(*)}^{(1)}
\end{aligned}
$$

This leads to a cancellation between lines 1,2,5,6 and the last two lines in the formula for $\mathcal{L}_4\big(A^{\rm var}\big)$. Inserting the decomposition $\Delta\Phi_{(*)}^{(3)} = \varphi_{(*)}^{(c)} + \varphi_{(*)}^{(l)}(\delta\mathcal{V})$ we get

$$
\mathcal{L}_4\big(A^{\rm var}(\psi_*,\psi,\delta\mu_n,\delta\mathcal{V})\big) = A_{\delta\mathcal{V}}(\psi_*,\psi,\delta\mathcal{V}) - B(\psi_*,\psi)
$$

with

$$
\begin{aligned}
A_{\delta\mathcal{V}}(\psi_*,\psi,\delta\mathcal{V}) &= \big\langle \Delta\Phi_*^{(1)}, Q_{n+1}^*\mathfrak{Q}_{n+1}Q_{n+1}\varphi^{(l)}(\delta\mathcal{V})\big\rangle_0\\
&\quad + \big\langle \varphi_*^{(l)}(\delta\mathcal{V}), Q_{n+1}^*\mathfrak{Q}_{n+1}Q_{n+1}\Delta\Phi^{(1)}\big\rangle_0\\
&\quad - L^2\mu_n\Big(\big\langle \varphi_*^{(l)}(\delta\mathcal{V}),\ \Delta\Phi^{(1)}\big\rangle_{n+1} + \big\langle \Delta\Phi_*^{(1)},\ \varphi^{(l)}(\delta\mathcal{V})\big\rangle_{n+1}\Big)\\
&\quad - \delta\mu_n\Big(\big\langle \varphi_*^{(l)}(\delta\mathcal{V}),\ \Phi^{(1)}+\Delta\Phi^{(1)}\big\rangle_{n+1} + \big\langle \Phi_*^{(1)}+\Delta\Phi_*^{(1)},\ \varphi^{(l)}(\delta\mathcal{V})\big\rangle_{n+1}\Big)\\
&\quad + \big\langle \varphi_*^{(l)}(\delta\mathcal{V}), D_{n+1}\Delta\Phi^{(1)}\big\rangle_{n+1} + \big\langle D_{n+1}^*\Delta\Phi_*^{(1)}, \varphi^{(l)}(\delta\mathcal{V})\big\rangle_{n+1}
\end{aligned}
$$

linear in $\delta\mathcal{V}$ and

$$\begin{aligned}B(\psi_*,\psi) = &-\big\langle \Delta\Phi_*^{(1)}\ ,\ Q_{n+1}^*\mathfrak{Q}_{n+1}Q_{n+1}\varphi^{(c)}\big\rangle_0 - \big\langle \varphi_*^{(c)}\ ,\ Q_{n+1}^*\mathfrak{Q}_{n+1}Q_{n+1}\Delta\Phi^{(1)}\big\rangle_0\\ &- \big\langle Q_{n+1}\Delta\Phi_*^{(1)}\ ,\ \mathfrak{Q}_{n+1}Q_{n+1}\Phi^{(3)}\big\rangle_0 - \big\langle \Phi_*^{(3)}\ ,\ Q_{n+1}^*\mathfrak{Q}_{n+1}Q_{n+1}\Delta\Phi^{(1)}\big\rangle_{n+1}\\ &- \mathbb{S}\mathcal{V}_n(\phi_*,\phi)\Big|_{\phi_{(*)}=\Phi_{(*)}^{(1)}}^{\phi_{(*)}=\Phi_{(*)}^{(1)}+\Delta\Phi_{(*)}^{(1)}}\\ &+ L^2\mu_n\Big(\big\langle \Phi_*^{(3)}+\varphi_*^{(c)},\ \Delta\Phi^{(1)}\big\rangle_{n+1} + \big\langle \Delta\Phi_*^{(1)},\ \Phi^{(3)}+\varphi^{(c)}\big\rangle_{n+1}\Big)\\ &+ \delta\mu_n\Big(\big\langle \Phi_*^{(3)}+\varphi_*^{(c)},\ \Phi^{(1)}+\Delta\Phi^{(1)}\big\rangle_{n+1} + \big\langle \Phi_*^{(1)}+\Delta\Phi_*^{(1)},\ \Phi^{(3)}+\varphi^{(c)}\big\rangle_{n+1}\Big)\\ &- \big\langle \Phi_*^{(3)}+\varphi_*^{(c)}, D_{n+1}\Delta\Phi^{(1)}\big\rangle_{n+1} - \big\langle D_{n+1}^*\Delta\Phi_*^{(1)}, (\Phi^{(3)}+\varphi^{(c)})\big\rangle_{n+1}\end{aligned}$$

independent of $\delta\mathcal{V}$. By (6.11) and (6.12), there is a constant c_1 such that

$$\bar{\|}A_{\delta\mathcal{V}}\bar{\|} \le c_1|\delta\mu_n|\,\|\delta V\|_{2m}\,\bar\kappa^4 \qquad\qquad \bar{\|}B\bar{\|} \le c_1|\delta\mu_n|\tfrac{\mathfrak{v}_n}{L}\,\bar\kappa^4$$

Therefore, by Lemma 6.3, (6.3) and the estimate on $\delta\mu_n$ in Lemma 6.2

$$\|\Omega_4(A_{\delta\mathcal{V}})\|_{2m} \le c_2\|\delta V\|_{2m}\,\tfrac{\mathfrak{e}_{\mathfrak{l}}(n)}{\bar\kappa^2} \qquad\qquad \|\Omega_4(B)\|_{2m} \le c_2\tfrac{\mathfrak{v}_n}{L}\tfrac{\mathfrak{e}_{\mathfrak{l}}(n)}{\bar\kappa^2}$$
$$\|\Omega_4(\mathcal{L}_4(\tilde{\mathcal{E}}_{\mathfrak{l}}))\|_{2m} \le c_2\tfrac{\mathfrak{e}_{\mathfrak{l}}(n)}{\bar\kappa^4}$$

Assuming that $\mathfrak{v}_0$ is small enough, the linear operator $\delta\mathcal{V}\mapsto\Omega_4(A_{\delta\mathcal{V}})$ has operator norm at most $\frac{1}{2}$ with respect to the norm $\|\cdot\|_{2m}$. Therefore the operator

$$\delta\mathcal{V}\mapsto\delta\mathcal{V}+\Omega_4(A_{\delta\mathcal{V}})$$

has an inverse $I_{\delta V}$ whose operator norm is bounded by 2. Set

$$\delta\mathcal{V}_n = I_{\delta V}\Big(\Omega_4\big(B-\mathcal{L}_4(\tilde{\mathcal{E}}_{\mathfrak{l}})\big)\Big)$$

By [7, (C.1.b)],

$$\|\delta\mathcal{V}_n\|_{2m} \le 2\,c_2\tfrac{\mathfrak{e}_{\mathfrak{l}}(n)}{\bar\kappa^2}\big(\tfrac{\mathfrak{v}_n}{L}+\tfrac{1}{\bar\kappa^2}\big) \le 2(1+2\rho_{\rm bg})\,c_2\tfrac{\mathfrak{e}_{\mathfrak{l}}(n)}{\bar\kappa^4}$$

(b) By [7, (C.3), Remark 1.18] and part (a),

$$\begin{aligned}\|\mathcal{V}_{n+1}-\mathcal{V}_{n+1}^{(u)}\|_{2m} &\le \big\|\mathbb{S}\big(\mathcal{V}_n-\mathcal{V}_n^{(u)}\big)\big\|_{2m} + \|\delta\mathcal{V}_n\|_{2m}\\ &\le \tfrac{1}{L}\big\|\mathcal{V}_n-\mathcal{V}_n^{(u)}\big\|_{2m} + \|\delta\mathcal{V}_n\|_{2m}\\ &\le \tfrac{C_{\delta\mathcal{V}}}{L^{n+1}}\sum_{\ell=1}^{n}\tfrac{L^\ell}{\kappa(\ell)^4}\mathfrak{e}_{\mathfrak{l}}(\ell-1) + c_{\delta\mathcal{V}}\tfrac{\mathfrak{e}_{\mathfrak{l}}(n)}{\kappa(n+1)^4}\end{aligned}$$

□

Lemma 6.5 (Garbage Collection from $\mathcal{R}$)**.** *There is a constant $c_{\rm gar}$, depending only on $\Gamma_{\rm op}$, $K_{\rm bg}$ and $\rho_{\rm bg}$, such that the following holds. There are*

$$\begin{aligned}&\mathcal{P}_{\mathcal{R}} = \sum_{\vec p\in\mathfrak{D}} \mathcal{P}_{\mathcal{R}}^{\vec p} \qquad \textit{with}\quad \mathcal{P}_{\mathcal{R}}^{\vec p}\in\mathfrak{P}_{\vec p} \textit{ for each } \vec p\in\mathfrak{D}\\ &\mathcal{I}_{\mathcal{R}}\in\mathfrak{P}_{\rm irr}\end{aligned}$$

such that

$$(\mathbb{S}\mathcal{R}_n)(\phi_*,\phi)\Big|_{\phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)}^{\phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,\mu_{n+1},\mathcal{V}_{n+1})} = \Big[\mathcal{P}_R\big(\tilde\psi_*,\tilde\psi\big)+\mathcal{I}_R\big(\tilde\psi_*,\tilde\psi\big)\Big]_{\tilde\psi_{(*)}=(\psi_{(*)},\{\partial_\nu\psi_{(*)}\})}$$

Furthermore

$$\begin{aligned}\underline{\|}\mathcal{P}_{\mathcal{R}}^{\vec p}\underline{\|} &\le c_{\rm gar}|\delta\mu_n|\big\|\mathbb{S}\tilde{\mathcal{R}}_n^{\vec p}\big\|_m\bar\kappa^{\vec p}\\ \underline{\|}\mathcal{I}_{\mathcal{R}}\underline{\|} &\le\ c_{\rm gar}\left(|\delta\mu_n|\mathfrak{v}_{n+1}+\|\delta\mathcal{V}_n\|_{2m}\right)\bar\kappa^2\,\mathfrak{r}(n,C_{\mathcal{R}})\end{aligned}$$

where

$$\mathfrak{r}(n,C) = \sum_{\vec p\in\mathfrak{D}} L^{5-\Delta(\vec p)}\,\bar\kappa^{\vec p}\,\mathfrak{r}_{\vec p}(n,C)$$

Proof. Similar to (6.10) we write

$$\begin{aligned}\Phi_{(*)} &= \phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\\ \Delta\Phi_{(*)} &= \phi_{(*)n+1}(\psi_*,\psi,\mu_{n+1},\mathcal{V}_{n+1}) - \phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\end{aligned}$$

and, for each $\vec p\in\mathfrak{D}$,

$$\mathcal{R}_{\rm var}^{\vec p}(\tilde\psi_*,\tilde\psi) = \big(\mathbb{S}\tilde{\mathcal{R}}_n^{\vec p}\big)(\tilde\phi_*,\tilde\phi)\Big|_{\phi_{(*)}=\Phi_{(*)},\quad \phi_{(*)\nu}=\partial_\nu\Phi_{(*)}}^{\phi_{(*)}=\Phi_{(*)}+\Delta\Phi_{(*)},\quad \phi_{(*)\nu}=\partial_\nu\Phi_{(*)}+\partial_\nu\Delta\Phi_{(*)}}$$

As in the proof of Lemma 6.4 we decompose

$$\begin{aligned}\Phi_{(*)}(\psi_*,\psi) &= \Phi_{(*)}^{(1)}(\psi_*,\psi)+\Phi_{(*)}^{(\ge3)}(\psi_*,\psi)\\ \Delta\Phi_{(*)}(\psi_*,\psi) &= \Delta\Phi_{(*)}^{(1)}(\psi_*,\psi)+\Delta\Phi_{(*)}^{(\ge3)}(\psi_*,\psi)\\ \partial_\nu\Phi_{(*)}(\psi_*,\psi) &= \partial_\nu\Phi_{(*)}^{(1)}(\psi_*,\psi)+\partial_\nu\Phi_{(*)}^{(\ge3)}(\psi_*,\psi)\\ \partial_\nu\Delta\Phi_{(*)}(\psi_*,\psi) &= \partial_\nu\Delta\Phi_{(*)}^{(1)}(\psi_*,\psi)+\partial_\nu\Delta\Phi_{(*)}^{(\ge3)}(\psi_*,\psi)\end{aligned}$$

where the superscript "(1)" signifies the part that is of degree precisely one in $\psi_{(*)(\nu)}$ and the superscript "(≥ 3)" signifies the part that is of degree at least three in $\psi_{(*)(\nu)}$. By [8, Propositions 2.1 and 4.1],

$$\begin{aligned}
&\left|\!\left|\!\left| \Phi^{(1)}_{(*)} \right|\!\right|\!\right| \le K_\Phi \bar\kappa && \left|\!\left|\!\left| \Phi^{(\ge 3)}_{(*)} \right|\!\right|\!\right| \le K_\Phi \mathfrak{v}_{n+1} \bar\kappa^3 \\
&\left|\!\left|\!\left| \Delta\Phi^{(1)}_{(*)} \right|\!\right|\!\right| \le K_\Phi |\delta\mu_n| \bar\kappa && \left|\!\left|\!\left| \Delta\Phi^{(\ge 3)}_{(*)} \right|\!\right|\!\right| \le K_\Phi \big(|\delta\mu_n|\mathfrak{v}_{n+1} + \|\delta\mathcal{V}_n\|_{2m}\big)\bar\kappa^3 \\
&\left|\!\left|\!\left| \partial_\nu\Phi^{(1)}_{(*)} \right|\!\right|\!\right| \le K_\Phi \bar\kappa' && \left|\!\left|\!\left| \partial_\nu\Phi^{(\ge 3)}_{(*)} \right|\!\right|\!\right| \le K_\Phi \mathfrak{v}_{n+1} \bar\kappa^2\bar\kappa' \\
&\left|\!\left|\!\left| \partial_\nu\Delta\Phi^{(1)}_{(*)} \right|\!\right|\!\right| \le K_\Phi |\delta\mu_n| \bar\kappa' && \left|\!\left|\!\left| \partial_\nu\Delta\Phi^{(\ge 3)}_{(*)} \right|\!\right|\!\right| \le K_\Phi \big(|\delta\mu_n|\mathfrak{v}_{n+1} + \|\delta\mathcal{V}_n\|_{2m}\big)\bar\kappa^2\bar\kappa'
\end{aligned} \tag{6.13}$$

We correspondingly decompose

$$\mathcal{R}^{\vec p}_{\rm var}(\tilde\psi_*, \tilde\psi) = \mathcal{R}^{\vec p}_{\rm l.o.}(\tilde\psi_*, \tilde\psi) + \mathcal{R}^{\vec p}_{\rm h.o.}(\tilde\psi_*, \tilde\psi)$$

where

$$\mathcal{R}^{\vec p}_{\rm l.o.}(\tilde\psi_*, \tilde\psi) = \big(\mathbb{S}\tilde{\mathcal{R}}^{\vec p}_n\big)(\tilde\phi_*, \tilde\phi)\Big|^{\phi_{(*)}=\Phi^{(1)}_{(*)}+\Delta\Phi^{(1)}_{(*)},\ \ \phi_{(*)\nu}=\partial_\nu\Phi^{(1)}_{(*)}+\partial_\nu\Delta\Phi^{(1)}_{(*)}}_{\phi_{(*)}=\Phi^{(1)}_{(*)},\ \ \phi_{(*)\nu}=\partial_\nu\Phi^{(1)}_{(*)}}$$

Clearly $\mathcal{R}^{\vec p}_{\rm l.o.} \in \mathfrak{P}_{\vec p}$ and

$$\begin{aligned}
\underline{\|}\mathcal{R}^{\vec p}_{\rm l.o.}\underline{\|} &\le c_3\, \|\mathbb{S}\tilde{\mathcal{R}}^{\vec p}_n\|_m\, |\delta\mu_n|\, \bar\kappa^{\vec p} \\
\underline{\|}\mathcal{R}^{\vec p}_{\rm h.o.}\underline{\|} &\le c_3\, \|\mathbb{S}\tilde{\mathcal{R}}^{\vec p}_n\|_m\, \big(|\delta\mu_n|\mathfrak{v}_{n+1} + \|\delta\mathcal{V}_n\|_{2m}\big)\, \bar\kappa^2\ \bar\kappa^{\vec p}
\end{aligned}$$

Set

$$\mathcal{P}^{\vec p}_{\mathcal{R}} = \mathcal{R}^{\vec p}_{\rm l.o.} \qquad\qquad \mathcal{I}_{\mathcal{R}} = \sum_{\vec p\in\mathfrak{D}} \mathcal{R}^{\vec p}_{\rm h.o.}$$

The estimates follow by Lemma 5.5.c and the bound $\big\|\tilde{\mathcal{R}}^{(\vec p)}_n\big\|_m \le \mathfrak{r}_{\vec p}(n, C_{\mathcal{R}})$ of [7, Remark 1.18]. ☐

Lemma 6.6. *There exist*

- *a polynomial* $\tilde{\mathcal{R}}_{n+1}(\tilde\phi_*, \tilde\phi) = \sum_{\vec p\in\mathfrak{D}} \tilde{\mathcal{R}}^{(\vec p)}_{n+1}(\tilde\phi_*, \tilde\phi)$ *on* $\tilde{\mathcal{H}}^{(0)}_{n+1} \times \tilde{\mathcal{H}}^{(0)}_{n+1}$*, with each* $\tilde{\mathcal{R}}^{(\vec p)}_{n+1}$ *being an* $\mathfrak{S}$ *invariant polynomial of type* $\vec p$*, and*
- *an* $\mathfrak{S}$ *invariant analytic function* $\tilde{\mathcal{E}}_{n+1,2}(\tilde\psi_*, \tilde\psi)$ *on a neighbourhood of the origin in* $\tilde{\mathcal{H}}^{(n+1)}_0 \times \tilde{\mathcal{H}}^{(n+1)}_0$ *with* $\tilde{\mathcal{E}}_{n+1,2}(0,0) = 0$*, whose power series expansion does not contain scaling/weight relevant monomials*

such that

$$\begin{aligned}
\Big[-A_{n+1}(\psi_*,\psi,\phi_*,\phi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)+(\mathbb{S}\mathcal{R}_n)(\phi_*,\phi)\Big]_{\phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)}&\\
+\tilde{\mathcal{E}}_{\mathfrak{l}}(\tilde\psi_*,\tilde\psi)\Big|_{\tilde\psi_{(*)}=(\psi_{(*)},\{\partial_\nu\psi_{(*)}\})}&\\
=\Big[-A_{n+1}(\psi_*,\psi,\phi_*,\phi,\mu_{n+1},\mathcal{V}_{n+1})+\mathcal{R}_{n+1}(\phi_*,\phi)\Big]_{\phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,\mu_{n+1},\mathcal{V}_{n+1})}&\\
+\tilde{\mathcal{E}}_{n+1,2}(\tilde\psi_*,\tilde\psi)\Big|_{\tilde\psi_{(*)}=(\psi_{(*)},\{\partial_\nu\psi_{(*)}\})}&
\end{aligned}\tag{6.14}$$

where

$$\mathcal{R}_{n+1}(\phi_*,\phi)=\tilde{\mathcal{R}}_{n+1}\big((\phi_*,\{\partial_\nu\phi_*\}),(\phi,\{\partial_\nu\phi\})\big)$$

Furthermore

(a) there exists a constant $C_{\mathcal{R}}$, depending only on $\Gamma_{\rm op}$, $K_{\rm bg}$, $\rho_{\rm bg}$ and m, such that if [7, (1.22)] holds for n, then

$$\big\|\tilde{\mathcal{R}}^{(\vec p)}_{n+1}\big\|_m\le\mathfrak{r}_{\vec p}(n+1,C_{\mathcal{R}})$$

(b) there exists a constant $C_{\rm ren}$, depending only on $\Gamma_{\rm op}$, $K_{\rm bg}$, $\rho_{\rm bg}$ and m, such that

$$\underline{\|}\tilde{\mathcal{E}}_{n+1,2}\underline{\|}\le C_{\rm ren}\,\mathfrak{e}_{\mathfrak{l}}(n)$$

Part (a) provides our choice for the $C_{\mathcal{R}}$ of [7, Remark 1.18].

Proof. Set

$$\begin{aligned}
\tilde A^{\rm var}(\psi_*,\psi)&=A_{n+1}(\psi_*,\psi,\phi_*,\phi,\mu_{n+1},\mathcal{V}_{n+1})\Big|_{\phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,\mu_{n+1},\mathcal{V}_{n+1})}\\
&\qquad-A_{n+1}(\psi_*,\psi,\phi_*,\phi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)\Big|_{\phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)}\\
&=A^{\rm var}(\psi_*,\psi,\delta\mu_n,\delta\mathcal{V}_n)+\delta\mathcal{V}_n(\phi_*,\phi)\Big|_{\phi_{(*)}=\phi_{(*)n+1}(\psi_*,\psi,\mu_{n+1},\mathcal{V}_{n+1})}
\end{aligned}$$

and

$$\mathcal{P}_A=\mathcal{L}_{\mathfrak{D}}(\tilde A^{\rm var})\qquad\qquad\mathcal{I}_A=\mathcal{I}(\tilde A^{\rm var})$$

By Corollary B.6.a and Lemmas 6.2, 6.3 and 6.4,

$$\begin{aligned}
\tilde A^{\rm var}(\psi_*,\psi)&=-\ell(\tilde{\mathcal{E}}_{\mathfrak{l}})\int dx\ \psi_*(x)\psi(x)-\mathcal{L}_4(\tilde{\mathcal{E}}_{\mathfrak{l}})\\
&\qquad+\Big[\mathcal{P}_A\big(\tilde\psi_*,\tilde\psi\big)+\mathcal{I}_A\big(\tilde\psi_*,\tilde\psi\big)\Big]_{\tilde\psi_{(*)}=(\psi_{(*)},\{\partial_\nu\psi_{(*)}\})}
\end{aligned}\tag{6.15}$$

By (6.7) and [8, Propositions 2.1 and 4.1]

$$\begin{aligned} \big\|A_2^{\rm var}\big\| &\le c_A\big[1+L^2|\mu_n|\big]|\delta\mu_n|\bar\kappa^2 \\ \big\|\tilde A^{\rm var}-A_2^{\rm var}-\mathcal{L}_4(\tilde A^{\rm var})\big\| &\le c_A(\mathfrak{v}_{n+1}|\delta\mu_n|+\|\delta\mathcal{V}_n\|_{2m})\bar\kappa^4 \end{aligned} \tag{6.16}$$

To prove the second bound, use (6.11), (6.13) and

$$\Big|\Big|\Big| D_{n+1}^{(*)}\Phi_{(*)}^{(\ge 3)}\Big|\Big|\Big| \le K_\Phi\mathfrak{v}_{n+1}\bar\kappa^3 \qquad \Big|\Big|\Big| D_{n+1}^{(*)}\Delta\Phi_{(*)}^{(\ge 3)}\Big|\Big|\Big| \le K_5\,\big(|\delta\mu_n|\mathfrak{v}_{n+1}+\|\delta\mathcal{V}_n\|_{2m}\big)\bar\kappa^3$$

and also the observation that if one substitutes

$$\phi_{(*)n+1}(\psi_*,\psi,\mu_{n+1},\mathcal{V}_{n+1})=\Phi_{(*)}^{(1)}+\Phi_{(*)}^{(\ge 3)}+\Delta\Phi_{(*)}^{(1)}+\Delta\Phi_{(*)}^{(\ge 3)}$$

and $\phi_{(*)n+1}(\psi_*,\psi,L^2\mu_n,\mathbb{S}\mathcal{V}_n)=\Phi_{(*)}^{(1)}+\Phi_{(*)}^{(\ge 3)}$ into $\tilde A^{\rm var}-A_2^{\rm var}-\mathcal{L}_4(\tilde A^{\rm var})$ and expands out, then

- every surviving term must contain at least one $\Phi_{(*)}^{(\ge 3)}$ or $\Delta\Phi_{(*)}^{(\ge 3)}$ and
- every surviving term, except for those coming from $\delta\mu_n\,\langle\phi_*,\phi\rangle_{n+1}$ and $\delta\mathcal{V}_n(\phi_*,\phi)$, must contain at least one $\Delta\Phi_{(*)}^{(1)}$ or $\Delta\Phi_{(*)}^{(\ge 3)}$

So, if we write $\mathcal{P}_A=\sum_{\vec p\in\mathfrak{D}}\mathcal{P}_A^{\vec p}$ with $\mathcal{P}_A^{\vec p}\in\mathfrak{P}_{\vec p}$, then, by Proposition B.4,

$$\begin{aligned} \underline{\|}\mathcal{P}_A^{\vec p}\underline{\|} &\le 18c_{\rm loc}c_A|\delta\mu_n|\bar\kappa^{\vec p} \qquad \text{if } \vec p\ne(6,0,0) \\ \underline{\|}\mathcal{P}_A^{(6,0,0)}\underline{\|},\ \underline{\|}\mathcal{I}_A\underline{\|} &\le c_A(\mathfrak{v}_{n+1}|\delta\mu_n|+\|\delta\mathcal{V}_n\|_{2m})\bar\kappa^4 \end{aligned} \tag{6.17}$$

Set

$$\begin{aligned} \tilde{\mathcal{R}}_{n+1}^{(\vec p)} &= \mathbb{S}\tilde{\mathcal{R}}_n^{(\vec p)}+\Omega_{\vec p}\big(\mathcal{L}_{\mathfrak{D}}(\tilde{\mathcal{E}}_{\mathfrak{l}})+\mathcal{P}_A-\mathcal{P}_{\mathcal{R}}\big) \qquad \text{for } \vec p\in\mathfrak{D} \\ \tilde{\mathcal{E}}_{n+1,2} &= \mathcal{I}(\tilde{\mathcal{E}}_{\mathfrak{l}})+\mathcal{I}_A-\mathcal{I}_{\mathcal{R}}+\Omega_{\rm irr}\big(\mathcal{L}_{\mathfrak{D}}(\tilde{\mathcal{E}}_{\mathfrak{l}})+\mathcal{P}_A-\mathcal{P}_{\mathcal{R}},\delta\mathcal{V}_n\big) \end{aligned}$$

Here $\Omega_{\vec p}(\mathcal{P})$ is the part of $\Omega(\mathcal{P})$ in $\mathfrak{R}_{\vec p}$. The identity (6.14) now follows from (6.3), (6.4), (6.15) and Lemmas 6.3 and 6.5.

Write $\mathcal{L}_{\mathfrak{D}}(\tilde{\mathcal{E}}_{\mathfrak{l}})=\sum_{\vec p\in\mathfrak{D}}\mathcal{P}_{\mathcal{E}}^{\vec p}$ with each $\mathcal{P}_{\mathcal{E}}^{\vec p}\in\mathfrak{P}^{\vec p}$. By (6.3), (6.17), Lemma 6.5, and the estimate $|\delta\mu_n|\le\frac{3}{\bar\kappa^2}\mathfrak{e}_{\mathfrak{l}}(n)$ of Lemma 6.2,

$$\begin{aligned} \underline{\|}\mathcal{P}_{\mathcal{E}}^{\vec p}+\mathcal{P}_A^{\vec p}-\mathcal{P}_{\mathcal{R}}^{\vec p}\underline{\|} &\le \Big[1+c_4\tfrac{\bar\kappa'}{\bar\kappa}\Big]^2\mathfrak{e}_{\mathfrak{l}}(n)+3\,c_{\rm gar}\tfrac{\bar\kappa^{\vec p}}{\bar\kappa^2}\|\mathbb{S}\tilde{\mathcal{R}}_n^{\vec p}\|_m\mathfrak{e}_{\mathfrak{l}}(n) \\ &\qquad + c_A\delta_{\vec p,(6,0,0)}(3\mathfrak{v}_{n+1}\bar\kappa^2\mathfrak{e}_{\mathfrak{l}}(n)+\|\delta\mathcal{V}_n\|_{2m}\bar\kappa^4) \\ &\le \Big\{\Big[1+c_5\big(\tfrac{\bar\kappa'}{\bar\kappa}+\mathfrak{v}_{n+1}\bar\kappa^2\big)\Big]^2+c_5\tfrac{\bar\kappa^{\vec p}}{\bar\kappa^2}\|\mathbb{S}\tilde{\mathcal{R}}_n^{\vec p}\|_m\Big\}\mathfrak{e}_{\mathfrak{l}}(n) \end{aligned}$$

with a constant c_5 depends only on c_4, $c_{\rm gar}$, c_A and $c_{\delta\mathcal{V}}$. For the second inequality we used Lemma 6.4. Using Lemma 5.5.c and the bound $\big\|\tilde{\mathcal{R}}_n^{(\vec{p})}\big\|_m \le \mathfrak{r}_{\vec{p}}(n, C_{\mathcal{R}})$ of [7, (1.22)],

$$\underline{\big\|}\mathcal{L}_{\mathfrak{D}}(\tilde{\mathcal{E}}_{\mathfrak{l}}) + \mathcal{P}_A - \mathcal{P}_{\mathcal{R}}\underline{\big\|} \le \Big\{4\Big[1 + c_5\big(\tfrac{\bar\kappa'}{\bar\kappa} + \mathfrak{v}_{n+1}\bar\kappa^2\big)\Big]^2 + \tfrac{c_5}{\bar\kappa^2}\mathfrak{r}(n, C_{\mathcal{R}})\Big\}\mathfrak{e}_{\mathfrak{l}}(n) \tag{6.18}$$

(a) By Lemmas 6.3 and 5.5.c and [7, (1.22)],

$$\begin{aligned}
\big\|\tilde{\mathcal{R}}_{n+1}^{(\vec{p})}\big\|_m &\le \big\|\mathbb{S}\tilde{\mathcal{R}}_n^{(\vec{p})}\big\|_m + \tfrac{c_\Omega}{\bar\kappa^{\vec{p}}}\underline{\big\|}\mathcal{P}_{\mathcal{E}}^{\vec{p}} + \mathcal{P}_A^{\vec{p}} - \mathcal{P}_{\mathcal{R}}^{\vec{p}}\underline{\big\|} \\
&\le \big(1 + \tfrac{c_\Omega c_5}{\bar\kappa^2}\mathfrak{e}_{\mathfrak{l}}(n)\big)\big\|\mathbb{S}\tilde{\mathcal{R}}_n^{(\vec{p})}\big\|_m + \tfrac{c_\Omega}{\bar\kappa^{\vec{p}}}\Big[1 + c_5\big(\tfrac{\bar\kappa'}{\bar\kappa} + \mathfrak{v}_{n+1}\bar\kappa^2\big)\Big]^2 \mathfrak{e}_{\mathfrak{l}}(n) \\
&\le \big(1 + \tfrac{c_\Omega c_5}{\bar\kappa^2}\mathfrak{e}_{\mathfrak{l}}(n)\big)L^{5-\Delta(\vec{p})}\mathfrak{r}_{\vec{p}}(n, C_{\mathcal{R}}) + \tfrac{c_\Omega}{\bar\kappa^{\vec{p}}}\Big[1 + c_5\big(\tfrac{\bar\kappa'}{\bar\kappa} + \mathfrak{v}_{n+1}\bar\kappa^2\big)\Big]^2 \mathfrak{e}_{\mathfrak{l}}(n)
\end{aligned}$$

If $C_{\mathcal{R}}$ is large enough, depending only on c_Ω, c_5 and $\rho_{\rm bg}$, then by [7, (C.2)],

$$\begin{aligned}
\big\|\tilde{\mathcal{R}}_{n+1}^{(\vec{p})}\big\|_m &\le \big(1 + \tfrac{C_{\mathcal{R}}}{\bar\kappa^2}\mathfrak{e}_{\mathfrak{l}}(n)\big)L^{5-\Delta(\vec{p})}\mathfrak{r}_{\vec{p}}(n, C_{\mathcal{R}}) + C_{\mathcal{R}}\tfrac{\mathfrak{e}_{\mathfrak{l}}(n)}{\kappa^{\vec{p}}(n+1)} \\
&\le \mathfrak{r}_{\vec{p}}(0)L^{(5-\Delta(\vec{p}))(n+1)}\,\Pi_0^{n+1}(C_{\mathcal{R}}) + C_{\mathcal{R}}\sum_{\ell=1}^{n+1} L^{(5-\Delta(\vec{p}))(n+1-\ell)}\tfrac{\mathfrak{e}_{\mathfrak{l}}(\ell-1)}{\kappa^{\vec{p}}(\ell)}\,\Pi_\ell^{n+1}(C_{\mathcal{R}}) \\
&= \mathfrak{r}_{\vec{p}}(n+1, C_{\mathcal{R}})
\end{aligned}$$

(b) We have, by (6.3), (6.17), Lemmas 6.5 and 6.3 and (6.18),

$$\begin{aligned}
\underline{\|}\tilde{\mathcal{E}}_{n+1,2}\underline{\|} &\le \Big[1 + 18c_{\rm loc}\tfrac{\bar\kappa'}{\bar\kappa}\Big]^2 \mathfrak{e}_{\mathfrak{l}}(n) + c_A\,(\mathfrak{v}_{n+1}|\delta\mu_n| + \|\delta\mathcal{V}_n\|_{2m})\bar\kappa^4 \\
&\qquad + c_{\rm gar}(\mathfrak{v}_{n+1}|\delta\mu_n| + \|\delta\mathcal{V}_n\|_{2m})\bar\kappa^2\,\mathfrak{r}(n, C_{\mathcal{R}}) \\
&\qquad + c_\Omega\mathfrak{v}_{n+1}\bar\kappa^2\Big\{4\Big[1 + c_5\big(\tfrac{\bar\kappa'}{\bar\kappa} + \mathfrak{v}_{n+1}\bar\kappa^2\big)\Big]^2 + \tfrac{c_5}{\bar\kappa^2}\mathfrak{r}(n, C_{\mathcal{R}})\Big\}\mathfrak{e}_{\mathfrak{l}}(n) \\
&\le \tfrac{1}{2}C_{\rm ren}\Big\{1 + (\mathfrak{v}_{n+1} + \tfrac{1}{\bar\kappa^2})\mathfrak{r}(n, C_{\mathcal{R}})\Big\}\,\mathfrak{e}_{\mathfrak{l}}(n) \\
&\le C_{\rm ren}\,\mathfrak{e}_{\mathfrak{l}}(n)
\end{aligned}$$

by [7, (C.1.b) and Lemma C.2.a], provided $\mathfrak{v}_0$ is small enough that the hypothesis

$$\epsilon|\log\mathfrak{v}_0| \ge 2\log(1 + C_{\mathcal{R}})\,\Pi_0^\infty(C_{\mathcal{R}})$$

of [7, Lemma C.2] is satisfied. □

Lemma 6.7 (Properties of $\delta\mu_n$)**.** *There is a constant $\Lambda_{\delta\mu}$, depending only on L, $\Gamma_{\rm op}$, $K_{\rm bg}$, $\rho_{\rm bg}$ and m, such that the following holds. Set $\mu_0^* = \mu_0$ and, for $n \geq 0$, $\delta\mu_n^* = \mu_{n+1}^* - L^2\mu_n^*$. Then, for $n \geq 0$,*

(a) $\big|\ell\big(\tilde{\mathcal{E}}_{\mathfrak{l}}\big) - \delta\mu_n^*\big| \leq \Lambda_{\delta\mu}\ \mathfrak{v}_0\big(\mathfrak{v}_0^{\frac{1}{3}-5\epsilon} + |\mu_n| + r_n^2 e^{-r_n^2}\big)\bar{\kappa}_{\mathfrak{l}}^6$

(b) $|\delta\mu_n - \delta\mu_n^*| \leq \Lambda_{\delta\mu}\ \mathfrak{v}_0^{1-7\epsilon}\big(\mathfrak{v}_0^{\frac{1}{3}-5\epsilon} + L^{2n}(\mu_0 - \mu_*)\big)L^{3\epsilon(n+1)}$

(c) $|\mu_{n+1} - \mu_{n+1}^*| \leq L^{2(n+1)}\ \mathfrak{v}_0^{1-8\epsilon}\sum_{\ell=1}^{n+1}\frac{1}{L^{(2-3\epsilon)\ell}}\big[\mathfrak{v}_0^{\frac{1}{3}-5\epsilon} + L^{2\ell}(\mu_0 - \mu_*)\big]$

The bound on $|\mu_{n+1} - \mu_{n+1}^*|$ in part (c) is exactly the bound of [7, Remark 1.18] with n replaced by $n+1$.

Proof. (a) The monomials $M_n(\psi_*, \psi)$ of Proposition 5.8.b are translation invariant with respect to $\mathcal{X}_0^{(n+1)}$, despite the fact that $C^{(n)}$ is only translation invariant with respect to $\mathcal{X}_{-1}^{(n+1)}$. Using Corollary B.6.b and Remark B.7 and using $1_0^{(1)}$ to denote the constant function on $\mathcal{X}_0^{(1)}$ that always takes the value 1, and using $M_0(x_2', x_3')$ to denote the kernel of M_0,

$$\begin{aligned}
\ell(M_0) &= \tfrac{1}{|\mathcal{X}_0^{(1)}|}\int_{\mathcal{X}_0^{(1)}} dx_2' dx_3'\ M_0(x_2', x_3') \\
&= -\tfrac{2}{L^3|\mathcal{X}_0^{(1)}|}\int_{\mathcal{X}_0} dx_1\cdots dx_4\ V_0(x_1,x_2,x_3,x_4)\ (S_1(L^2\mu_0)Q_1^*\mathfrak{Q}_1 1_0^{(1)})(\mathbb{L}^{-1}x_2) \\
&\qquad\qquad (S_1(L^2\mu_0)^* Q_1^*\mathfrak{Q}_1 1_0^{(1)})(\mathbb{L}^{-1}x_3)\ C^{(0)}(x_4, x_1) \\
&= -\tfrac{2}{L^3|\mathcal{X}_0^{(1)}|}\big(\tfrac{a_1}{a_1 - L^2\mu_0}\big)^2\int_{\mathcal{X}_0} dx_1\cdots dx_4\ V_0(x_1,x_2,x_3,x_4)\ C^{(0)}(x_4,x_1) \\
&= -\tfrac{2}{L^3|\mathcal{X}_0^{(1)}|}\int_{\mathcal{X}_0} dx_1\cdots dx_4\ V_0(x_1,x_2,x_3,x_4)\ C^{(0)}(x_4,x_1) + O\big(|\mu_0|\mathfrak{v}_0\big)
\end{aligned}$$

Recalling that

$$S_1 = (D_1 + Q_1^*\mathfrak{Q}_1 Q_1)^{-1} \quad \mathfrak{Q}_1 = a\mathbb{1} \quad D_1 = L^2\ \mathbb{L}_*^{-1}D_0\mathbb{L}_* \quad Q_1 = \mathbb{L}_*^{-1}\,Q\,\mathbb{L}_*$$

we have

$$\tfrac{1}{L^2}C^{(0)} = (aQ^*Q + L^2D_0)^{-1} = \big(\mathbb{L}_*Q_1^*\mathfrak{Q}_1Q_1\mathbb{L}_*^{-1} + \mathbb{L}_*D_1\mathbb{L}_*^{-1}\big)^{-1} = \mathbb{L}_*S_1\mathbb{L}_*^{-1}$$

or, in terms of kernels,

$$C^{(0)}(x_4, x_1) = \tfrac{1}{L^3}S_1(\mathbb{L}^{-1}x_4, \mathbb{L}^{-1}x_1)$$

by [4, Lemma 15.a]. So

$$\begin{aligned}
\ell(M_0) &= -\tfrac{2}{L^6|\mathcal{X}_0^{(1)}|}\int_{\mathcal{X}_0} dx_1\cdots dx_4\ V_0(x_1,x_2,x_3,x_4)\ S_1(\mathbb{L}^{-1}x_4,\mathbb{L}^{-1}x_1) + O\big(|\mu_0|\mathfrak{v}_0\big)\\
&= -\tfrac{2}{|\mathcal{X}_0^{(1)}|}\int_{\mathcal{X}_1} du_1\cdots du_4\ V_1^{(u)}(u_1,u_2,u_3,u_4)\ S_1(u_4,u_1) + O\big(|\mu_0|\mathfrak{v}_0\big)\\
&= \delta\mu_0^* + O\big(|\mu_0|\mathfrak{v}_0\big)
\end{aligned}$$

Similarly, for $n \ge 1$, using $1_0^{(n+1)}$ to denote the constant function on $\mathcal{X}_0^{(n+1)}$ that always takes the value 1, and using $M_n(x_2,x_3)$ to denote the kernel of M_n,

$$\begin{aligned}
\ell(M_n) &= \tfrac{1}{|\mathcal{X}_0^{(n+1)}|}\int_{\mathcal{X}_0^{(n+1)}} dx_2 dx_3\ M_n(x_2,x_3)\\
&= -\tfrac{2}{L^3|\mathcal{X}_0^{(n+1)}|}\int_{\mathcal{X}_n} du_1\cdots du_4\ V_n(u_1,u_2,u_3,u_4)\\
&\qquad\qquad \big(S_n(\mu_n)Q_n^*\mathfrak{Q}_n\, C^{(n)}\mathfrak{Q}_n\, Q_n S_n(\mu_n)\big)(u_4,u_1)\\
&\qquad\qquad \big(S_{n+1}(L^2\mu_n)Q_{n+1}^*\mathfrak{Q}_{n+1}1_0^{(n+1)}\big)(\mathbb{L}^{-1}u_2)\\
&\qquad\qquad \big(S_{n+1}(L^2\mu_n)^*Q_{n+1}^*\mathfrak{Q}_{n+1}1_0^{(n+1)}\big)(\mathbb{L}^{-1}u_3)\\
&= -\tfrac{2}{L^3|\mathcal{X}_0^{(n+1)}|}\big(\tfrac{a_{n+1}}{a_{n+1}-L^2\mu_n}\big)^2\int_{\mathcal{X}_n} du_1\cdots du_4\ V_n(u_1,u_2,u_3,u_4)\\
&\qquad\qquad \big(S_n(\mu_n)Q_n^*\mathfrak{Q}_n\, C^{(n)}\mathfrak{Q}_n\, Q_n S_n(\mu_n)\big)(u_4,u_1)\\
&= -\tfrac{2}{L^3|\mathcal{X}_0^{(n+1)}|}\int_{\mathcal{X}_n} du_1\cdots du_4\ V_n(u_1,u_2,u_3,u_4)\big(S_n Q_n^*\mathfrak{Q}_n\, C^{(n)}\mathfrak{Q}_n\, Q_n S_n\big)(u_4,u_1)\\
&\qquad\qquad + O\big(|\mu_n|\mathfrak{v}_n\big)\\
&= -\tfrac{2}{L^3|\mathcal{X}_0^{(n+1)}|}\int_{\mathcal{X}_n} du_1\cdots du_4\ V_n^{(u)}(u_1,u_2,u_3,u_4)(S_n Q_n^*\mathfrak{Q}_n\, C^{(n)}\mathfrak{Q}_n\, Q_n S_n)(u_4,u_1)\\
&\qquad\qquad + O\big(\big[|\mu_n| + \mathfrak{v}_0^{\frac{2}{3}-6\epsilon}\big]\mathfrak{v}_n\big)
\end{aligned}$$

since $\frac{a_{n+1}}{a_{n+1}-L^2\mu_n} = 1 + O(|\mu_n|)$ and $\|S_n(\mu_n) - S_n\|_{2m} \le \Gamma_{\rm op}|\mu_n|$, by [4, Proposition 5.1], and $\|V_n - V_n^{(u)}\|_{2m} \le C_{\delta\mathcal{V}}\mathfrak{v}_0^{\frac{2}{3}-6\epsilon}\mathfrak{v}_n$, by [7, Remark 1.18 and Lemma C.3.b]. By [6, Remark 10.c]

$$S_n Q_n^*\mathfrak{Q}_n C^{(n)}\mathfrak{Q}_n Q_n S_n = L^2\,\mathbb{S}^{-1}S_{n+1}\mathbb{S} - S_n$$

In terms of kernels, by [4, Lemma 15.a],

$$\big(S_n Q_n^*\mathfrak{Q}_n C^{(n)}\mathfrak{Q}_n Q_n S_n\big)(u_4,u_1) = \tfrac{1}{L^3}S_{n+1}(\mathbb{L}^{-1}u_4,\mathbb{L}^{-1}u_1) - S_n(u_4,u_1)$$

Now, by [7, Definition 1.5.a,b],

$$\tfrac{2}{L^3|\mathcal{X}_0^{(n+1)}|}\int_{\mathcal{X}_n} du_1\cdots du_4\ V_n^{(u)}(u_1,u_2,u_3,u_4)\ \tfrac{1}{L^3}S_{n+1}(\mathbb{L}^{-1}u_4,\mathbb{L}^{-1}u_1)$$
$$= \tfrac{2}{|\mathcal{X}_0^{(n+1)}|}\int_{\mathcal{X}_{n+1}} dv_1\cdots dv_4\ V_{n+1}^{(u)}(v_1,v_2,v_3,v_4)\ S_{n+1}(v_4,v_1)$$

so that

$$\begin{aligned}\ell(M_n) &= -\tfrac{2}{|\mathcal{X}_0^{(n+1)}|}\int_{\mathcal{X}_{n+1}} dv_1\cdots dv_4\ V_{n+1}^{(u)}(v_1,v_2,v_3,v_4)\ S_{n+1}(v_4,v_1)\\ &\qquad + \tfrac{2}{L^3|\mathcal{X}_0^{(n+1)}|}\int_{\mathcal{X}_n} du_1\cdots du_4\ V_n^{(u)}(u_1,u_2,u_3,u_4)\, S_n(u_4,u_1)\\ &\qquad\qquad + O\big(\big[|\mu_n| + \mathfrak{v}_0^{\frac{2}{3}-6\epsilon}\big]\mathfrak{v}_n\big)\\ &= \delta\mu_n^* + O\big(\big[|\mu_n| + \mathfrak{v}_0^{\frac{2}{3}-6\epsilon}\big]\mathfrak{v}_n\big)\end{aligned}$$

since $|\mathcal{X}_0^{(n+1)}| = \frac{1}{L^5}\mathcal{X}_0^{(n)}$. Using Corollary B.6.b, Proposition 5.8.a,b gives the claim.

(b) Recall, from Lemma 6.2 and (6.9), that $\delta\mu_n$ obeys

$$\frac{a_{n+1}^2}{a_{n+1} - L^2\mu_n - \delta\mu_n} - \frac{a_{n+1}^2}{a_{n+1} - L^2\mu_n} = \ell(\tilde{\mathcal{E}}_{\mathfrak{l}})$$

As

$$\begin{aligned}\frac{a_{n+1}^2}{a_{n+1} - L^2\mu_n - \delta\mu} &= \frac{a_{n+1}}{1 - \frac{L^2\mu_n+\delta\mu}{a_{n+1}}} = a_{n+1}\left[1 + \tfrac{L^2\mu_n+\delta\mu}{a_{n+1}} + \frac{\left(\frac{L^2\mu_n+\delta\mu}{a_{n+1}}\right)^2}{1 - \frac{L^2\mu_n+\delta\mu}{a_{n+1}}}\right]\\ &= a_{n+1} + L^2\mu_n + \delta\mu + \frac{(L^2\mu_n+\delta\mu)^2}{a_{n+1} - L^2\mu_n - \delta\mu}\end{aligned}$$

the left hand side

$$\begin{aligned}\frac{a_{n+1}^2}{a_{n+1} - L^2\mu_n - \delta\mu}\Bigg|_{\delta\mu=0}^{\delta\mu=\delta\mu_n} &= \delta\mu_n + \frac{(L^2\mu_n+\delta\mu_n)^2}{a_{n+1} - L^2\mu_n - \delta\mu_n} - \frac{(L^2\mu_n)^2}{a_{n+1} - L^2\mu_n}\\ &= \delta\mu_n + \frac{\delta\mu_n\big[a_{n+1}(2L^2\mu_n+\delta\mu_n) - (L^2\mu_n+\delta\mu_n)L^2\mu_n\big]}{\big(a_{n+1} - L^2\mu_n - \delta\mu_n\big)\big(a_{n+1} - L^2\mu_n\big)}\end{aligned}$$

As $L^2|\mu_n|, |\delta\mu_n| \le \frac{1}{4} a_{n+1}$,

$$\begin{aligned} |\delta\mu_n - \ell(\tilde{\mathcal{E}}_{\mathfrak{l}})| &= \left|\delta\mu_n \frac{a_{n+1}(2L^2\mu_n + \delta\mu_n) - (L^2\mu_n + \delta\mu_n)L^2\mu_n}{\big(a_{n+1} - L^2\mu_n - \delta\mu_n\big)\big(a_{n+1} - L^2\mu_n\big)}\right| \\ &\le \tfrac{12L^2}{a_{n+1}} |\delta\mu_n| \big(|\mu_n| + |\delta\mu_n|\big) \\ &\le \tfrac{12L^2}{a_{n+1}} \tfrac{3}{\bar\kappa^2} \mathfrak{e}_{\mathfrak{l}}(n) \big(|\mu_n| + \tfrac{3}{\bar\kappa^2}\mathfrak{e}_{\mathfrak{l}}(n)\big) \qquad \text{(by Lemma 6.2)} \end{aligned}$$

Part (a) and

$$\tfrac{\mathfrak{e}_{\mathfrak{l}}(n)}{\bar\kappa^2} = L^{-(2\eta-\eta_{\mathfrak{l}})n} \mathfrak{v}_0^{1-4\epsilon} \le \mathfrak{v}_0^{1-4\epsilon}$$

now implies, using Remark 5.1, that

$$\begin{aligned} |\delta\mu_n - \delta\mu_n^*| &\le \Lambda'_{\delta\mu}\ \mathfrak{v}_0^{1-4\epsilon} \big(\mathfrak{v}_0^{\frac{1}{3}-5\epsilon} + |\mu_n| + r_n^2 e^{-r_n^2}\big) \bar\kappa_{\mathfrak{l}}^6 \\ &\le \Lambda_{\delta\mu}\ \mathfrak{v}_0^{1-7\epsilon} \big(\mathfrak{v}_0^{\frac{1}{3}-5\epsilon} + L^{2n}(\mu_0 - \mu_*)\big) L^{3\epsilon(n+1)} \end{aligned}$$

with a new $\Lambda_{\delta\mu}$.

(c) Since $\mu_{n+1} = L^2\mu_n + \delta\mu_n$ and $\mu_{n+1}^* = L^2\mu_n^* + \delta\mu_n^*$, we have

$$\begin{aligned} \big|\mu_{n+1} - \mu_{n+1}^*\big| &\le L^2\big|\mu_n - \mu_n^*\big| + \big|\delta\mu_n - \delta\mu_n^*\big| \\ &\le L^{2(n+1)}\ \mathfrak{v}_0^{1-8\epsilon} \sum_{\ell=1}^{n+1} \tfrac{1}{L^{(2-3\epsilon)\ell}} \big[\mathfrak{v}_0^{\frac{1}{3}-5\epsilon} + L^{2\ell}(\mu_0 - \mu_*)\big] \end{aligned}$$

by [7, Remark 1.18] and part (b). □

Completion of the Inductive Proof of [7, Theorem 1.17 and Remark 1.18]

Proof. Set

$$\begin{aligned} \mathcal{R}_{n+1}(\phi_*, \phi) &= \tilde{\mathcal{R}}_{n+1}\big((\phi_*, \{\partial_\nu\phi_*\}), (\phi, \{\partial_\nu\phi\})\big) \\ \tilde{\mathcal{E}}_{n+1}(\tilde\psi_*, \tilde\psi) &= \tilde{\mathcal{E}}_{n+1,1}(\tilde\psi_*, \tilde\psi) + \tilde{\mathcal{E}}_{n+1,2}(\tilde\psi_*, \tilde\psi) \\ \mathcal{E}_{n+1}(\psi_*, \psi) &= \tilde{\mathcal{E}}_{n+1}\big((\psi_*, \{\partial_\nu\psi_*\}), (\psi, \{\partial_\nu\psi\})\big) \\ \mathcal{Z}_{n+1} &= \mathcal{Z}_n \tilde{N}_{\mathbb{T}}^{(n)} \mathcal{Z}'_n \end{aligned}$$

where $\tilde{\mathcal{E}}_{n+1,1}$ was defined in Lemma 5.5, $\tilde{\mathcal{R}}_{n+1}$ and $\tilde{\mathcal{E}}_{n+1,2}$ were defined in Lemma 6.6, $\tilde{N}_{\mathbb{T}}^{(n)}$ was defined in [7, Definition 1.6] and $\mathcal{Z}'_n$ was defined in Proposition 5.6. Then, by [7, Corollary 4.3], Lemma 5.5.b,c, Proposition 5.6 and Lemma 6.6,

$$
\begin{aligned}
&\Big((\mathbb{ST}_n^{(SF)})\circ(\mathbb{ST}_{n-1}^{(SF)})\circ\cdots\circ(\mathbb{ST}_0^{(SF)})\Big)\Big(e^{\mathcal{A}_0(\psi^*,\psi)}\Big)\\
&\qquad\qquad = \tfrac{1}{\mathcal{Z}_{n+1}}\exp\Big\{-A_{n+1}(\psi_*,\psi,\phi_{*n+1},\phi_{n+1},\ \mu_{n+1},\mathcal{V}_{n+1})\\
&\qquad\qquad\qquad\qquad\qquad + \mathcal{R}_{n+1}(\phi_{*n+1},\phi_{n+1}) + \mathcal{E}_{n+1}(\psi_*,\psi)\Big\}
\end{aligned}
$$

The bounds on $|\mu_{n+1}-\mu^*_{n+1}|$, $\big\|\mathcal{V}_{n+1}-\mathcal{V}^{(u)}_{n+1}\big\|_{2m}$ and $\big\|\tilde{\mathcal{R}}^{(\vec p)}_{n+1}\big\|_m$ required by [7, Remark 1.18] were proven in Lemmas 6.7, 6.4 and 6.6. That these bounds in turn imply the bounds on $\big|\mu_{n+1}-L^{2(n+1)}(\mu_0-\mu_*)\big|$, $\|\mathcal{V}_{n+1}-\mathcal{V}^{(u)}_{n+1}\|_{2m}$ and $\big\|\tilde{\mathcal{R}}^{(\vec p)}_{n+1}\big\|_m$ specified in [7, Theorem 1.17] was pointed out in [7, Remark 1.18].

By Lemma 5.5.b, Lemma 6.6 and [7, Theorem 1.17], $\tilde{\mathcal{E}}_{n+1}$ does not contain any scaling/weight relevant monomials and

$$
\|\tilde{\mathcal{E}}_{n+1}\|^{(n+1)} \le L^5\,\mathrm{sdf}(C_{\mathfrak{l}})\,\mathfrak{v}_0^{\epsilon} + C_{\mathrm{ren}}\,\mathfrak{e}_{\mathfrak{l}}(n) \le \mathfrak{v}_0^{\epsilon}
$$

by Remark 5.3 (with L chosen big enough that $L^5\,\mathrm{sdf}(C_{\mathfrak{l}})\le\frac12$) and [7, (C.1.a)]. □

A The Limiting Behaviour of μ_n^*

In [7, §1.5], we defined

$$\mu_* = 2\int_{((\mathbb{Z}/L_{\rm tp}\mathbb{Z})\times\mathbb{Z}^3)^3} dx_1\cdots dx_3\ \mathbf{V}_0(0,x_1,x_2,x_3)\ \mathbf{D}_0^{-1}(x_3,0)$$

with $\mathbf{D}_0 = \mathbb{1} - e^{-\mathbf{h}_0} - e^{-\mathbf{h}_0}\partial_0$ and in [7, Remark 1.18], we defined, for $n\ge 1$,

$$\mu_n^* = L^{2n}\mu_0 - \tfrac{2}{|\mathcal{X}_0^{(n)}|}\int_{\mathcal{X}_n^4} du_1\cdots du_4\ V_n^{(u)}(u_1,u_2,u_3,u_4)\ S_n(u_4,u_1)$$

From [7, (1.5)], we see that there will be a well developed potential well when μ_n is sufficiently positive for large n. As $\mu_n\approx\mu_n^*$ (see [7, (1.21)]), the following lemma shows that this is the case if $\mu_0-\mu_*$ is sufficiently positive. That is why we expect μ_* to be the critical μ, to leading order in the coupling constant.

Lemma A.1. *There is a constant c_{μ_*}, depending only on $\Gamma_{\rm op}$ and m, such that*

$$\big|L^{2n}\big(\mu_0-\mu_*\big)-\mu_n^*\big|\le c_{\mu_*}\mathfrak{v}_0$$

for all $1\le n\le n_p$.

Proof. In [4, (5.5)] we defined, on $\mathcal{X}_n$, the operator

$$S_n' = \big[D_n + a_n\exp\{-\Delta_n\}\big]^{-1} \qquad \text{where}$$
$$\Delta_n = \partial_0^*\partial_0 + \big(\partial_1^*\partial_1+\partial_2^*\partial_2+\partial_3^*\partial_3\big) \quad\text{and}\quad a_n = a\Big(1+\sum_{j=1}^{n-1}\tfrac{1}{L^{2j}}\Big)^{-1}$$

It is fully translation invariant with respect to $\mathcal{X}_n$, is exponentially decaying, and has the same local singularity as S_n. Precisely, we proved in [4, Lemma 5.4.d] that

$$\big|S_n(u,u')-S_n'(u,u')\big|\le\Gamma_{\rm op}e^{-2m|u-u'|}$$

so that

$$\begin{aligned}\tfrac{2}{|\mathcal{X}_0^{(n)}|}\Big|\int_{\mathcal{X}_n^4} du_1\cdots du_4\ V_n^{(u)}(u_1,u_2,u_3,u_4)\ \big\{S_n(u_4,u_1)-S_n'(u_4,u_1)\big\}\Big|&\\ \le\tfrac{2\Gamma_{\rm op}}{|\mathcal{X}_0^{(n)}|}\int_{\mathcal{X}_n^4} du_1\cdots du_4\ |V_n^{(u)}(u_1,u_2,u_3,u_4)|\le 2\Gamma_{\rm op}\tfrac{\mathfrak{v}_0}{L^n}&\end{aligned}$$

and, by [7, Definition 1.5.b],

$$\begin{aligned}
L^{2n}\mu_0 - \mu_n^* &= \tfrac{2}{|\mathcal{X}_0^{(n)}|}\int_{\mathcal{X}_n^4} du_1\cdots du_4\ V_n^{(u)}(u_1,u_2,u_3,u_4)\ S_n'(u_4,u_1) + O\big(\tfrac{\mathfrak{v}_0}{L^n}\big)\\
&= \tfrac{2L^{14n}}{|\mathcal{X}_0^{(n)}|}\int_{\mathcal{X}_n^4} du_1\cdots du_4\ V_0(\mathbb{L}^n u_1,\mathbb{L}^n u_2,\mathbb{L}^n u_3,\mathbb{L}^n u_4)\ S_n'(u_4,u_1) + O\big(\tfrac{\mathfrak{v}_0}{L^n}\big)\\
&= \tfrac{2}{L^{6n}|\mathcal{X}_0^{(n)}|}\int_{\mathcal{X}_0^4} dx_1\cdots dx_4\ V_0(x_1,x_2,x_3,x_4)\ S_n'(\mathbb{L}^{-n}x_4,\mathbb{L}^{-n}x_1) + O\big(\tfrac{\mathfrak{v}_0}{L^n}\big)\\
&= \tfrac{2}{L^{n}|\mathcal{X}_0|}\int_{\mathcal{X}_0^4} dx_1\cdots dx_4\ V_0(x_1,x_2,x_3,x_4)\ S_n'(\mathbb{L}^{-n}x_4,\mathbb{L}^{-n}x_1) + O\big(\tfrac{\mathfrak{v}_0}{L^n}\big)\\
&= \tfrac{2}{L^{n}}\int_{\mathcal{X}_0^3} dx_1\cdots dx_3\ V_0(0,x_1,x_2,x_3)\, S_n'(\mathbb{L}^{-n}x_3,0) + O\big(\tfrac{\mathfrak{v}_0}{L^n}\big) \qquad \text{(A.1)}
\end{aligned}$$

The operator S_n' acts on on $L^2(\mathcal{X}_n)$ with, as in [7, Definition 1.5.a],

$$\mathcal{X}_n = \big(\varepsilon_n^2\mathbb{Z}/L_{\rm tp}\varepsilon_n^2\mathbb{Z}\big)\times\big(\varepsilon_n\mathbb{Z}^3/L_{\rm sp}\varepsilon_n\mathbb{Z}^3\big) \qquad \text{where} \quad \varepsilon_n = \tfrac{1}{L^n}$$

It may be expressed as the spatial periodization of an operator $\mathbf{S}_n'$ on $L^2(\boldsymbol{\mathcal{X}}_n)$ where

$$\boldsymbol{\mathcal{X}}_n = \big(\varepsilon_n^2\mathbb{Z}/L_{\rm tp}\varepsilon_n^2\mathbb{Z}\big)\times\varepsilon_n\mathbb{Z}^3$$

We define $\mathbf{S}_n'$ in terms of its Fourier transform

$$\mathbf{S}_n'(u,u') = \int_{\hat{\boldsymbol{\mathcal{X}}}_n} \hat{\mathbf{S}}_n'(p)\ e^{ip\cdot(u-u')}\,\tfrac{d^4p}{(2\pi)^4}$$

where the dual space

$$\hat{\boldsymbol{\mathcal{X}}}_n = \big(\tfrac{2\pi}{L_{\rm tp}}L^{2n}\mathbb{Z}/2\pi L^{2n}\mathbb{Z}\big)\times\big(\mathbb{R}^3/2\pi L^n\mathbb{Z}^3\big)$$

and the integral

$$\int_{\hat{\boldsymbol{\mathcal{X}}}_n} f(p)\ \tfrac{d^4p}{(2\pi)^4} = \sum_{p_0\in\frac{2\pi}{L_{\rm tp}}L^{2n}\mathbb{Z}/2\pi L^{2n}\mathbb{Z}} \tfrac{L^{2n}}{L_{\rm tp}}\ \int_{\mathbb{R}^3/2\pi L^n\mathbb{Z}^3} f(p_0,p_1,p_2,p_3)\ \tfrac{dp_1dp_2dp_3}{(2\pi)^3}$$

The Fourier transform

$$\hat{\mathbf{S}}_n'(p) = \big[\widehat{\mathbf{D}}_n(p)+a_n\exp\{-\boldsymbol{\Delta}_n(p)\}\big]^{-1} \quad \text{with} \quad \boldsymbol{\Delta}_n(p_0,\mathbf{p}) = \Big[\tfrac{\sin\frac{1}{2}\varepsilon_n^2p_0}{\frac{1}{2}\varepsilon_n^2}\Big]^2+\sum_{\nu=1}^{3}\Big[\tfrac{\sin\frac{1}{2}\varepsilon_n\mathbf{p}_\nu}{\frac{1}{2}\varepsilon_n}\Big]^2$$

and

$$\widehat{\mathbf{D}}_n(p_0,\mathbf{p}) = \tfrac{1}{2}\varepsilon_n^2 e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})}\left[\frac{\sin\frac{1}{2}\varepsilon_n^2 p_0}{\frac{1}{2}\varepsilon_n^2}\right]^2 + \frac{1-e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})}}{\varepsilon_n^2} - ie^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})}\frac{\sin\varepsilon_n^2 p_0}{\varepsilon_n^2}$$

Define, for $u_3, u_4 \in \mathcal{X}_n$, $\tilde{\mathbf{S}}'_n(u_3,u_4) = \mathbf{S}'_n(\tilde{u}_3,\tilde{u}_4)$, where $\tilde{u}_3$ and $\tilde{u}_4$ are representatives of $u_3,\ u_4$ in $\boldsymbol{\mathcal{X}}_n$ that minimize the magnitude of each spatial component of $\tilde{u}_3-\tilde{u}_4$. Thanks to the exponential decay of $\mathbf{S}'_n$ proven in [4, Lemma 5.4.d], the difference $S'_n(\mathbb{L}^{-n}x_3,\mathbb{L}^{-n}x_4) - \tilde{\mathbf{S}}'_n(\mathbb{L}^{-n}x_3,\mathbb{L}^{-n}x_4)$ is bounded, uniformly in n. Hence

$$\tfrac{2}{L^n}\left|\int_{\mathcal{X}_0} dx_1\cdots dx_3\ V_0(0,x_1,x_2,x_3)\left\{S'_n(\mathbb{L}^{-n}x_3,0) - \tilde{\mathbf{S}}'_n(\mathbb{L}^{-n}x_3,0)\right\}\right| = O\big(\tfrac{\mathfrak{v}_0}{L^n}\big) \quad \text{(A.2)}$$

So we consider

$$\lim_{n\to\infty}\tfrac{2}{L^n}\int_{\mathcal{X}_0} dx_1\cdots dx_3\ V_0(0,x_1,x_2,x_3)\,\tilde{\mathbf{S}}'_n(\mathbb{L}^{-n}x_3,0)$$

If $\tilde{x}_3 \in \boldsymbol{\mathcal{X}}_0$ is the representative of $x_3 \in \mathcal{X}_0$ whose spatial components have minimum magnitude, then

$$\begin{aligned}\tfrac{1}{L^n}\tilde{\mathbf{S}}'_n(\mathbb{L}^{-n}x_3,0) &= \tfrac{1}{L^n}\int_{\hat{\boldsymbol{\mathcal{X}}}_n}\hat{\mathbf{S}}'_n(p)e^{ip\cdot(\mathbb{L}^{-n}\tilde{x}_3)}\tfrac{d^4p}{(2\pi)^4}\\ &= L^{4n}\int_{\hat{\boldsymbol{\mathcal{X}}}_0}\hat{\mathbf{S}}'_n(\mathbb{L}^n k)e^{ik\cdot\tilde{x}_3}\tfrac{d^4k}{(2\pi)^4} \qquad \text{with } k=\mathbb{L}^{-n}p\end{aligned}$$

Observe that

$$L^{2n}\hat{\mathbf{S}}'_n(\mathbb{L}^n k) = \left\{2e^{-\hat{\mathbf{h}}_0(\mathbf{k})}\sin^2\tfrac{1}{2}k_0 + \big(1-e^{-\hat{\mathbf{h}}_0(\mathbf{k})}\big) - i\,e^{-\hat{\mathbf{h}}_0(\mathbf{k})}\sin k_0 + \tfrac{a_n}{L^{2n}}e^{-\boldsymbol{\Delta}_n(\mathbb{L}^n k)}\right\}^{-1}$$

converges pointwise, as $n\to\infty$, to

$$\widehat{\mathbf{D}}_0(k)^{-1} = \left\{2e^{-\hat{\mathbf{h}}_0(\mathbf{k})}\sin^2\tfrac{1}{2}k_0 + \big(1-e^{-\hat{\mathbf{h}}_0(\mathbf{k})}\big) - i\,e^{-\hat{\mathbf{h}}_0(\mathbf{k})}\sin k_0\right\}^{-1} \qquad \text{(A.3)}$$

and is bounded, uniformly in n, by $\big|\widehat{\mathbf{D}}_0(k)\big|^{-1} \in L^1(\hat{\boldsymbol{\mathcal{X}}}_0)$. Hence $\tfrac{1}{L^{3n}}\tilde{\mathbf{S}}'_n(\mathbb{L}^{-n}x_3,0)$ is bounded, uniformly in n and x_3 and converges pointwise, as $n\to\infty$, to

$$\mathbf{D}_0^{-1}(\tilde{x}_3,0) = \int_{\hat{\boldsymbol{\mathcal{X}}}_0}\widehat{\mathbf{D}}_0(k)^{-1}e^{ik\cdot\tilde{x}_3}\tfrac{d^4k}{(2\pi)^4}$$

Hence, by (A.1) and (A.2),

$$\big(\mu_0 - \mu_*\big) - \tfrac{1}{L^{2n}}\mu_n^* = 2\int_F d\tilde{x}_1 \cdots d\tilde{x}_3 \; \mathbf{V}_0(0,\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)\left[\tfrac{1}{L^{3n}}\mathbf{S}_n'(\mathbb{L}^{-n}\tilde{x}_3,0) - \mathbf{D}_0^{-1}(\tilde{x}_3,0)\right] + O\big(\tfrac{\mathfrak{v}_0}{L^{3n}}\big) + O\big(\mathfrak{v}_0 e^{-mL_{\rm sp}}\big) \tag{A.4}$$

where $F = \Big\{ (\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)^3 \in \boldsymbol{\mathcal{X}}_0^3 \;\Big|\; \frac{L_{\rm sp}}{2} < \tilde{x}_{i,j} < \frac{L_{\rm sp}}{2} \text{ for all } i,j = 1,2,3 \Big\}$. To bound the right hand side, observe that

$$\big|\tfrac{1}{L^{3n}}\mathbf{S}_n'(\mathbb{L}^{-n}\tilde{x}_3,0) - \mathbf{D}_0^{-1}(\tilde{x}_3,0)\big| \le \tfrac{a_n}{L^{2n}}\int_{\hat{\boldsymbol{\mathcal{X}}}_0} \tfrac{1}{|\widehat{\mathbf{D}}_0(k)+\frac{a_n}{L^{2n}}e^{-\boldsymbol{\Delta}_n(\mathbb{L}^n k)}|\;|\widehat{\mathbf{D}}_0(k)|}\tfrac{d^4k}{(2\pi)^4}$$

As $\hat{\mathcal{X}}_0$ is a compact set, both terms $2e^{-\hat{\mathbf{h}}_0(\mathbf{k})}\sin^2\frac{1}{2}k_0$ and $\big(1-e^{-\hat{\mathbf{h}}_0(\mathbf{k})}\big)$ of the real part of $\widehat{\mathbf{D}}_0(k)$ are nonnegative, and $\widehat{\mathbf{D}}_0(k)$ is bounded away from zero outside of any neignbourhood of $k_0 = 0$, $\mathbf{k} = 0$ we have

$$\begin{aligned}\big|\widehat{\mathbf{D}}_0(k)\big| &\ge \text{const}\big|ik_0 + \mathbf{k}^2\big|\\ \big|\widehat{\mathbf{D}}_0(k) + \tfrac{a_n}{L^{2n}}e^{-\boldsymbol{\Delta}_n(\mathbb{L}^n k)}\big| &\ge \text{const}\big|ik_0 + \mathbf{k}^2 + \tfrac{a_n}{L^{2n}}e^{-\text{const}L^{2n}[k_0^2+\mathbf{k}^2]}\big|\end{aligned}$$

For the part of the integral with $k_0 \ne 0$,

$$\begin{aligned}\tfrac{a_n}{L^{2n}}\int_{\substack{\hat{\boldsymbol{\mathcal{X}}}_0\\ k_0\ne 0}} \tfrac{1}{|\widehat{\mathbf{D}}_0(k)+\frac{a_n}{L^{2n}}e^{-\boldsymbol{\Delta}_n(\mathbb{L}^n k)}|\;|\widehat{\mathbf{D}}_0(k)|}\tfrac{d^4k}{(2\pi)^4} &\le \tfrac{\text{const}}{L^{2n}}\int_{-\pi}^{\pi} dk_0\;\tfrac{1}{|k_0|^{3/4}}\int_{|\mathbf{k}|\le 2\pi} d^3\mathbf{k}\;\tfrac{1}{|\mathbf{k}|^{5/2}}\\ &\le \tfrac{\text{const}}{L^{2n}}\end{aligned}$$

For the part of the integral with $k_0 = 0$, scaling $\mathbf{k} = \frac{\mathbf{p}}{L^n}$,

$$\begin{aligned}\tfrac{a_n}{L^{2n}}\int_{\substack{\hat{\boldsymbol{\mathcal{X}}}_0\\ k_0= 0}} \tfrac{1}{|\widehat{\mathbf{D}}_0(k)+\frac{a_n}{L^{2n}}e^{-\boldsymbol{\Delta}_n(\mathbb{L}^n k)}|\;|\widehat{\mathbf{D}}_0(k)|}\tfrac{d^4k}{(2\pi)^4} &\le \tfrac{\text{const}}{L^{2n}L_{\rm tp}}\int_{|\mathbf{k}|\le 2\pi} \tfrac{1}{[\mathbf{k}^2+\frac{1}{L^{2n}}e^{-\text{const}L^{2n}\mathbf{k}^2}]\mathbf{k}^2}d^3\mathbf{k}\\ &= \tfrac{\text{const}}{L^{2n}L_{\rm tp}}L^n\int_{|\mathbf{p}|\le 2\pi L^n} \tfrac{1}{[\mathbf{p}^2+e^{-\text{const}\mathbf{p}^2}]\mathbf{p}^2}d^3\mathbf{p}\\ &\le \tfrac{\text{const}}{L^{2n}}\tfrac{L^n}{L_{\rm tp}}\\ &\le \tfrac{\text{const}}{L^{2n}} \qquad \text{for all } n \le n_p\end{aligned}$$

Putting these bounds into (A.4),

$$\tfrac{1}{L^{2n}}\big[L^{2n}\big(\mu_0-\mu_*\big) - \mu_n^*\big] \le O\big(\tfrac{\mathfrak{v}_0}{L^{2n}}\big) + O\big(\tfrac{\mathfrak{v}_0}{L^{3n}}\big) + O\big(\mathfrak{v}_0 e^{-mL_{\rm sp}}\big)$$

□

B Localization

Fix masses $\mathfrak{m} \geq 0$ and $\bar{\mathfrak{m}} > \mathfrak{m}$.

Lemma B.1. *Let $0 \leq j \leq n$. For each point u of the fine lattice $\mathcal{X}_j^{(n-j)}$, we use $X(u)$ to denote the point of the unit lattice $\mathcal{X}_0^{(n)}$ nearest to u. There exists a constant $C_{\mathfrak{m},\bar{\mathfrak{m}}}$, depending only on $\mathfrak{m}$ and $\bar{\mathfrak{m}}$, such that the following holds. For each linear tansformation $B : \mathcal{H}_0^{(n)} \to \mathcal{H}_j^{(n-j)}$ there are linear maps B_ν, $0 \leq \nu \leq 3$, such that*

$$\sum_{x \in \mathcal{X}_0^{(n)}} B(u,x) \big[\psi(x) - \psi\big(X(u)\big)\big] = \sum_{\nu=0}^{3} B_\nu(\partial_\nu \psi)(u) \qquad \textit{for all } u \in \mathcal{X}_j^{(n-j)}$$

and

$$\|B_\nu\|_{\mathfrak{m}} \leq C_{\mathfrak{m},\bar{\mathfrak{m}}} \|B\|_{\bar{\mathfrak{m}}} \qquad 0 \leq \nu \leq 3$$

Proof. Define

- $\mathcal{B}_0^{(n)}$ to be the set of (oriented) bonds on the lattice $\mathcal{X}_0^{(n)}$.
- For any bond $b = \langle x_1, x_2 \rangle \in \mathcal{B}_0^{(n)}$, $\nabla\psi(b) = \frac{\psi(x_2)-\psi(x_1)}{|x_2-x_1|} = \psi(x_2) - \psi(x_1)$.
- Given fields ψ_ν, $0 \leq \nu \leq 3$ on $\mathcal{X}_0^{(n)}$, we write, for each $\langle x_1, x_2 \rangle \in \mathcal{B}_0^{(n)}$

$$\psi_\nabla(\{\psi_\nu\})(\langle x_1, x_2\rangle) = \begin{cases} \psi_\nu(x_1) & \text{if } x_2 - x_1 = |x_2 - x_1| e_\nu \\ -\psi_\nu(x_2) & \text{if } x_1 - x_2 = |x_2 - x_1| e_\nu \end{cases} \tag{B.1}$$

  where e_ν is the usual unit vector in direction ν. Observe that $\psi_\nabla(\{\partial_\nu\psi\})(b) = \nabla\psi(b)$.
- If $x, x' \in \mathcal{X}_0^{(n)}$ we select by any reasonable algorithm a set $\Pi(x,x') \subset \mathcal{B}_0^{(n)}$ of bonds forming a path from x, x'. This algorithm must be such that no bond ever appears more than once, even ignoring orientation, in any $\Pi(x,x')$ and such that if z is any point on a path $\Pi(x,x')$, then $|x-z|, |z-x'| \leq |x-x'|$. We have $\psi(x') - \psi(x) = \sum_{b \in \Pi(x,x')} \nabla\psi(b)$.

Using this notation,

$$\sum_{x \in \mathcal{X}_0^{(n)}} B(u,x) \big[\psi(x) - \psi\big(X(u)\big)\big] = \sum_{x \in \mathcal{X}_0^{(n)}} \sum_{b \in \Pi(X(u),x)} B(u,x)\, \nabla\psi(b)$$

$$= \sum_{x \in \mathcal{X}_0^{(n)}} \sum_{b \in \Pi(X(u),x)} B(u,x)\, \psi_\nabla(\{\partial_\nu\psi\})(b) = \sum_{\nu=0}^{3} \sum_{z \in \mathcal{X}_0^{(n)}} B_\nu(u;z)\, \partial_\nu\psi(z)$$

with

$$B_\nu(u;z) = \sum_{\substack{x\in\mathcal{X}_0^{(n)} \\ \langle z,z+e_\nu\rangle\in\Pi(X(u),x)}} B(u,x) \quad - \sum_{\substack{x\in\mathcal{X}_0^{(n)} \\ \langle z+e_\nu,z\rangle\in\Pi(X(u),x)}} B(u,x)$$

Recall, from [7, Definition 1.9], that

$$\|B_\nu\|_{\mathfrak{m}} = \max\Big\{ \sup_{u\in\mathcal{X}_j^{(n-j)}} \mathrm{vol}_0 \sum_{z\in\mathcal{X}_0^{(n)}} |B_\nu(u;z)|e^{\mathfrak{m}|u-z|} \ , \quad \sup_{z\in\mathcal{X}_0^{(n)}} \mathrm{vol}_j \sum_{u\in\mathcal{X}_j^{(n-j)}} |B_\nu(u;z)|e^{\mathfrak{m}|u-z|} \Big\}$$

Now, writing $m' = \frac{1}{2}(\bar{\mathfrak{m}} - \mathfrak{m})$,

$$\begin{aligned}
|B_\nu(u;z)|e^{\mathfrak{m}|u-z|} &\le 2 \sum_{\substack{x\in\mathcal{X}_0^{(n)} \\ z \text{ on } \Pi(X(u),x)}} \big|B(u,x)\big|e^{\mathfrak{m}|u-z|} \\
&\le 2e^{2\mathfrak{m}} \sum_{\substack{x\in\mathcal{X}_0^{(n)} \\ z \text{ on } \Pi(X(u),x)}} \big|B(u,x)\big|e^{\bar{\mathfrak{m}}|u-x|-(\bar{\mathfrak{m}}-\mathfrak{m})|u-x|} \\
&\le 2e^{2\bar{\mathfrak{m}}} \sum_{x\in\mathcal{X}_0^{(n)}} \big|B(u,x)\big|e^{\bar{\mathfrak{m}}|u-x|-m'|u-z|-m'|z-x|}
\end{aligned}$$

so that

$$\|B_\nu\|_{\mathfrak{m}} \le 2e^{2\bar{\mathfrak{m}}}\Big(\sup_{u\in\mathcal{X}_j^{(n-j)}} \sum_{z\in\mathcal{X}_0^{(n)}} e^{-m'|z-u|}\Big)\|B\|_{\bar{\mathfrak{m}}}$$

□

Corollary B.2. *Let*

$$\mathcal{P}(\gamma,\psi) = \int dx\,dy\ \gamma(x)\,K(x,y)\,\psi(y)$$

be a bilinear form on $\mathcal{H}_0^{(n)}$ with translation invariant kernel K. Then there exist bilinear forms $\mathcal{P}_\nu\big(\gamma,\psi_\nu\big)$, $0\le\nu\le 3$, such that

$$\mathcal{P}(\gamma,\psi) = \mathcal{K}\int dx\ \gamma(x)\psi(x) + \sum_{\nu=0}^{3}\mathcal{P}_\nu\big(\gamma,\partial_\nu\psi\big)$$

where $\mathcal{K} = \int dy\ K(0,y)$. Furthermore, for each $0\le\nu\le 3$, the kernel K_ν of $\mathcal{P}_\nu$ obeys $\|K_\nu\|_{\mathfrak{m}} \le C_{\mathfrak{m},\bar{\mathfrak{m}}}\|K\|_{\bar{\mathfrak{m}}}$.

Proof. Write

$$\mathcal{P}(\gamma,\psi) = \int dx\,dy\ \gamma(x)\,K(x,y)\,[\psi(y)-\psi(x)] + \mathcal{K}\int dx\ \gamma(x)\,\psi(x)$$

where $\mathcal{K} = \int dy\ K(x,y)$ is independent of x. Lemma B.1, with $j=0$, and thus $X(x)=x$, gives kernels K_ν, $0\le\nu\le 3$ such that

$$\int dy\ K(x,y)\,[\psi(y)-\psi(x)] = \sum_{\nu=0}^{3}\int dy\ K_\nu(x,y)\,\partial_\nu\psi(y)$$

and $\|K_\nu\|_{\mathfrak{m}} \le C_{\mathfrak{m},\bar{\mathfrak{m}}}\|K\|_{\bar{\mathfrak{m}}}$, $0\le\nu\le 3$. Setting

$$\mathcal{P}_\nu\big(\gamma,\psi_\nu\big) = \sum_{x,y\in\mathcal{X}_0^{(n)}} \gamma(x)\,K_\nu(x,y)\,\psi_\nu(y)$$

the corollary follows. □

Lemma B.3. *There is a constant* $c_{\rm loc}$*, depending only on* $\mathfrak{m}$ *and* $\bar{\mathfrak{m}}$*, such that the following holds.*

(a) Let $1\le\nu\le 3$ *and let*

$$\mathcal{P}(\psi_{*\nu},\psi) = \int_{\mathcal{X}_0^{(n)}} dx\,dy\ \psi_{*\nu}(x)\,K(x,y)\,\psi(y)$$

be invariant under $\mathfrak{S}_{\rm spatial}$*. Then there exists a bilinear form*

$$\mathcal{P}_{\rm ren}\big(\psi_{*\nu},\{\psi_{\nu'}\}_{\nu'=0}^{3}\big)$$

that is also invariant under $\mathfrak{S}_{\rm spatial}$*, such that*

$$\mathcal{P}(\partial_\nu\psi_*,\psi) = \mathcal{P}_{\rm ren}\big(\partial_\nu\psi_*,\{\partial_{\nu'}\psi\}_{\nu'=0}^{3}\big) \qquad \text{and} \qquad \|\mathcal{P}_{\rm ren}\|_{\mathfrak{m}} \le c_{\rm loc}\|\mathcal{P}\|_{\bar{\mathfrak{m}}}$$

(b) Let $1\le\nu\le 3$ *and let*

$$\mathcal{P}(\psi_*,\psi,\psi_\nu) = \int dx_1\cdots dx_4\ K(x_1,x_2,x_3,x_4)\ \psi_*(x_1)\psi(x_2)\psi_*(x_3)\psi_\nu(x_4)$$

be invariant under $\mathfrak{S}_{\rm spatial}$*. Then there exists, in the notation of Definition 6.1,*

$$\mathcal{P}_{\rm ren}\big((\psi_*,\{\psi_{*\nu'}\})\,,\,(\psi,\{,\psi_{\nu'}\})\big) \in \mathfrak{P}_{(2,1,1)}\oplus\mathfrak{P}_{(2,0,2)}$$

that

- *is of degree at least one in* ψ_ν *and*
- *obeys* $\mathcal{P}(\psi_*, \psi, \partial_\nu \psi) = \mathcal{P}_{\rm ren}\big((\psi_*, \{\partial_{\nu'}\psi_*\})\,,\,(\psi, \{\partial_{\nu'}\psi\})\big)$ *and with*
- *each monomial in* $\mathcal{P}_{\rm ren}$ *having* $\|\ \cdot\ \|_{\mathfrak{m}}$ *norm bounded by* $c_{\rm loc}\|\mathcal{P}\|_{\bar{\mathfrak{m}}}$

(c) Let

$$\mathcal{P}(\psi_*, \psi) = \int_{\mathcal{X}_0^{(n)}} dx\, dy\ \psi_*(x)\, K(x,y)\, \psi(y)$$

be invariant under $\mathfrak{S}$*. Then there exists*

$$\mathcal{P}_{\rm ren}\big(\psi_*, \psi, \psi_{*\nu}, \psi_\nu\big) \in \mathfrak{P}_{(1,1,0)} \oplus \mathfrak{P}_{(0,1,1)} \oplus \mathfrak{P}_{(0,0,2)}$$

such that each monomial in $\mathcal{P}_{\rm ren}$ *has* $\|\ \cdot\ \|_{\mathfrak{m}}$ *norm bounded by* $c_{\rm loc}\|\mathcal{P}\|_{\bar{\mathfrak{m}}}$ *and*

$$\mathcal{P}(\psi_*, \psi) = \delta\mu \int dx\ \psi_*(x)\, \psi(x) + \mathcal{P}_{\rm ren}\big(\psi_*, \psi, \partial_\nu\psi_*, \partial_\nu\psi\big)$$

where

$$\delta\mu = \int dy\ K(0,y)$$

is real and obeys $|\delta\mu| \le \|K\|_{m=0}$.

Proof. (a) By Corollary B.2, with $\gamma = \psi_{*\nu}$,

$$\mathcal{P}(\psi_{*\nu}, \psi) = \mathcal{K} \int dx\ \psi_{*\nu}(x)\psi(x) + \sum_{\nu'=0}^{3} \mathcal{P}_{\nu'}\big(\psi_{*\nu}, \partial_{\nu'}\psi\big)$$

We have $K(x,y) = -K(R_\nu x - e_\nu, R_\nu y)$, by [7, Lemma B.4], so that

$$\mathcal{K} = \int dy\ K(0,y) = -\int dy\ K(-e_\nu, R_\nu y) = -\int dy\ K(-e_\nu, y) = -\mathcal{K}$$

yielding $\mathcal{K} = 0$. Set

$$\mathcal{P}'_{\rm ren}\big(\psi_{*\nu}, \{\psi_{\nu'}\}\big) = \sum_{\nu'=0}^{3} \mathcal{P}_{\nu'}\big(\psi_{*\nu}, \psi_{\nu'}\big)$$

It has all of the properties required of $\mathcal{P}_{\rm ren}$, with the possible exception of invariance under $\mathfrak{S}_{\rm spatial}$. To recover invariance under $\mathfrak{S}_{\rm spatial}$ we define $\mathcal{P}_{\rm ren}$ by averaging over $\mathfrak{S}_{\rm spatial}$.

$$\mathcal{P}_{\rm ren} = \tfrac{1}{|\mathfrak{S}_{\rm spatial}|} \sum_{g \in \mathfrak{S}_{\rm spatial}} g\mathcal{P}'_{\rm ren}$$

The claim follows by [7, Remark B.5].

(b) Write

$$\mathcal{P}(\psi_*, \psi, \psi_\nu) = \mathcal{K} \int dx\ \psi_*(x)\psi(x)\psi_*(x)\psi_\nu(x) + \delta\mathcal{P}(\psi_*, \psi, \psi_\nu)$$

where

$$\mathcal{K} = \int dx_1 dx_2 dx_3\ K(x_1, x_2, x_3, 0)$$

and

$$\begin{aligned}
&\delta\mathcal{P}(\psi_*, \psi, \psi_\nu) \\
&\quad = \int dx_1\, dx_2\, dx_3\, dx\ K(x_1, x_2, x_3, x)\ \big[\psi_*(x_1)\psi(x_2)\psi_*(x_3) - \psi_*(x)\psi(x)\psi_*(x)\big]\psi_\nu(x)
\end{aligned}$$

As $K(x_1, x_2, x_3, x_4) = -K(R_\nu x_1, R_\nu x_2, R_\nu x_3, R_\nu x_4 - e_\nu)$, by [7, Lemma B.4], we have

$$\begin{aligned}
\mathcal{K} &= \int dx_1 dx_2 dx_3\ K(x_1, x_2, x_3, 0) \\
&= -\int dx_1 dx_2 dx_3\ K(x_1, x_2, x_3, -e_\nu) \\
&= -\int dx_1 dx_2 dx_3\ K(x_1 + e_\nu, x_2 + e_\nu, x_3 + e_\nu, 0) \\
&= -\mathcal{K}
\end{aligned}$$

so that $\mathcal{K} = 0$. As in Lemma B.1,

$$\begin{aligned}
&\delta\mathcal{P}(\psi_*, \psi, \psi_\nu) \\
&\quad = \int dx_1\, dx_2\, dx_3\, dx\ K(x_1, x_2, x_3, x)\ \psi_*(x_1)\psi(x_2)\big[\psi_*(x_3) - \psi_*(x)\big]\psi_\nu(x) \\
&\qquad + \int dx_1\, dx_2\, dx_3\, dx\ K(x_1, x_2, x_3, x)\ \psi_*(x_1)\big[\psi(x_2) - \psi(x)\big]\psi_*(x)\psi_\nu(x) \\
&\qquad + \int dx_1\, dx_2\, dx_3\, dx\ K(x_1, x_2, x_3, x)\ \big[\psi_*(x_1) - \psi_*(x)\big]\psi(x)\psi_*(x)\psi_\nu(x) \\
&\quad = \mathcal{P}'_1(\psi_*, \psi, \{\partial_{\nu'}\psi_*\}, \psi_\nu) + \mathcal{P}'_2(\psi_*, \psi, \{\partial_{\nu'}\psi\}, \psi_\nu) + \mathcal{P}'_3(\psi_*, \psi, \{\partial_{\nu'}\psi\}, \psi_\nu)
\end{aligned}$$

where

$$\mathcal{P}'_1(\psi_*,\psi,\{\psi_{*\nu'}\},\psi_\nu)=\sum_{b\in\Pi(x,x_3)}\int dx_1\,dx_2\,dx_3\,dx\;K(x_1,x_2,x_3,x)\qquad\psi_*(x_1)\,\psi(x_2)\,\psi_{*\triangledown}\big(\{\psi_{*\nu'}\big)(b)\,\psi_\nu(x)$$

$$\mathcal{P}'_2(\psi_*,\psi,\{\psi_{*\nu'}\},\psi_\nu)=\sum_{b\in\Pi(x,x_2)}\int dx_1\,dx_2\,dx_3\,dx\;K(x_1,x_2,x_3,x)\qquad\psi_*(x_1)\,\psi_{\triangledown}\big(\{\psi_{\nu'}\big)(b)\,\psi_*(x)\,\psi_\nu(x)$$

$$\mathcal{P}'_3(\psi_*,\psi,\{\psi_{*\nu'}\},\psi_\nu)=\sum_{b\in\Pi(x,x_1)}\int dx_1\,dx_2\,dx_3\,dx\;K(x_1,x_2,x_3,x)\qquad\psi_{*\triangledown}\big(\{\psi_{*\nu'}\big)(b)\,\psi(x)\,\psi_*(x)\,\psi_\nu(x)$$

For each $i=1,2,3$, we may write

$$\mathcal{P}'_i(\psi_*,\psi,\{\psi_{(*)\nu'}\},\psi_\nu)=\sum_{\nu'=0}^{3}\mathcal{P}_{i\nu'}(\psi_*,\psi,\psi_{(*)\nu'},\psi_\nu)$$

and bound $\mathcal{P}_{\nu'}$ just as $\mathcal{P}_\nu$ was bounded in Lemma B.1. Then it suffices to set

$$\mathcal{P}_{\rm ren}\big((\psi_*,\{\psi_{*\nu'}\})\,,\,(\psi,\{,\psi_{\nu'}\})\big)=\tfrac{1}{|\mathfrak{S}_{\rm spatial}|}\sum_{g\in\mathfrak{S}_{\rm spatial}}\sum_{\nu'=0}^{3}\sum_{i=1}^{3}g\mathcal{P}_{i\nu'}(\psi_*,\psi,\psi_{(*)\nu'},\psi_\nu)$$

(c) By Corollary B.2, with $\gamma=\psi_*$,

$$\mathcal{P}(\psi_*,\psi)=\delta\mu\int dx\;\psi_*(x)\psi(x)+\sum_{\nu=0}^{3}\mathcal{P}_\nu\big(\psi_*,\partial_\nu\psi\big)$$

with

$$\|\mathcal{P}_\nu\|_{(\mathfrak{m}+\bar{\mathfrak{m}})/2}\le C'_r\|\mathcal{P}\|_{\bar{\mathfrak{m}}}$$

We have $\overline{K(R_0y,R_0x)}=K(x,y)$, by [7, Example B.3], so that

$$\delta\mu=\int dy\;K(0,y)=\int dy\;\overline{K(R_0y,0)}=\int dy\;\overline{K(0,-R_0y)}=\int dy\;\overline{K(0,y)}=\overline{\delta\mu}$$

so that $\delta\mu$ is real. By averaging as in part (a), we may assume that each $\mathcal{P}_\nu\big(\psi_*,\psi_\nu\big)$ is invariant under $\mathfrak{S}_{\rm spatial}$. It now suffices to apply part (a) to each $\mathcal{P}_\nu$, $1\le\nu\le3$, and average over $\mathfrak{S}$. ☐

We fix any $\mathfrak{k}$, $\mathfrak{k}'_0$ and $\mathfrak{k}'_{\rm sp}$ and use norms $\overline{\|}\mathcal{F}(\tilde\psi_*,\tilde\psi)\overline{\|}$ and $\underline{\|}\mathcal{F}(\tilde\psi_*,\tilde\psi)\underline{\|}$ which associate the weight factor $\mathfrak{k}$ to the fields $\psi_{(*)}$, the weight factor $\mathfrak{k}'_0$ to the fields $\psi_{(*)0}$, and the weight factor $\mathfrak{k}'_{\rm sp}$ to the fields $\psi_{(*)\nu}$, $1\le\nu\le 3$. The norm $\overline{\|}\ \cdot\ \overline{\|}$ has mass $\bar{\mathfrak{m}}$ and the norm $\underline{\|}\ \cdot\ \underline{\|}$ has mass $\mathfrak{m}$.

Let $\mathfrak{P}_{\rm rel}$, $\mathfrak{P}_{\mathfrak{D}}$ and $\mathfrak{P}_{\rm irr}$ be the spaces of Definition 6.1 and, as in Definition 6.1, denote by $\mathfrak{P}_{(4,0,0)}$ the space of quartic monomials in ψ_*,ψ that are $\mathfrak{S}$ invariant and particle number preserving.

Proposition B.4. *There exist linear maps*

$$\ell:\mathfrak{P}_{\rm rel}\to\mathbb{C}\qquad \mathcal{L}_4:\mathfrak{P}_{\rm rel}\to\mathfrak{P}_{(4,0,0)}\qquad \mathcal{L}_{\mathfrak{D}}:\mathfrak{P}_{\rm rel}\to\mathfrak{P}_{\mathfrak{D}}\qquad \mathcal{I}:\mathfrak{P}_{\rm rel}\to\mathfrak{P}_{\rm irr}$$

such that, for all $\mathcal{P}\in\mathfrak{P}_{\rm rel}$,

$$\begin{aligned}\mathcal{P}\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\big)=\ell(\mathcal{P})\int dx\ \psi_*(x)\psi(x)&+\mathcal{L}_4(\mathcal{P})(\psi_*,\psi)\\ &+\mathcal{L}_{\mathfrak{D}}(\mathcal{P})\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\big)\\ &+\mathcal{I}(\mathcal{P})\big((\psi_*,\{\partial_\nu\psi_*\})\,,\,(\psi,\{\partial_\nu\psi\})\big)\end{aligned}$$

and

- *for $\mathcal{P}\in\mathfrak{P}_{\mathfrak{D}}$,*

$$\ell(\mathcal{P})=0\qquad \mathcal{L}_4(\mathcal{P})=0\qquad \mathcal{L}_{\mathfrak{D}}(\mathcal{P})=\mathcal{P}\qquad \mathcal{I}(\mathcal{P})=0$$

- *for $\mathcal{P}\in\mathfrak{P}_{(4,0,0)}$,*

$$\ell(\mathcal{P})=0\qquad \mathcal{L}_4(\mathcal{P})=\mathcal{P}\qquad \mathcal{L}_{\mathfrak{D}}(\mathcal{P})=0\qquad \mathcal{I}(\mathcal{P})=0$$

- *for $\mathcal{P}=\int dx\,dx'\ \psi_*(x)K(x,x')\psi(x')\in\mathfrak{P}_{(2,0,0)}$*
  - $\ell(\mathcal{P})=\int dx'\ K(0,x')$
  - $\mathcal{L}_4(\mathcal{P})=0$
  - $\mathcal{L}_{\mathfrak{D}}(\mathcal{P})=\mathcal{L}_{(1,1,0)}(\mathcal{P})+\mathcal{L}_{(0,1,1)}(\mathcal{P})+\mathcal{L}_{(0,0,2)}(\mathcal{P})$ *with*

$$\begin{aligned}&\mathcal{L}_{(1,1,0)}(\mathcal{P})\in\mathfrak{P}_{(1,1,0)} &&\underline{\|}\mathcal{L}_{(1,1,0)}(\mathcal{P})\underline{\|}\le 2c_{\rm loc}\tfrac{\mathfrak{k}'_0}{\mathfrak{k}}\overline{\|}\mathcal{P}\overline{\|}\\ &\mathcal{L}_{(0,1,1)}(\mathcal{P})\in\mathfrak{P}_{(0,1,1)} &&\underline{\|}\mathcal{L}_{(0,1,1)}(\mathcal{P})\underline{\|}\le 6c_{\rm loc}\tfrac{\mathfrak{k}'_0\mathfrak{k}'_{\rm sp}}{\mathfrak{k}^2}\overline{\|}\mathcal{P}\overline{\|}\\ &\mathcal{L}_{(0,0,2)}(\mathcal{P})\in\mathfrak{P}_{(0,0,2)} &&\underline{\|}\mathcal{L}_{(0,0,2)}(\mathcal{P})\underline{\|}\le 9c_{\rm loc}\tfrac{{\mathfrak{k}'_{\rm sp}}^2}{\mathfrak{k}^2}\overline{\|}\mathcal{P}\overline{\|}\end{aligned}$$

  - $\mathcal{I}(\mathcal{P})=0$

- *for* $\mathcal{P} \in \mathfrak{P}_{(1,0,1)}$*, we have* $\ell(\mathcal{P}) = 0$*,* $\mathcal{L}_4(\mathcal{P}) = 0$*,* $\mathcal{I}(\mathcal{P}) = 0$ *and* $\mathcal{L}_{\mathfrak{D}}(\mathcal{P}) = \mathcal{L}_{(0,1,1)}(\mathcal{P}) + \mathcal{L}_{(0,0,2)}(\mathcal{P})$ *with*

$$\begin{aligned} &\mathcal{L}_{(0,1,1)}(\mathcal{P}) \in \mathfrak{P}_{(0,1,1)} && \underline{\|}\mathcal{L}_{(0,1,1)}(\mathcal{P})\underline{\|} \le c_{\rm loc}\tfrac{\mathfrak{k}'_0}{\mathfrak{k}}\overline{\|}\mathcal{P}\overline{\|} \\ &\mathcal{L}_{(0,0,2)}(\mathcal{P}) \in \mathfrak{P}_{(0,0,2)} && \underline{\|}\mathcal{L}_{(0,0,2)}(\mathcal{P})\underline{\|} \le 3c_{\rm loc}\tfrac{\mathfrak{k}'_{\rm sp}}{\mathfrak{k}}\overline{\|}\mathcal{P}\overline{\|} \end{aligned}$$

- *for* $\mathcal{P} \in \mathfrak{P}_{(3,0,1)}$*, we have* $\ell(\mathcal{P}) = 0$*,* $\mathcal{L}_4(\mathcal{P}) = 0$*,* $\mathcal{L}_{\mathfrak{D}}(\mathcal{P}) = 0$ *and*

$$\underline{\|}\mathcal{I}(\mathcal{P})\underline{\|} \le 18c_{\rm loc}\big(\tfrac{\mathfrak{k}'_0}{\mathfrak{k}} + \tfrac{\mathfrak{k}'_{\rm sp}}{\mathfrak{k}}\big)\overline{\|}\mathcal{P}\overline{\|}$$

Proof. Just apply the previous lemma. □

Definition B.5. Let $\mathcal{F}(\tilde\psi_*, \tilde\psi)$ be an analytic function of the fields in a neighbourhood of the origin in $\tilde{\mathcal{H}}_0^{(n)} \times \tilde{\mathcal{H}}_0^{(n)}$ that obeys $\mathcal{F}(0,0) = 0$. Write $\mathcal{F} = \mathcal{F}_{\rm rel} + \mathcal{F}_{\rm irr}$ with $\mathcal{F}_{\rm rel} \in \mathfrak{P}_{\rm rel}$ and $\mathcal{F}_{\rm irr} \in \mathfrak{P}_{\rm irr}$. Define

$$\ell(\mathcal{F}) = \ell(\mathcal{F}_{\rm rel}) \quad \mathcal{L}_4(\mathcal{F}) = \mathcal{L}_4(\mathcal{F}_{\rm rel}) \quad \mathcal{L}_{\mathfrak{D}}(\mathcal{F}) = \mathcal{L}_{\mathfrak{D}}(\mathcal{F}_{\rm rel}) \quad \mathcal{I}(\mathcal{F}) = \mathcal{I}(\mathcal{F}_{\rm rel}) + \mathcal{F}_{\rm irr}$$

Corollary B.6. *Let* $\mathcal{F}(\tilde\psi_*, \tilde\psi)$ *be an analytic function of the fields in a neighbourhood of the origin in* $\tilde{\mathcal{H}}_0^{(n)} \times \tilde{\mathcal{H}}_0^{(n)}$ *that obeys* $\mathcal{F}(0,0) = 0$*.*

(a) Then

$$\begin{aligned} \mathcal{F}\big((\psi_*, \{\partial_\nu\psi_*\}) , (\psi, \{\partial_\nu\psi\})\big) = \ell(\mathcal{F})&\int dx\ \psi_*(x)\psi(x) + \mathcal{L}_4(\mathcal{F})(\psi_*, \psi) \\ &+ \mathcal{L}_{\mathfrak{D}}(\mathcal{F})\big((\psi_*, \{\partial_\nu\psi_*\}) , (\psi, \{\partial_\nu\psi\})\big) \\ &+ \mathcal{I}(\mathcal{F})\big((\psi_*, \{\partial_\nu\psi_*\}) , (\psi, \{\partial_\nu\psi\})\big) \end{aligned}$$

(b) If the monomial in $\mathcal{F}$ *of type* $(2,0,0)$ *is* $\mathcal{F}_2(\psi_*, \psi) = \int dx\, dx'\ \psi_*(x)K(x,x')\psi(x')$ *then*

$$\ell(\mathcal{F}) = \int dx\ K(0,x) = \frac{\mathcal{F}_2(1,1)}{\int dx} \qquad \text{and} \qquad |\ell(\mathcal{F})| \le \tfrac{1}{\mathfrak{k}^2}\underline{\|}\mathcal{F}\underline{\|}$$

(c) We have

$$\overline{\|}\mathcal{L}_4(\mathcal{F})\overline{\|}\ ,\ \underline{\|}\mathcal{L}_{\mathfrak{D}}(\mathcal{F})\underline{\|}\ ,\ \underline{\|}\mathcal{I}(\mathcal{F})\underline{\|}\ \le\ \Big[1 + 9c_{\rm loc}\big(\tfrac{\mathfrak{k}'_0}{\mathfrak{k}} + \tfrac{\mathfrak{k}'_{\rm sp}}{\mathfrak{k}}\big)\Big]^2\,\overline{\|}\mathcal{F}\overline{\|}$$

(d) *Define a partial ordering*[2] *on the set of vectors* $\vec{p} = (p_u, p_0, p_{\rm sp})$ *by*

$$(p_u, p_0, p_{\rm sp}) \lesssim (p'_u, p'_0, p'_{\rm sp}) \quad\iff\quad p_0 \le p'_0,\ \ p_{\rm sp} \le p'_{\rm sp},\ \ p_u+p_0+p_{\rm sp} \le p'_u+p'_0+p'_{\rm sp}$$

If $\mathcal{F} \in \mathfrak{P}_{\vec{p}}$ *then* $\mathcal{L}_{\mathfrak{D}}(\mathcal{F}) \in \bigoplus_{\vec{p}'\gtrsim\vec{p}} \mathfrak{P}_{\vec{p}'}$.

Remark B.7. The following are useful when exploiting Corollary B.6.b.

(a) Denote by 1,$1_{\rm crs}$ and $1_{\rm fin}$ the functions on $\mathcal{X}_0^{(n)}$, $\mathcal{X}_{-1}^{(n+1)}$ and $\mathcal{X}_n$, respectively, which alway take the value 1. Then

$$Q1 = 1_{\rm crs} \quad Q^*1_{\rm crs} = 1 \quad Q_n 1_{\rm fin} = 1 \quad Q_n^* 1 = 1_{\rm fin} \quad \mathfrak{Q}_n 1 = a_n 1 \quad D_n 1_{\rm fin} = 0$$

where, as in (6.8),

$$a_n = a\big(1 + \sum_{j=1}^{n-1} \tfrac{1}{L^{2j}}\big)^{-1}$$

(b) We have

$$\begin{array}{ll} S_n 1_{\rm fin} = S_n^* 1_{\rm fin} = \tfrac{1}{a_n} 1_{\rm fin} & S_n(\mu) 1_{\rm fin} = S_n(\mu)^* 1_{\rm fin} = \tfrac{1}{a_n-\mu} 1_{\rm fin} \\ S_n(\mu)^{(*)} Q_n^* \mathfrak{Q}_n 1 = \tfrac{a_n}{a_n-\mu} 1_{\rm fin} & \big(S_n(\mu)^{(*)} Q_n^* \mathfrak{Q}_n\big)^* 1_{\rm fin} = \tfrac{a_n}{a_n-\mu} 1 \\ B_{(*)n,\mu} 1 = \tfrac{a_n}{[a_n-\mu-\delta\mu][a_n-\mu]} 1_{\rm fin} & B^*_{(*)n,\mu} 1_{\rm fin} = \tfrac{a_n}{[a_n-\mu-\delta\mu][a_n-\mu]} 1_{\rm fin} \\ B_{(*)n,\mu,D} 1 = 0 & B^*_{(*)n,\mu,D} 1_{\rm fin} = 0 \\ B^{(-)}_{n,\mu,D} 1 = 0 & B^{(-)*}_{n,\mu,D} 1_{\rm fin} = 0 \end{array}$$

where $B_{(*)n,\mu}$ and $B_{(*)n,\mu,D}$ are the operators of [8, Proposition 4.1] and $B^{(-)}_{n,\mu,D}$ is the operator of [8, Proposition 2.1].

Proof. (a) Taking Fourier transforms, both of the equations

$$Q_n 1_{\rm fin} = 1 \quad \text{and} \quad Q_n^* 1 = 1_{\rm fin}$$

follow from the facts that the function $u_n(p)$ of [4, Remark 2.1.b] obeys $u_n(k+\ell) = 1$, when $k = \ell = 0$ and $u_n(k+\ell) = 0$ when $k = 0$ and $0 \ne \ell \in \hat{\mathcal{B}}_n$. See [4, Remark 2.1.e and Lemma 2.2.b,c]. Similarly, both of the equations $Q1 = 1_{\rm crs}$ and $Q^*1_{\rm crs} = 1$ follow from the facts that the function $u_+(p)$ of [4, (2.4)] obeys $u_+(\mathfrak{k}+\ell) = 1$, when $\mathfrak{k} = \ell = 0$ and $u_+(\mathfrak{k} + \ell) = 0$ when $\mathfrak{k} = 0$ and $0 \ne \ell \in \hat{\mathcal{B}}^+$. See [4, Remark 2.1.e and Lemma

[2] "Converting a nonderivative field to a derivative field" or "adding a field", increases $(p_u, p_0, p_{\rm sp})$ under this partial ordering.

2.3.c,d]. As $\mathfrak{Q}_n = a\big(\mathbb{1}+\sum_{j=1}^{n-1} \frac{1}{L^{2j}} Q_j Q_j^*\big)^{-1}$ the equality $\mathfrak{Q}_n 1 = a\big(1+\sum_{j=1}^{n-1} \frac{1}{L^{2j}}\big)^{-1} 1 = a_n 1$ follows. That $D_n 1_{\rm fin} = 0$ is true is trivial since discrete derivatives annihilate constant functions.

(b) follows from part (a) and the definitions

$$\begin{aligned}
S_n^{(*)\,-1} &= D_n + Q_n^* \mathfrak{Q}_n Q_n \\
S_n(\mu)^{(*)\,-1} &= D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu \\
B_{(*)n,\mu} &= S_n^{(*)} \left[\mathbb{1} - (\mu + \delta\mu) S_n^{(*)}\right]^{-1} S_n(\mu)^{(*)} Q_n^* \mathfrak{Q}_n \\
B_{(*)n,\mu,D} &= S_n(\mu)^{(*)} Q_n^* \mathfrak{Q}_n - \big(Q_n^* \mathfrak{Q}_n Q_n - \mu - \delta\mu\big) B_{(*)n,\mu} \\
B_{n,\mu,D}^{(-)} 1 &= \big[\mathbb{1} - (Q_n^* \mathfrak{Q}_n Q_n - \mu) S_n(\mu)\big] Q_n^* \mathfrak{Q}_n
\end{aligned}$$

□

C Scaling and Bounds

Let $n \ge 0$ and $0 \le i, j \le n+1$. In this appendix we consider the impact of scaling on norms of functions

$$\mathcal{F} : \tilde{\mathcal{H}}_{j-1}^{(n+1-j)} \times \tilde{\mathcal{H}}_{j-1}^{(n+1-j)} \to \mathbb{C}$$

and field maps

$$A : \tilde{\mathcal{H}}_{j-1}^{(n+1-j)} \times \tilde{\mathcal{H}}_{j-1}^{(n+1-j)} \times \mathcal{H}_{i-1}^{(n+1-i)} \times \mathcal{H}_{i-1}^{(n+1-i)} \to \mathcal{H}_n^{(0)}$$

Recall from [7, Definition 2.1.b], that $(\mathbb{S}\mathcal{F})(\tilde\beta_*, \tilde\beta) = \mathcal{F}\big(\mathbb{S}^{-1}\tilde\beta_*, \mathbb{S}^{-1}\tilde\beta)$ maps

$$\mathbb{S}\mathcal{F} : \tilde{\mathcal{H}}_j^{(n+1-j)} \times \tilde{\mathcal{H}}_j^{(n+1-j)} \to \mathbb{C}$$

Similarly, define the scaled field map

$$A^{(s)}(\tilde\beta_*, \tilde\beta, z_*, z) = \mathbb{L}_*^{-1}\big[A\big(\mathbb{S}^{-1}\tilde\beta_*, \mathbb{S}^{-1}\tilde\beta, \mathbb{S}^{-1}z_*, \mathbb{S}^{-1}z)\big] \tag{C.1}$$

with the $\mathbb{L}_*$ of [7, Definition 1.5.a]. It maps

$$A^{(s)} : \tilde{\mathcal{H}}_j^{(n+1-j)} \times \tilde{\mathcal{H}}_j^{(n+1-j)} \times \mathcal{H}_i^{(n+1-i)} \times \mathcal{H}_i^{(n+1-i)} \to \mathcal{H}_{n+1}^{(0)}$$

We fix any $\check{\mathfrak{m}}, \check{\mathfrak{k}}, \check{\mathfrak{k}}', \check{\mathfrak{k}}_{\mathfrak{l}} > 0$ and use the norms $\|\mathcal{F}(\tilde\alpha_*, \tilde\alpha)\|^{\check{}}$, $|||A(\tilde\alpha_*, \tilde\alpha, \zeta_*, \zeta)|||^{\check{}}$ with mass $\check{\mathfrak{m}} > 0$ and weight factors $\check{\mathfrak{k}}, \check{\mathfrak{k}}', \check{\mathfrak{k}}_{\mathfrak{l}}$ to measure the unscaled functions and field maps. See [7, Definition A.3]. The weight factor $\check{\mathfrak{k}}$ is used for the $\alpha_{(*)}$'s, the weight factor $\check{\mathfrak{k}}'$ is used for the $\alpha_{(*)\nu}$'s, $0 \le \nu \le 3$, and the weight factor $\check{\mathfrak{k}}_{\mathfrak{l}}$ is used for the $\zeta_{(*)}$'s.

Also, fix any $\mathfrak{m}$, $\mathfrak{k}$, $\mathfrak{k}'$, $\mathfrak{k}_{\mathfrak{l}} > 0$ and use the norms $\|(\mathbb{S}\mathcal{F})(\tilde{\beta}_*, \tilde{\beta})\|$, $|||A^{(s)}(\tilde{\beta}_*, \tilde{\beta}, z_*, z)|||$ with mass $\mathfrak{m} > 0$ and weight factors $\mathfrak{k}, \mathfrak{k}', \mathfrak{k}_{\mathfrak{l}}$ to measure the scaled functions and field maps. The weight factor $\mathfrak{k}$ is used for the $\beta_{(*)}$'s, the weight factor $\mathfrak{k}'$ is used for the $\beta_{(*)\nu}$'s, $0 \le \nu \le 3$, and the weight factor $\mathfrak{k}_{\mathfrak{l}}$ is used for the $z_{(*)}$'s.

Definition C.1 (Scaling Divergence Factor)**.**

(a) Let

$$\mathcal{M}\big(\,(\alpha_*, \{\alpha_{*\nu}\})\,,\,(\alpha, \{\alpha_\nu\})\,\big) = \int_{\mathcal{X}_{j-1}^{(n+1-j)}} dv_1 \cdots dv_p \; M(v_1, \cdots, v_p) \prod_{\ell=1}^{p} \alpha_{\sigma_\ell}(v_\ell)$$

be a monomial of degree p. Here each α_{σ_ℓ} is one of $\alpha_*, \alpha, \big\{\alpha_{*\nu}, \alpha_\nu\big\}_{\nu=0}^{3}$. Denote by

- p_u, the number of α_{σ_ℓ}'s that is either α_* or α and
- p_0, the number of α_{σ_ℓ}'s that is either α_{*0} or α_0 and
- $p_{\rm sp}$, the number of α_{σ_ℓ}'s that is one of $\big\{\alpha_{*\nu}, \alpha_\nu\big\}_{\nu=1}^{3}$.

Set

$$\mathrm{Sdf}(\mathcal{M}) = \big(\tfrac{1}{L^{3/2}}\tfrac{\mathfrak{k}}{\check{\mathfrak{k}}}\big)^{p_u}\big(\tfrac{1}{L^{7/2}}\tfrac{\mathfrak{k}'}{\check{\mathfrak{k}}'}\big)^{p_0}\big(\tfrac{1}{L^{5/2}}\tfrac{\mathfrak{k}'}{\check{\mathfrak{k}}'}\big)^{p_{\rm sp}}$$

(b) Let $\mathcal{F}$ be an analytic function on a neighbourhood of the origin in $\tilde{\mathcal{H}}_{j-1}^{(n+1-j)} \times \tilde{\mathcal{H}}_{j-1}^{(n+1-j)}$. Then $\mathrm{Sdf}(\mathcal{F})$ is the supremum of $\mathrm{Sdf}(\mathcal{M})$ with $\mathcal{M}$ running over the nonzero monomials in the power series representation of $\mathcal{F}$.

Lemma C.2. *Assume that* $\mathfrak{m} \le L\check{\mathfrak{m}}$.

(a) Let

$$\mathcal{M}\big(\,(\alpha_*, \{\alpha_{*\nu}\})\,,\,(\alpha, \{\alpha_\nu\})\,\big) = \int_{\mathcal{X}_{j-1}^{(n+1-j)}} dv_1 \cdots dv_p \; M(v_1, \cdots, v_p) \prod_{\ell=1}^{p} \alpha_{\sigma_\ell}(v_\ell)$$

be a monomial as in Definition C.1.a. Then the kernel of $\mathbb{S}\mathcal{M}$ *is*

$$M^{(s)}(u_1, \cdots, u_p) = L^{\frac{7}{2}p_u + \frac{3}{2}p_0 + \frac{5}{2}p_{\rm sp}} M(\mathbb{L}u_1, \cdots, \mathbb{L}u_p) \tag{C.2}$$

and

$$\big\|M^{(s)}\big\|_{\mathfrak{m}} \le L^5 L^{-\frac{3}{2}p_u - \frac{7}{2}p_0 - \frac{5}{2}p_{\rm sp}} \big\|M\big\|_{\check{\mathfrak{m}}} \tag{C.3}$$

(b) Let $\mathcal{F}$ be an analytic function on a neighbourhood of the origin in $\tilde{\mathcal{H}}_{j-1}^{(n+1-j)} \times \tilde{\mathcal{H}}_{j-1}^{(n+1-j)}$. Then

$$\left\|\mathbb{S}\mathcal{F}\right\| \le L^5 \,\mathrm{Sdf}(\mathcal{F})\, \|\mathcal{F}\|^{\check{}}$$

In the event that $\mathfrak{e} \le L^{3/2}\check{\mathfrak{e}}$ and $\mathfrak{e}' \le L^{5/2}\check{\mathfrak{e}}'$, then $\left\|\mathbb{S}\mathcal{F}\right\| \le L^5 \|\mathcal{F}\|^{\check{}}$.

(c) Assume that $\mathfrak{e} \le L^{3/2}\check{\mathfrak{e}}$, $\mathfrak{e}' \le L^{5/2}\check{\mathfrak{e}}'$ and $\mathfrak{e}_{\mathfrak{l}} \le L^{3/2}\check{\mathfrak{e}}_{\mathfrak{l}}$. Let A be a field map defined on a neighbourhood of the origin in $\tilde{\mathcal{H}}_{j-1}^{(n+1-j)} \times \tilde{\mathcal{H}}_{j-1}^{(n+1-j)} \times \mathcal{H}_{i-1}^{(n+1-i)} \times \mathcal{H}_{i-1}^{(n+1-i)}$ and taking values in $\mathcal{H}_n^{(0)}$. Then $|||A^{(s)}||| \le |||A|||^{\check{}}$.

Proof. (a) [7, Remark 2.2.h] gives (C.2). Then, introducing the local shorthand notation $\mathcal{X} = \mathcal{X}_j^{(n+1-j)}$ and $\check{\mathcal{X}} = \mathcal{X}_{j-1}^{(n+1-j)}$,

$$\begin{aligned}
&\left\|M^{(s)}\right\|_{\mathfrak{m}} \\
&= \max_{1\le i\le p} \max_{u_i} \int_{\mathcal{X}^{p-1}} du_1 \cdots du_{i-1}\, du_{i+1} \cdots du_p \; L^{\frac{7}{2}p_u + \frac{3}{2}p_0 + \frac{5}{2}p_{\rm sp}} M(\mathbb{L}u_1, \cdots, \mathbb{L}u_p)\, e^{\mathfrak{m}\tau(u_1,\cdots,u_p)} \\
&\le \tfrac{L^{\frac{7}{2}p_u + \frac{3}{2}p_0 + \frac{5}{2}p_{\rm sp}}}{L^{5(p-1)}} \max_{1\le i\le p} \max_{v_i} \int dv_1 \cdots dv_{i-1}\, dv_{i+1} \cdots dv_p\, M(v_1, \cdots, v_p)\, e^{L\check{\mathfrak{m}}\tau(\mathbb{L}^{-1}v_1,\cdots,\mathbb{L}^{-1}v_p)} \\
&\le \tfrac{L^{\frac{7}{2}p_u + \frac{3}{2}p_0 + \frac{5}{2}p_{\rm sp}}}{L^{5(p-1)}} \max_{1\le i\le p} \max_{v_i} \int dv_1 \cdots dv_{i-1}\, dv_{i+1} \cdots dv_p \; M(v_1, \cdots, v_p)\, e^{\check{\mathfrak{m}}\tau(v_1,\cdots,v_p)} \\
&= L^5 L^{-\frac{3}{2}p_u - \frac{7}{2}p_0 - \frac{5}{2}p_{\rm sp}} \left\|M\right\|_{\check{\mathfrak{m}}}
\end{aligned}$$

since, if $t(v_1, \cdots, v_p)$ is the length of a specific tree T that is minimal for $\tau(v_1, \cdots, v_p)$ and if $t_L(v_1, \cdots, v_p)$ is the length of the tree constructed from T by moving the location v of each vertex of T to $\mathbb{L}^{-1}v$,

$$L\tau(\mathbb{L}^{-1}v_1, \cdots, \mathbb{L}^{-1}v_p) \le L t_L(v_1, \cdots, v_p) \le t(v_1, \cdots, v_p) = \tau(v_1, \cdots, v_p)$$

(b) It suffices to consider the case that $\mathcal{F}$ is a monomial as in part (a). Then

$$\begin{aligned}
\|\mathbb{S}\mathcal{F}\| &= \left\|M^{(s)}\right\|_{\mathfrak{m}} \mathfrak{e}^{p_u}\, \mathfrak{e}'^{\,p_0+p_{\rm sp}} \\
&\le L^5 L^{-\frac{3}{2}p_u - \frac{7}{2}p_0 - \frac{5}{2}p_{\rm sp}} \left(\tfrac{\mathfrak{e}}{\check{\mathfrak{e}}}\right)^{p_u} \left(\tfrac{\mathfrak{e}'}{\check{\mathfrak{e}}'}\right)^{p_0+p_{\rm sp}} \left\|M\right\|_{\check{\mathfrak{m}}} \check{\mathfrak{e}}^{p_u}\, \check{\mathfrak{e}}'^{p_0+p_{\rm sp}} \\
&= L^5 \,\mathrm{Sdf}(\mathcal{F}) \|\mathcal{F}\|^{\check{}}
\end{aligned}$$

(c) Once again it suffices to consider monomials

$$A\big(\,(\alpha_*, \{\alpha_{*\nu}\})\,,\, (\alpha, \{\alpha_\nu\})\,,\, \zeta_*\,,\, \zeta\big)(v_0) = \int dv_1 \cdots dv_p \; M(v_0, v_1, \cdots, v_p) \prod_{\ell=1}^{p} \alpha_{\sigma_\ell}(v_\ell)$$

of degree p. Here each α_{σ_ℓ} is one of $\alpha_*, \alpha, \{\alpha_{*\nu}, \alpha_\nu\}_{\nu=0}^3, \zeta_*, \zeta$. If α_{σ_ℓ} is one of $\alpha_*, \alpha, \{\alpha_{*\nu}, \alpha_\nu\}_{\nu=0}^3$, then v_ℓ runs over $\mathcal{X}_{j-1}^{(n+1-j)}$. If α_{σ_ℓ} is one of ζ_*, ζ, then v_ℓ runs over $\mathcal{X}_{i-1}^{(n+1-i)}$. The argument v_0 runs over $\mathcal{X}_n$. We denote by

- p_u, the number of α_{σ_ℓ}'s that is one of $\alpha_*, \alpha, \zeta_*, \zeta$.
- p_0, the number of α_{σ_ℓ}'s that is either α_{*0} or α_0 and
- $p_{\rm sp}$, the number of α_{σ_ℓ}'s that is one of $\{\alpha_{*\nu}, \alpha_\nu\}_{\nu=1}^3$.

The analog of (C.2) for A is

$$M^{(s)}(u_0, u_1, \cdots, u_p) = L^{\frac{7}{2}p_u + \frac{3}{2}p_0 + \frac{5}{2}p_{\rm sp}} M(\mathbb{L}u_0, \mathbb{L}u_1, \cdots, \mathbb{L}u_p)$$

The analog of (C.3) for A is

$$\left\|M^{(s)}\right\|_{\mathfrak{m}} \le L^{-\frac{3}{2}p_u - \frac{7}{2}p_0 - \frac{5}{2}p_{\rm sp}} \left\|M\right\|_{\check{\mathfrak{m}}}$$

and the claim follows. □

D Notation

The references in the following tables are to [7] and this paper.

Notation	Definition	Comments
X	§1.1	spatial lattice
$\mathbf{h}$	§1.1	"kinetic energy" operator
h	§1.1	periodization of $\mathbf{h}$
$h_0 = \theta h$	after (1.4)	periodization of $\mathbf{h}_0$
$\mathbf{h}_0 = \nabla^* \mathbf{H} \nabla$	§1.5	
$L_{\rm sp}$	§1.1	spatial cutoff
$L_{\rm tp} = \frac{1}{\theta kT}$	after (1.2)	temporal cutoff
$\mathcal{X}_0 = \big(\mathbb{Z}/L_{\rm tp}\mathbb{Z}\big) \times \big(\mathbb{Z}^3/L_{\rm sp}\mathbb{Z}^3\big)$	after (1.2)	unit lattice
$\mathcal{X}_n = \big(\frac{1}{L^{2n}}\mathbb{Z}/\frac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\big) \times \big(\frac{1}{L^n}\mathbb{Z}^3/\frac{L_{\rm sp}}{L^n}\mathbb{Z}^3\big)$	Defn. 1.5.a	"fine" scaled lattice
$\mathcal{X}_0^{(n)} = \big(\mathbb{Z}/\frac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\big) \times \big(\mathbb{Z}^3/\frac{L_{\rm sp}}{L^n}\mathbb{Z}^3\big)$	before Defn. 1.1	unit blocked lattice
$\mathcal{X}_{-1}^{(n+1)} = \big(L^2\mathbb{Z}/\frac{L_{\rm tp}}{L^{2n}}\mathbb{Z}\big) \times \big(L\mathbb{Z}^3/\frac{L_{\rm sp}}{L^n}\mathbb{Z}^3\big)$	Defn. 1.1.a	"coarse" blocked lattice
$\mathcal{X}_j^{(n)}$	Defn. 1.5.a	blocked, scaled lattices
L	Theorem 1.17	scaling parameter
$\mathcal{H}_n = L^2(\mathcal{X}_n)$	Defn. 1.5.a	
$\mathcal{H}_0^{(n)} = L^2(\mathcal{X}_0^{(n)})$	Defn. 1.5.a	
$\mathcal{H}_j^{(n)} = L^2(\mathcal{X}_j^{(n)})$	Defn. 1.5.a	
$\langle \alpha_1, \alpha_2 \rangle_j = \int_{X_j^{(n)}} \alpha_1(u)\, \alpha_2(u)\ du$	Defn. 1.5.a	bilinear form for $\mathcal{H}_j^{(n)}$
$\int_{X_j^{(n)}} du = \frac{1}{L^{5j}} \sum_{u \in \mathcal{X}_j^{(n)}}$	Defn. 1.5.a	"integral" over $\mathcal{X}_j^{(n)}$
$\mathbb{L} : \mathcal{X}_j^{(n)} \to \mathcal{X}_{j-1}^{(n)}$	Defn. 1.5.a	$(u_0, \mathbf{u}) \mapsto (L^2 u_0, L\mathbf{u})$
$\mathbb{L}_* : \mathcal{H}_j^{(n)} \to \mathcal{H}_{j-1}^{(n)}$	Defn. 1.5.a	$\mathbb{L}_*(\alpha)(\mathbb{L}u) = \alpha(u)$
$\mathbb{S} = L^{3/2}\, \mathbb{L}_*^{-1} : \mathcal{H}_{j-1}^{(k)} \to \mathcal{H}_j^{(k)}$	Defn. 2.1.a	field scaling operator
$\mathbb{S}_\nu$	Defn. 2.1.a	scales differentiated fields
$Q : \mathcal{H}_0^{(n)} \to \mathcal{H}_{-1}^{(n+1)}$	Defn. 1.1.a	blockspin average
$Q_n : \mathcal{H}_n^{(0)} \to \mathcal{H}_0^{(n)}$	Defn. 1.5.a	blockspin average
$\check{Q}_n$	Lemma 2.4	$\check{Q}_n = \mathbb{S}^{-1} Q_n \mathbb{S} = Q Q_{n-1}$

Notation	Definition	Comments
q	Definition 1.11.d	block spin averaging profile
$\mathcal{A}_0(\psi_*, \psi)$	(1.20)	initial action
$A_0(\psi_*, \psi, \mu, \mathcal{V})$	Definition 1.5.b	dominant part of $\mathcal{A}_0$
$\mathcal{A}_n(\psi_*, \psi)$	Proposition 4.2.a	scale n action
$A_n(\psi_*, \psi, \phi_*, \phi, \mu, \mathcal{V})$	Definition 1.5.b	dominant part of $\mathcal{A}_n$
$D_0 = \mathbb{1} - e^{-h_0} - e^{-h_0}\partial_0$	(1.4)	
$\mathbf{D}_0$	§1.5	D_0 is the periodization of $\mathbf{D}_0$
$D_n = L^{2n}\, \mathbb{L}_*^{-n}\, D_0\, \mathbb{L}_*^n$	Definition 1.5.a	scaled D_0
$\mathbf{v}$	§1.1	original two–body interaction
$\mathcal{V}_0(\psi_*, \psi)$	(1.4), [7, Prop. D.1]	scale zero interaction
V_0	§1.5	kernel of $\mathcal{V}_0$
$\mathbf{V}_0$	§1.5	V_0 is the periodization of $\mathbf{V}_0$
$\mathrm{v}_0 = \sum\limits_{x_2,x_3,x_4} \mathbf{V}_0(0, x_2, x_3, x_4)$	§1.5	
$\mathfrak{v}_0 = 2\Vert\mathbf{V}_0\Vert_{2m}$	§1.5	
$\mathfrak{v}_n$	after (5.1)	$\frac{\mathfrak{v}_0}{L^n} = 2\Vert\mathcal{V}_n^{(u)}\Vert_{2m}$
$\mathcal{V}_n^{(u)}$	Definition 1.5.b	n–fold scaled $\mathcal{V}_0$
$V_n^{(u)}$	Definition 1.5.b	kernel of $\mathcal{V}_n^{(u)}$
$\mathcal{V}_n(\phi_*, \phi)$	Theorem 1.17	scale n interaction
$\mathcal{R}_0(\psi_*, \psi)$	(1.4), [7, Prop. D.1]	
$\mathcal{E}_0(\psi_*, \psi)$	(1.4), [7, Prop. D.1]	
μ	§1.1	original chemical potential
μ_0	(1.4), [7, Prop. D.1]	scale zero chemical potential
μ_*	(1.19)	$\mu_* + \mathfrak{v}_0^{\frac{4}{3}-16\epsilon} \le \mu_0 \le \mathfrak{v}_0^{\frac{8}{9}+\epsilon}$
μ_n	Theorem 1.17	scale n chemical potential
$\mathbb{T}$	Definition 1.1.b	block spin transformation
$a = 1$	Definition 1.1.b	block spin parameter
a_n	(6.8)	$a\big(1 + \sum_{j=1}^{n-1} \frac{1}{L^{2j}}\big)^{-1}$

Notation	Definition	Comments
$\mathfrak{Q}_n$	Definition 1.5.b	$a\big(1+\sum_{j=1}^{n-1}\frac{1}{L^{2j}}Q_jQ_j^*\big)^{-1}$ if $n\ge 2$
$\check{\mathfrak{Q}}_n$	Lemma 2.4	$\check{\mathfrak{Q}}_n=\frac{1}{L^2}\mathbb{S}^{-1}\mathfrak{Q}_n\mathbb{S}$
$N_{\mathbb{T}}^{(n)}$	Definition 1.1.b	normalization constant for $\mathbb{T}$
$\mathbb{T}_n^{(SF)}$	Definition 1.6	small field blockspin transformation
$\tilde{N}_{\mathbb{T}}^{(n)}$	Definition 1.6	normalization constant for $\mathbb{T}_n^{(SF)}$
$\phi_{(*)n}(\psi_*,\psi,\mu,\mathcal{V})$	Proposition 1.14	background fields
$\phi_{(*)n}^{(\ge 3)}(\psi_*,\psi,\mu,\mathcal{V})$	Proposition 1.14	part of $\phi_{(*)n}$ of degree at least 3
$\psi_{(*)n}(\theta_*,\theta,\mu_n,\mathcal{V}_n)$	Proposition 1.15	critical fields
$\psi_{(*)n}^{(\ge 3)}(\theta_*,\theta,\mu_n,\mathcal{V}_n)$	Proposition 1.15	part of $\psi_{(*)n}$ of degree at least 3
$\Delta^{(n)}$	(1.14)	
$C^{(n)}$	(1.15)	covariance
$D^{(n)}$	before (1.15)	square root of $C^{(n)}$
$C^{(n)}(\mu)$	Proposition 1.15	$C^{(n)}(\mu)=\big(\frac{a}{L^2}Q^*Q+\Delta^{(n)}(\mu)\big)^{-1}$
$\Delta^{(n)}(\mu)$	Proposition 1.15	
$\delta\psi_{(*)}$	(1.13)	fluctuation fields
$\delta\psi_{(*)}=D^{(n)(*)}\zeta_{(*)}$	after (1.15)	fluctuation fields
$z(w)=\zeta(\mathbb{L}w)$	before (4.10)	fluctuation field
$\tilde{\alpha}=\big(\alpha,\{\alpha_\nu\}_{\nu=0,1,2,3}\big)$	(1.17)	$\alpha,\alpha_\nu\in\mathcal{H}_j^{(n)}$
$\tilde{\mathcal{H}}_j^{(n)}$	(1.17)	$\big\{\tilde{\alpha}\big\}=\tilde{\mathcal{H}}_j^{(n)\oplus 4}$
$\vec{p}=(p_u,p_0,p_{\rm sp})$	Definition 1.8	monomial type
$\mathfrak{D}$	(1.18)	low degree watch list
$\mathfrak{D}_{\rm rel}$	Definition 1.16	scaling/weight relevant monomial types
$\mathrm{sdf}(\vec{p};C)$	Definition 5.2	Scaling divergence factor
$\mathrm{sdf}(C)$	Definition 5.2	$\sup_{\vec{p}\notin\mathfrak{D}_{\rm rel}}\mathrm{sdf}(\vec{p};C)$
$\Delta(\vec{p})$	Definition 5.2	$\frac{3}{2}p_u+\frac{7}{2}p_0+\frac{5}{2}p_{\rm sp}$ where $\vec{p}=(p_u,p_0,p_{\rm sp})$
$\tilde{\mathcal{R}}_0(\tilde{\psi}_*,\tilde{\psi})$	§1.5	$\mathcal{R}_0(\psi_*,\psi)=\tilde{\mathcal{R}}_0\big((\psi_*,\{\partial_\nu\psi_*\}),(\psi,\{\partial_\nu\psi\})\big)$
$\tilde{\mathcal{R}}_0^{(\vec{p})}$	§1.5	part of $\tilde{\mathcal{R}}_0$ of type $\vec{p}$
$\tilde{\mathcal{R}}_n^{(\vec{p})}(\tilde{\phi}_*,\tilde{\phi})$	Theorem 1.17	polynomial of type $\vec{p}$

Notation	Definition	Comments
$\tilde{\mathcal{R}}_n(\tilde\phi_*,\tilde\phi)$	Thm. 1.17	$\tilde{\mathcal{R}}_n(\tilde\phi_*,\tilde\phi)=\sum_{\vec p\in\mathfrak{D}}\tilde{\mathcal{R}}_n^{(\vec p)}(\tilde\phi_*,\tilde\phi)$
$\mathcal{R}_n(\phi_*,\phi)$	Thm. 1.17	$\mathcal{R}_n(\phi_*,\phi)=\tilde{\mathcal{R}}_n\big((\phi_*,\{\partial_\nu\phi_*\}),(\phi,\{\partial_\nu\phi\})\big)$
$\tilde{\mathcal{E}}_n(\tilde\psi_*,\tilde\psi)$	Thm. 1.17	scaling/weight irrelevant function
$\mathcal{E}_n(\psi_*,\psi)$	Thm. 1.17	$\mathcal{E}_n(\psi_*,\psi)=\tilde{\mathcal{E}}_n\big((\psi_*,\{\partial_\nu\psi_*\}),(\psi,\{\partial_\nu\psi\})\big)$
$\mathcal{Z}_n$	Thm. 1.17	normalization constant
$\tilde{\mathcal{Z}}_n$	(1.6)	$\tilde{\mathcal{Z}}_n=\prod_{j=1}^n L^{3\lvert\mathcal{X}_0^{(j)}\rvert}$
$\mathcal{Z}'_n$	Prop. 5.6	normalization constant
$\lVert f\rVert_{\mathfrak{m}}$	Defn. 1.9	ℓ^1–ℓ^∞ norm with mass $\mathfrak{m}$ of $f:\mathcal{X}\to\mathbb{C}$
	Defn. 1.10	norm with mass $\mathfrak{m}$ and weights $\kappa_1,\cdots,\kappa_s$
$\lvert\lvert\lvert A\rvert\rvert\rvert$	[7, Defn. A.3]	field–map norm of mass $\mathfrak{m}$ and weights κ_j
$\kappa(n)=\frac{L^{\eta n}}{\mathfrak{v}_0^{1/3-\epsilon}}$	Defn. 1.11.a	weight for $\psi_{(*)}$ in the $n^{\rm th}$ step
$\eta=\frac12+\frac13\frac{\log\mathfrak{v}_0}{\log(\mu_0-\mu_*)}$	Defn. 1.11.a	$\frac34+2\epsilon<\eta<\frac78-\frac\epsilon3$
$\kappa'(n)=\frac{L^{\eta' n}}{\mathfrak{v}_0^{1/3-\epsilon}}$	Defn. 1.11.a	weight for $\partial_\nu\psi_{(*)}$ in the $n^{\rm th}$ step
$\eta'=\frac32-\frac{\log\mathfrak{v}_0}{\log(\mu_0-\mu_*)}-\epsilon$	Defn. 1.11.a	$\frac38<\eta'<\frac34-8\epsilon$
$\mathfrak{e}_{\mathfrak{l}}(n)=L^{\eta_{\mathfrak{l}}n}\,\mathfrak{v}_0^{\frac13-2\epsilon}$	Defn. 1.11.a	bound on fluctuation integral of $n^{\rm th}$ step
$\eta_{\mathfrak{l}}=\big(\frac23-4\epsilon\big)\frac{\log\mathfrak{v}_0}{\log(\mu_0-\mu_*)}$	Defn. 1.11.a	
$\bar\kappa$	(5.1)	$\kappa(n+1)$
$\bar\kappa'$	(5.1)	$\kappa'(n+1)$
$\bar\kappa_{\mathfrak{l}}$	(5.1)	$\kappa_{\mathfrak{l}}(n+1)=4r_n$
$\bar\kappa^{\vec p}$	Lemma 5.5.c	$\bar\kappa^{\vec p}=\bar\kappa^{p_u}\bar\kappa'^{p_0+p_{\rm sp}}$ where $\vec p=(p_u,p_0,p_{\rm sp})$
$\lVert\tilde{\mathcal{E}}(\tilde\psi_*,\tilde\psi)\rVert^{(n)}$	Defn. 1.11.a	norm with mass m and weights $\kappa(n)$, $\kappa'(n)$
$\lVert\tilde{\mathcal{E}}(\tilde\psi_*,\tilde\psi)\rVert_m$	Defn. 1.11.a	norm with mass m and weights all one
$n_p\le\log_L\frac{1}{\mathfrak{v}_0^{\frac23-8\epsilon}}$	Defn. 1.11.b	number of steps in the "parabolic flow"
$r_n=\frac14\kappa_{\mathfrak{l}}(n+1)$	Defn. 1.11.c	radius of domain of integration in $n^{\rm th}$ step
$\kappa_{\mathfrak{l}}(n)=\big(\frac{L^n}{\mathfrak{v}_0}\big)^{\epsilon/2}$	Defn. 1.11.c	
$\mathfrak{r}_{\vec p}(n,C)$	Remark 1.18	$\big\lVert\tilde{\mathcal{R}}_n^{(\vec p)}\big\rVert_m\le\mathfrak{r}_{\vec p}(n,C_{\mathcal{R}})$
$\Pi_\ell^n(C)$	Remark 1.18	

Notation	Definition	Comments
$C_{\mathcal{R}}$	Remark 1.18	n, L, independent constant
$C_{\delta\mathcal{V}}$	Remark 1.18	n, L, independent constant
$C_{\mathfrak{l}}$	Lemma 5.5	n, L, independent constant
C_{ren}	Lemma 6.6	n, L, independent constant
Γ_{op}	Convention 1.2	n, L, independent constant
μ_{up}	Convention 1.2	n, L, independent constant
$K_1, K_2, \cdots$	Convention 1.2	n, L, independent constants
K_{bg}	Convention 1.2	$\max_j K_j$
$\rho_1, \rho_2, \cdots$	Convention 1.2	n, L, independent constants
ρ_{bg}	Convention 1.2	$\min\left\{\frac{1}{8}, \min_j \rho_j\right\}$
$\Lambda_{\delta\mu}$	Lemma 6.7	n independent, L dependent constant
c_{loc}	Lemma B.3	n, L, independent constant
c_A	(6.7)	n, L, independent constant
c_Ω	Lemma 6.3	n, L, independent constant
$c_{\delta\mathcal{V}}$	Lemma 6.4	n, L, independent constant
K_Φ	(6.11), (6.12)	n, L, independent constant
c_{gar}	Lemma 6.5	n, L, independent constant
c_{μ_*}	Lemma A.1	n, L, independent constant
$S_n = (D_n + Q_n^* \mathfrak{Q}_n Q_n)^{-1}$	Theorem 1.13	Green's functions
$S_n(\mu)$	Theorem 1.13	$S_n(\mu) = (D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu)^{-1}$
$e_{\mathrm{r}},\ e_{\mathrm{R}},\ e_\mu,\ e'_{\mathrm{R}}$	before [7, (D.1)]	parameters in [2, Hypothesis 2.14]
$\mathcal{Z}_\theta$	[7, (D.2)]	normalization constant
$\mathcal{Z}_{\mathrm{in}}$	[7, Prop. D.1]	$\mathcal{Z}_{\mathrm{in}} = \mathcal{Z}_\theta e^{-\theta\mu}$
$j(t) = e^{-t(h-\mu)}$	[7, (D.2)]	
$V_\theta(\alpha^*, \beta)$	[7, (D.2)]	interaction output from [2]
$\mathcal{R}_\theta(\alpha_*, \beta)$	[7, (D.2)]	degree two output from [2]
$\mathcal{E}_\theta(\alpha_*, \beta)$	[7, (D.2)]	higher degree output from [2]

Notation	Definition	Comments
$\mathcal{D}_\theta(\alpha_*, \beta)$	[7, (D.2)]	$\mathcal{D}_\theta(\alpha_*, \beta) = \mathcal{R}_\theta(\alpha_*, \beta) + \mathcal{E}_\theta(\alpha_*, \beta)$
$\check{A}_n(\theta_*, \theta, \check{\phi}_*, \check{\phi}, \mu, \mathcal{V})$	Definition 2.3	$A_n(\mathbb{S}\theta_*, \mathbb{S}\theta, \mathbb{S}\check{\phi}_*, \mathbb{S}\check{\phi}, L^2\mu, \mathbb{S}\mathcal{V})$
$\check{\phi}_{(*)n}(\theta_*, \theta, \mu, \mathcal{V})$	Definition 3.2	$\mathbb{S}^{-1}\big[\phi_{(*)n}(\mathbb{S}\theta_*, \mathbb{S}\theta, L^2\mu, \mathbb{S}\mathcal{V})\big]$
$\delta\phi_{(*)n}\big(\psi_*, \psi, \delta\psi_*, \delta\psi, \mu, \mathcal{V}\big)$	Definition 3.5	
$\delta\check{\phi}_{(*)n}\big(\theta_*, \theta, \delta\psi_*, \delta\psi, \mu, \mathcal{V}\big)$	Definition 3.5	
$\delta\check{\phi}^{(+)}_{(*)n}\big(\theta_*, \theta; \delta\psi_*, \delta\psi, \mu, \mathcal{V}\big)$	Definition 3.5	
$\hat{\psi}_{(*)n}(\psi_*, \psi, \mu, \mathcal{V})$	(4.3)	$\mathbb{S}\big[\psi_{*n}(\mathbb{S}^{-1}\psi_*, \mathbb{S}^{-1}\psi, \mu, \mathcal{V})\big]$
$\delta\hat{\phi}_{(*)n+1}(\psi_*, \psi, z_*, z)$	(4.7)	
$\delta\hat{\phi}^{(+)}_{(*)n+1}(\psi_*, \psi, z_*, z)$	(4.9)	
$\check{\mathcal{C}}_n(\theta_*, \theta)$	beginning §4	
$\check{\mathcal{F}}_n(\theta_*, \theta)$	beginning §4	
$\check{\mathcal{E}}_{n+1,1}(\theta_*, \theta)$	beginning §4	$\mathcal{E}_n\big(\psi_{*n}(\theta_*, \theta, \mu_n, \mathcal{V}_n), \psi_n(\theta_*, \theta, \mu_n, \mathcal{V}_n)\big)$
$\delta\check{\mathcal{E}}_n(\theta_*, \theta, \delta\psi_*, \delta\psi)$	beginning §4	
$\delta\check{\mathcal{R}}_n(\theta_*, \theta, \delta\psi_*, \delta\psi)$	beginning §4	
$\delta\check{A}_n(\theta_*, \theta, \delta\psi_*, \delta\psi)$	beginning §4	
$\mathcal{C}_n(\psi_*, \psi)$	before (4.3)	
$\mathcal{F}_n(\psi_*, \psi)$	(4.4)	Also see Proposition 5.6
$\mathcal{E}_{n+1,1}(\psi_*, \psi)$	(4.3)	$(\mathbb{S}\mathcal{E}_n)\big(\hat{\psi}_{(*)n}(\psi_*, \psi, \mu_n, \mathcal{V}_n)\big)$
$\delta\mathcal{E}_n(\psi_*, \psi, z_*, z)$	(4.5)	
$\delta\mathcal{R}_n(\psi_*, \psi, z_*, z)$	(4.6)	
$\delta A_n(\psi_*, \psi, z_*, z)$	(4.8)	
$\delta A_n^{(2)},\ \delta A_n^{(\ge 3)}$	Lemma 5.5.a	
$\tilde{\mathcal{E}}_{n+1,1}(\tilde{\psi}_*, \tilde{\psi})$	Lemma 5.5.b	$\mathcal{E}_{n+1,1}(\psi_*, \psi) = \tilde{\mathcal{E}}_{n+1,1}\big((\psi_{(*)}, \{\partial_\nu\psi_{(*)}\})\big)$
$\tilde{\mathcal{E}}_{n+1,2}(\tilde{\psi}_*, \tilde{\psi})$	Lemma 6.6	$\tilde{\mathcal{E}}_{n+1} = \tilde{\mathcal{E}}_{n+1,1} + \tilde{\mathcal{E}}_{n+1,2}$
$\delta\tilde{\mathcal{E}}_n(\tilde{\psi}_*, \tilde{\psi}, z_*, z)$	Lemma 5.5.b	
$\delta\tilde{\mathcal{R}}_n^{(\vec{p})}(\tilde{\psi}_*, \tilde{\psi}, z_*, z)$	Lemma 5.5.c	
$\delta\tilde{\mathcal{R}}_n(\psi_*, \psi, z_*, z)$	Lemma 5.5.c	

Notation	Definition	Comments
$\sigma_n(\vec{p})$	Lemma 5.5.c	
$\tilde{\mathcal{E}}_{\mathfrak{l}}(\tilde{\psi}_*, \tilde{\psi})$	Proposition 5.6	
$\tilde{\mathcal{D}}(\tilde{\psi}_*, \tilde{\psi}, z_*, z)$	before Lemma 5.7	
P_2^ψ	before Lemma 5.7	degree 1 in each of ψ_*, ψ, any degree in $z_{(*)}$
P_1^ψ	before Lemma 5.7	extracts degree 1 in $\psi_{(*)}$, any degree in $z_{(*)}$
P_0^ψ	before Lemma 5.7	degree 0 in $\psi_{(*)}$, $\psi_{(*)\nu}$, any degree in $z_{(*)}$
$\mathcal{M}_n$	Lemma 5.7.a	
$P_{\psi_*\psi}$	Proposition 5.8.a	degree 1 in each of ψ_*, ψ, degree 0 in $\psi_{(*)\nu}$
M'_n	Proposition 5.8.a	
M_n	Proposition 5.8.b	

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