Operators for Parabolic Block Spin Transformations

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Abstract

This paper is a contribution to a program to see symmetry breaking in a weakly interacting many Boson system on a three dimensional lattice at low temperature. It is part of an analysis of the "small field" approximation to the "parabolic flow" which exhibits the formation of a "Mexican hat" potential well. Bounds on the fluctuation integral covariance, as well as on some other linear operators, are an important ingredient in the renormalization group step analysis of [7, 8]. These bounds are proven here.

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1 Introduction

In [7, 8], we exhibit, for a many particle system of weakly interacting Bosons in three space dimensions, the formation of a potential well of the type that typically leads to symmetry breaking in the thermodynamic limit. To do so, we use the block spin renormalization group approach. In previous papers [4, 1, 2, 3] (followed by a simple change of variables) we have written the partition function of such a system on a discrete torus[1] in terms of a functional integral on a $1+3$ dimensional space

$$\mathcal{X}_0 = \big(\mathbb{Z}/L_{\rm tp}\mathbb{Z}\big) \times \big(\mathbb{Z}^3/L_{\rm sp}\mathbb{Z}^3\big)$$

with positive integers $L_{\rm tp}$, $L_{\rm sp}$. Up to corrections which are exponentially small in the coupling constant, and up to a multiplicative normalization factor, this representation is of the form

$$\int \Big[\prod_{x\in\mathcal{X}_0} \tfrac{d\psi(x)^*\wedge d\psi(x)}{2\pi\imath} \Big] \, e^{\mathcal{A}_0(\psi^*,\psi)} \chi_0(\psi) \tag{1.1}$$

with an action $\mathcal{A}_0$ of the form

$$\mathcal{A}_0(\psi_*,\psi) = -\left\langle \psi_*,\, D_0\psi\right\rangle_0 - \mathcal{V}_0(\psi_*,\psi) + \mu_0 \left\langle \psi_*,\, \psi\right\rangle_0 + \mathcal{E}_0'(\psi_*,\psi) \tag{1.2}$$

Here

- $D_0 = \mathbb{1} - e^{-h_0} - e^{-h_0}\partial_0$, where ∂_0 the forward time derivative (see (2.9) below and h_0 is – up to a scaling – the single particle Hamiltonian.
- $\mathcal{V}_0(\psi_*,\psi)$ is a quartic monomial that describes the coupling between the particles
- μ_0 is related to the chemical potential of the system
- $\mathcal{E}_0'(\psi_*,\psi)$ is perturbatively small
- $\chi_0(\psi)$ is a "small field cut off function".

See [7, (1.3), (1.4)].

For the block spin renormalization group action, we pick a "block rectangle" of length L^2 in the "time direction" and L in "space directions", where L is a sufficiently large odd positive integer, and a corresponding nonnegative, compactly supported function $q(x)$ on $\mathbb{Z}\times\mathbb{Z}^3$ (the averaging profile). The choice of this kind of rectangle is characteristic of "parabolic scaling". See [7, Definition 1.3, Remark 1.4, Definition 1.11.d]. For simplicity we assume that $L_{\rm sp}$ and $L_{\rm tp}$ are powers of L.

The block spin averaging operator, which we denote Q, maps functions on the lattice $\mathcal{X}_0$ to functions on the "coarse" lattice $\mathcal{X}_{-1}^{(1)} = \big(L^2\mathbb{Z}/L_{\rm tp}\mathbb{Z}\big) \times \big(L\mathbb{Z}^3/L_{\rm sp}\mathbb{Z}^3\big)$.

[1] All bounds achieved so far are uniform in the volume of this torus.

After each renormalization group step we scale, to again give functions on a unit lattice. After the first RG step this unit lattice is $\mathcal{X}_0^{(1)} = \big(\mathbb{Z}/\tfrac{1}{L^2}L_{\rm tp}\mathbb{Z}\big)\times\big(\mathbb{Z}^3/\tfrac{1}{L}L_{\rm sp}\mathbb{Z}^3\big)$. The "scaled" block spin averaging operator maps functions on the lattice $\mathcal{X}_1 = \big(\tfrac{1}{L^2}\mathbb{Z}/\tfrac{1}{L^2}L_{\rm tp}\mathbb{Z}\big)\times\big(\tfrac{1}{L}\mathbb{Z}^3/\tfrac{1}{L}L_{\rm sp}\mathbb{Z}^3\big)$ to functions on the unit lattice $\mathcal{X}_0^{(1)}$.

In the $n^{\rm th}$ renormalization group step, we end up considering functions on the chain of lattices

$$\mathcal{X}_{-1}^{(n+1)} \subset \mathcal{X}_0^{(n)} \subset \mathcal{X}_1^{(n-1)} \subset \cdots \subset \mathcal{X}_{n-1}^{(1)} \subset \mathcal{X}_n^{(0)}$$

where, for integers $j \ge -1$ and $n \ge 0$,

$$\mathcal{X}_j^{(n)} = \big(\varepsilon_j^2\mathbb{Z}/\varepsilon_{n+j}^2 L_{\rm tp}\mathbb{Z}\big)\times\big(\varepsilon_j\mathbb{Z}^3/\varepsilon_{n+j}L_{\rm sp}\mathbb{Z}^3\big) \qquad \text{with } \varepsilon_j = \tfrac{1}{L^j}$$

The subscript in $\mathcal{X}_j^{(n)}$ determines the "coarseness" of the lattice — nearest neighbour points are a distance $\varepsilon_{2j} = \frac{1}{L^{2j}}$ apart in the time direction and a distance $\varepsilon_j = \frac{1}{L^j}$ apart in spatial directions. The superscript in $\mathcal{X}_j^{(n)}$ determines the number of points in the lattice — $|\mathcal{X}_j^{(n)}| = |\mathcal{X}_0|/L^{5n}$ for all j. We usually write $\mathcal{X}_n^{(0)} = \mathcal{X}_n$. See [7, Definition 1.5.a or Appendix A.1].

The $(n+1)^{\rm st}$ block spin transformation involves the passage from $\mathcal{X}_0^{(n)}$ to its sublattice $\mathcal{X}_{-1}^{(n+1)}$. The averaging operations determine linear maps

$$Q : \mathcal{H}_0^{(n)} \mapsto \mathcal{H}_{-1}^{(n+1)} \qquad \text{and} \qquad Q_n : \mathcal{H}_n = \mathcal{H}_n^{(0)} \mapsto \mathcal{H}_0^{(n)}$$

where $\mathcal{H}_j^{(n)} = L^2\big(\mathcal{X}_j^{(n)}\big)$ denotes the (finite dimensional) Hilbert space of functions on $\mathcal{X}_j^{(n)}$ with integral $\int_{X_j^{(n)}} du = \varepsilon_j^5 \sum_{u\in\mathcal{X}_j^{(n)}}$ and the *real* inner product

$$\langle \alpha_1, \alpha_2\rangle_j = \int_{X_j^{(n)}} \alpha_1(u)\,\alpha_2(u)\; du$$

Again see [7, Definition 1.5.a or Appendix A.3]. In §2 we pick a specific averaging profile and give bounds on the operators Q, Q_n, their Fourier transforms, and related operators.

Scaling is performed by the linear isomorphisms

$$\mathbb{L} : \mathcal{X}_j^{(n)} \to \mathcal{X}_{j-1}^{(n)} \qquad\qquad (u_0, \mathbf{u}) \mapsto (L^2 u_0, L\mathbf{u})$$

For a function $\alpha \in \mathcal{H}_j^{(n)}$, define the function $\mathbb{L}_*(\alpha) \in \mathcal{H}_{j-1}^{(n)}$ by $\mathbb{L}_*(\alpha)(\mathbb{L}u) = \alpha(u)$. See [7, Appendix A.2]. In particular, after rescaling and multiplication with the "scaling factor" L^{2n}, the differential operator D_0 in (1.2) becomes the operator

$$D_n = L^{2n}\, \mathbb{L}_*^{-n}\big(\mathbb{1} - e^{-h_0} - e^{-h_0}\partial_0\big)\mathbb{L}_*^n$$

on $\mathcal{H}_n$. This operator is discussed in §3.

As mentioned above, the passage from a functional integral on $\mathcal{X}_0^{(n)}$ to a functional integral on $\mathcal{X}_{-1}^{(n+1)}$ is an averaging procedure over, roughly speaking, a rectangle of size L^2 in the time direction and size L in the spatial directions. This passage is analyzed using stationary phase techniques that involve

- the determination of critical fields on $\mathcal{X}_0^{(n)}$ (that are functions of external fields on $\mathcal{X}_{-1}^{(n+1)}$) for an appropriate action, and
- a functional integral over "fluctuation fields" around the critical field.

The covariance for the integral over the fluctuation fields has been identified in [7, (1.15)] and is bounded in §4.

The composition of the critical fields of n renormalization group steps is – after rescaling – a field on $\mathcal{X}_n$, called the "background field", that is a function of an external field on $\mathcal{X}_0^{(n)}$. It is crucial in our representation of the partition function. See [7, Theorem 1.17]. The "leading order" part of the background field is linear in the external field and has been identified in [7, Proposition 1.14]. It is the composition of an operator, from $\mathcal{H}_0^{(n)}$ to $\mathcal{H}_n$ determined by the averaging profile q, and an operator S_n on $\mathcal{H}_n$ which can be viewed as a Green's function for the differential operator D_n (plus a mass term). This operator, S_n, is discussed in §5.

To get bounds on the critical fields in the fluctuation integral at step $n+1$, we use a well known algebraic relation between these critical fields and the background fields at step $n+1$ given in [6, Proposition 9] and [7, Proposition 3.4.a]. The operators in the linearization of this relation, and various other linearizations, are studied in §6.

By construction, many of the operators discussed in this paper are linear operators defined on the Hilbert space of functions on a lattice that are invariant under translations with respect to a sublattice. It is natural to use Bloch/Floquet decompositions and Fourier transforms for an analysis of such operators. We use the abstract basic results about such decompositions given in [5].

Most operator estimates we obtain in this paper are with respect to a norm of the following kind.

Definition 1.1. For any operator $A : \mathcal{H}_j^{(n-j)} \to \mathcal{H}_k^{(n-k)}$, with kernel $A(u,u')$, and for any mass $m \ge 0$, we define the norm

$$\|A\|_m = \max\Big\{ \sup_{u\in\mathcal{X}_k^{(n-k)}} \int_{\mathcal{X}_j^{(n-j)}} du'\ e^{m|u-u'|}|A(u,u')|\ ,\ \sup_{u'\in\mathcal{X}_j^{(n-j)}} \int_{\mathcal{X}_k^{(n-k)}} du\ e^{m|u-u'|}|A(u,u')| \Big\}$$

In the special case that $m=0$, this is just the usual ℓ^1–ℓ^∞ norm of the kernel.

As we point out in [5, Lemmas 12 and 13] this norm is related to the analyticity properties of the Fourier transform. In this paper we use the following Fourier transform conventions.
The dual lattice of $\mathcal{X}_j^{(n)}$ is

$$\hat{\mathcal{X}}_j^{(n)} = \big(\tfrac{2\pi}{\varepsilon_{n+j}^2 L_{\rm tp}}\mathbb{Z}/\tfrac{2\pi}{\varepsilon_j^2}\mathbb{Z}\big)\times\big(\tfrac{2\pi}{\varepsilon_{n+j} L_{\rm sp}}\mathbb{Z}^3/\tfrac{2\pi}{\varepsilon_j}\mathbb{Z}^3\big)$$

For a function $\alpha \in \mathcal{H}_j^{(n)}$

$$\hat{\alpha}(p) = \int_{\mathcal{X}_j^{(n)}} \alpha(u) e^{-ip\cdot u}\, du \qquad \alpha(u) = \int_{\hat{\mathcal{X}}_j^{(n)}} \hat{\alpha}(p) e^{iu\cdot p}\, \tfrac{dp}{(2\pi)^4}$$

where $\int_{\hat{\mathcal{X}}_j^{(n)}} \frac{dp}{(2\pi)^4} = \frac{1}{\varepsilon_{n+j}^5 L_{\rm tp} L_{\rm sp}^3} \sum_{p\in\hat{\mathcal{X}}_j^{(n)}}$. The maps

$$\mathbb{L} : \hat{\mathcal{X}}_{j-1}^{(n)} \to \hat{\mathcal{X}}_j^{(n)} \qquad\qquad (q_0, \mathbf{q}) \mapsto (L^2 q_0, L\mathbf{q})$$

are again linear isomorphisms, and, for a function $\alpha \in \mathcal{H}_j^{(n)}$,

$$\widehat{\mathbb{L}_*(\alpha)}(q) = L^5 \hat{\alpha}(\mathbb{L}q) \tag{1.3}$$

The quotient map dual to the inclusion $\mathcal{X}_j^{(n)} \subset \mathcal{X}_{j+k}^{(n-k)}$ is

$$\hat{\pi}_{n+j}^{(j+k,j)} : \hat{\mathcal{X}}_{j+k}^{(n-k)} \to \hat{\mathcal{X}}_j^{(n)} \tag{1.4}$$

When the indices are clear from the context we suppress them and write $\hat{\pi}$.

The estimates of this paper are used in [7, 8]. In particular, the construction of the background fields and the critical fields in [9] uses a contraction mapping argument around the linearizations of §6 and §5 in this paper.

For the readers' convenience we have included, in Appendix B, a list of most of the operators and lattices that appear in this paper.

Convention 1.2. Most estimates in this paper are bounds on norms of operators as in Definition 1.1. The (finite number of) constants that appear in these bounds are consecutively labelled $\Gamma_1, \Gamma_2, \cdots, \gamma_1, \gamma_2, \cdots, m_1, m_2, \cdots$. All of these constants Γ_j, γ_j, m_j are independent of L and the scale index n. We define $\Gamma_{\rm op}$ to be the maximum of the Γ_j's, and, in [7, 8, 9], refer to the estimates using only this constant $\Gamma_{\rm op}$.

2 Block Spin Operators

In this chapter, we analyze the block spin "averaging operators" Q of [7, Definitions 1.1.a and 1.11.d] and Q_n of [7, Definition 1.11.d] as well as the operator $\mathfrak{Q}_n$ of [7, Definition 1.5.b]. Recall that $Q : \mathcal{H}_0^{(n)} \to \mathcal{H}_{-1}^{(n+1)}$ is defined by

$$(Q\psi)(y) = \sum_{x\in\mathbb{Z}\times\mathbb{Z}^3} q(x)\psi(y+[x]) \tag{2.1}$$

where $[x]$ denotes the class of $x \in \mathbb{Z}\times\mathbb{Z}^3$ in the quotient space $\mathcal{X}_0^{(n)}$. The averaging profile q is the $\mathfrak{q}$–fold convolution of the characteristic function, $1_{\square}(x)$, of the rectangle $\big[-\frac{L^2-1}{2}, \frac{L^2-1}{2}\big] \times \big[-\frac{L-1}{2}, \frac{L-1}{2}\big]^3$, normalized to have integral one. That is,

$$q = \tfrac{1}{L^{5\mathfrak{q}}} \overbrace{1_{\square} * 1_{\square} * \cdots * 1_{\square}}^{\mathfrak{q} \text{ times}}$$

See [5, Example 8 and Remark 11]. Except where otherwise stated, we shall assume that $\mathfrak{q} \ge 4$ is a fixed even natural number.[2]

The operator

$$Q_n = Q^{(1)} \cdots Q^{(n)} = \big(\mathbb{L}_*^{-1} Q\big)^n \mathbb{L}_*^n : \mathcal{H}_n = \mathcal{H}_n^{(0)} \to \mathcal{H}_0^{(n)} \tag{2.2}$$

where $Q^{(j)} = \mathbb{L}_*^{-j}\, Q\, \mathbb{L}_*^j : \mathcal{H}_j^{(n-j)} \to \mathcal{H}_{j-1}^{(n-j+1)}$. The operator

$$\mathfrak{Q}_n = a\Big(\mathbb{1} + \sum_{j=1}^{n-1} \tfrac{1}{L^{2j}} Q_j Q_j^*\Big)^{-1}$$

The Fourier transform of the characteristic function $1_{\square}$ is $\frac{\sigma(\mathbb{L}k)}{\sigma(k)}$ with $k \in \hat{\mathcal{X}}_0^{(n)}$ and with

$$\sigma(k) = \sin\big(\tfrac{1}{2}k_0\big) \prod_{\nu=1}^{3} \sin\big(\tfrac{1}{2}\mathbf{k}_\nu\big) \tag{2.3}$$

Therefore

$$\hat{q}(k) = u_+(k)^{\mathfrak{q}} \qquad \text{with} \qquad u_+(k) = \frac{\sigma(\mathbb{L}k)}{L^5\sigma(k)} \tag{2.4}$$

and, by [5, Lemma 9.a]

$$\widehat{(Q\psi)}(\mathfrak{k}) = \sum_{\substack{k\in\hat{\mathcal{X}}_0^{(n)} \\ \hat{\pi}(k)=\mathfrak{k}}} \hat{q}(k)\hat{\psi}(k) \tag{2.5}$$

for all $\psi \in \mathcal{H}_0^{(n)}$ and $\mathfrak{k} \in \hat{\mathcal{X}}_{-1}^{(n+1)}$.

[2] See Remark 2.7 for a discussion of the condition $\mathfrak{q} > 2$. The condition that $\mathfrak{q}$ be even is imposed purely for convenience.

Remark 2.1.

(a) Since $\mathfrak{q}$ is even, $\sigma(k)^{\mathfrak{q}}$ is an entire function of $k \in \mathbb{C} \times \mathbb{C}^3$ that is periodic with respect to the lattice $2\pi(\mathbb{Z} \times \mathbb{Z}^3)$. Also

$$\sigma(p_j)^{\mathfrak{q}} = \sigma\big(\hat{\pi}_n^{(j,0)}(p_j)\big)^{\mathfrak{q}} \qquad \text{for all } p_j \in \hat{\mathcal{X}}_j^{(n-j)}$$

(b) For all $\phi \in \mathcal{H}_n$ and $k \in \hat{\mathcal{X}}_0^{(n)}$,

$$\widehat{(Q_n\phi)}(k) = \sum_{\substack{p \in \hat{\mathcal{X}}_n \\ \hat{\pi}(p)=k}} u_n(p)^{\mathfrak{q}} \hat{\phi}(p) \qquad \text{with} \qquad u_n(p) = \varepsilon_n^5 \frac{\sigma(p)}{\sigma(\mathbb{L}^{-n}p)}$$

(c) For all $\psi \in \mathcal{H}_0^{(n)}$ and $k \in \hat{\mathcal{X}}_0^{(n)}$, $\widehat{\mathfrak{Q}_n\psi}(k) = \hat{\mathfrak{Q}}_n(k)\hat{\psi}(k)$ where

$$\hat{\mathfrak{Q}}_n(k) = a\Bigg[1 + \sum_{j=1}^{n-1} \sum_{\substack{p_j \in \hat{\mathcal{X}}_j^{(n-j)} \\ \hat{\pi}(p_j)=k}} \tfrac{1}{L^{2j}} u_j(p_j)^{2\mathfrak{q}}\Bigg]^{-1}$$

(d) The functions $u_n(p)$ and $u_+(p)$ are entire in p and are invariant under $p_\nu \to -p_\nu$ for each $0 \le \nu \le 3$ and under $p_\nu \leftrightarrow p_{\nu'}$ for all $1 \le \nu, \nu' \le 3$.

(e) Set, with the notation of (1.4), the “single period” lattices and their duals

$$\begin{aligned} \mathcal{B}^+ &= \big(\mathbb{Z}/L^2\mathbb{Z}\big) \times \big(\mathbb{Z}^3/L\mathbb{Z}^3\big) & \hat{\mathcal{B}}^+ &= \big(\tfrac{2\pi}{L^2}\mathbb{Z}/2\pi\mathbb{Z}\big) \times \big(\tfrac{2\pi}{L}\mathbb{Z}^3/2\pi\mathbb{Z}^3\big) = \ker \hat{\pi}_{n-1}^{(0,-1)} \\ \mathcal{B}_j &= \big(\varepsilon_j^2\mathbb{Z}/\mathbb{Z}\big) \times \big(\varepsilon_j\mathbb{Z}^3/\mathbb{Z}^3\big) & \hat{\mathcal{B}}_j &= \big(2\pi\mathbb{Z}/\tfrac{2\pi}{\varepsilon_j^2}\mathbb{Z}\big) \times \big(2\pi\mathbb{Z}^3/\tfrac{2\pi}{\varepsilon_j}\mathbb{Z}^3\big) = \ker \hat{\pi}_n^{(j,0)} \end{aligned}$$

for each integer $j \ge 0$. In this notation, the representations of Q, Q_n and $\mathfrak{Q}_n$ of (2.5) and parts (b) and (c) are

$$\begin{aligned} \widehat{(Q\psi)}(\mathfrak{k}) &= \sum_{\ell \in \hat{\mathcal{B}}^+} u_+(\mathfrak{k}+\ell)^{\mathfrak{q}} \hat{\psi}(\mathfrak{k}+\ell) \\ \widehat{(Q_n\phi)}(k) &= \sum_{\ell \in \hat{\mathcal{B}}_n} u_n(k+\ell)^{\mathfrak{q}} \hat{\phi}(k+\ell) \\ \hat{\mathfrak{Q}}_n(k) &= a\Bigg[1 + \sum_{j=1}^{n-1} \sum_{\ell_j \in \hat{\mathcal{B}}_j} \tfrac{1}{L^{2j}} u_j(k+\ell_j)^{2\mathfrak{q}}\Bigg]^{-1} \end{aligned}$$

Here in $\widehat{(Q\psi)}(\mathfrak{k}) = \sum_{\ell\in\hat{\mathcal{B}}^+} u_+(\mathfrak{k}+\ell)^{\mathfrak{q}}\hat{\psi}(\mathfrak{k}+\ell)$, for example, $\mathfrak{k}\in\hat{\mathcal{X}}_{-1}^{(n+1)}$ is represented by the element of $\frac{2\pi}{\varepsilon_n^2 L_{\rm tp}}\mathbb{Z}\times\frac{2\pi}{\varepsilon_n L_{\rm sp}}\mathbb{Z}^3$ having minimal components and ℓ is represented by the element of $\frac{2\pi}{\varepsilon_{-1}^2}\mathbb{Z}\times\frac{2\pi}{\varepsilon_{-1}}\mathbb{Z}^3$ having minimal components. Similarly

$$\big(\widehat{Q^*\theta}\big)(\mathfrak{k}+\ell) = u_+(\mathfrak{k}+\ell)^{\mathfrak{q}}\hat{\theta}(\mathfrak{k})$$
$$\big(\widehat{Q_n^*\psi}\big)(k+\ell_n) = u_n(k+\ell_n)^{\mathfrak{q}}\hat{\psi}(k)$$

Proof. (a) Any two points of $\hat{\mathcal{X}}_j^{(n-j)}$ with the same image in $\hat{\mathcal{X}}_0^{(n)}$ under $\hat{\pi}_n^{(j,0)}$ differ by 2π times an integer vector. The formula follows.

(b) By (1.3) and (2.5), we have, for $\alpha\in\mathcal{H}_j^{(n-j)}$ and $p_{j-1}\in\hat{\mathcal{X}}_{j-1}^{(n-j+1)}$

$$\begin{aligned}
\widehat{(Q^{(j)}\alpha)}(p_{j-1}) &= \tfrac{1}{L^{5j}}\widehat{(QL_*^j\alpha)}(\mathbb{L}^{-j}p_{j-1}) = \tfrac{1}{L^{5j}}\sum_{\substack{k\in\hat{\mathcal{X}}_0^{(n-j)}\\ \hat{\pi}(k)=\mathbb{L}^{-j}p_{j-1}}}\hat{q}(k)\widehat{(\mathbb{L}_*^j\alpha)}(k)\\
&= \frac{1}{L^{5\mathfrak{q}}}\sum_{\substack{k\in\hat{\mathcal{X}}_0^{(n-j)}\\ \hat{\pi}(k)=\mathbb{L}^{-j}p_{j-1}}}\frac{\sigma(\mathbb{L}k)^{\mathfrak{q}}}{\sigma(k)^{\mathfrak{q}}}\hat{\alpha}\big(\mathbb{L}^j k\big) = \frac{1}{L^{5\mathfrak{q}}}\sum_{\substack{p_j\in\hat{\mathcal{X}}_j^{(n-j)}\\ \hat{\pi}(p_j)=p_{j-1}}}\frac{\sigma(\mathbb{L}^{-j+1}p_j)^{\mathfrak{q}}}{\sigma(\mathbb{L}^{-j}p_j)^{\mathfrak{q}}}\hat{\alpha}(p_j)
\end{aligned}$$

so that, by part (a),

$$\begin{aligned}
\widehat{(Q_n\phi)}(p_0) &= \frac{1}{L^{5\mathfrak{q}n}}\sum_{\substack{p_j\in\hat{\mathcal{X}}_j^{(n-j)}\\ \hat{\pi}(p_j)=p_{j-1}\\ 1\le j\le n}}\frac{\sigma(p_1)^{\mathfrak{q}}}{\sigma(\mathbb{L}^{-1}p_1)^{\mathfrak{q}}}\,\frac{\sigma(\mathbb{L}^{-1}p_2)^{\mathfrak{q}}}{\sigma(\mathbb{L}^{-2}p_2)^{\mathfrak{q}}}\cdots\frac{\sigma(\mathbb{L}^{-n+1}p_n)^{\mathfrak{q}}}{\sigma(\mathbb{L}^{-n}p_n)^{\mathfrak{q}}}\;\hat{\phi}(p_n)\\
&= \varepsilon_n^{5\mathfrak{q}}\sum_{\substack{p_j\in\hat{\mathcal{X}}_j^{(n-j)}\\ \hat{\pi}(p_j)=p_{j-1}\\ 1\le j\le n}}\frac{\sigma(p_n)^{\mathfrak{q}}}{\sigma(\mathbb{L}^{-1}p_n)^{\mathfrak{q}}}\,\frac{\sigma(\mathbb{L}^{-1}p_n)^{\mathfrak{q}}}{\sigma(\mathbb{L}^{-2}p_n)^{\mathfrak{q}}}\cdots\frac{\sigma(\mathbb{L}^{-n+1}p_n)^{\mathfrak{q}}}{\sigma(\mathbb{L}^{-n}p_n)^{\mathfrak{q}}}\;\hat{\phi}(p_n)\\
&= \varepsilon_n^{5\mathfrak{q}}\sum_{\substack{p_n\in\hat{\mathcal{X}}_n\\ \hat{\pi}(p_n)=p_0}}\frac{\sigma(p_n)^{\mathfrak{q}}}{\sigma(\mathbb{L}^{-n}p_n)^{\mathfrak{q}}}\;\hat{\phi}(p_n)
\end{aligned}$$

(c) follows from part (b) and [5, Lemma 9.a].

(d) is obvious since $\frac{\sin z}{\sin\frac{z}{m}}$ is even and entire for any nonzero integer m. $\square$

In Lemmas 2.2 and 2.3 we derive a number of bounds on the kernels u_n and u_+ that appear in the representations for Q_n and Q of Remark 2.1.e. In Proposition 2.4 we analyze the operator $\hat{\mathfrak{Q}}_n$. Then in Remark 2.5 and Lemma 2.6 we study how to move derivatives past Q and Q_n.

When dealing with the asymmetry between "temporal" and "spatial" scaling we set, for convenience,

$$L_\nu = \begin{Bmatrix} L^2 & \text{for } \nu = 0 \\ L & \text{for } \nu = 1,2,3 \end{Bmatrix} \qquad \varepsilon_{n,\nu} = \begin{Bmatrix} \varepsilon_n^2 = \frac{1}{L_0^n} = \frac{1}{L^{2n}} & \text{for } \nu = 0 \\ \varepsilon_n = \frac{1}{L_\nu^n} = \frac{1}{L^n} & \text{for } \nu = 1,2,3 \end{Bmatrix} \tag{2.6}$$

Lemma 2.2. *Let* $\mathfrak{q} \in \mathbb{N}$*. Assume that* $|\mathrm{Re}\, k_\nu| \le \pi$, $|\mathrm{Im}\, k_\nu| \le 2$ *for each* $0 \le \nu \le 3$*.*

(a) $\big|u_n(k+\ell)\big| \le \prod_{\nu=0}^3 \frac{24}{|\ell_\nu|+\pi}$ *for all* $\ell \in \hat{\mathcal{B}}_n$*. We use* $|\ell_\nu|$ *to denote the magnitude of the smallest representative of* ℓ_ν *in its equivalence class, as an element of* $\hat{\mathcal{B}}_n$*. There is a constant* Γ_1*, depending only on* $\mathfrak{q}$*, such that* $\|Q_n\|_{m=1} \le \Gamma_1$*.*

(b) $\big|u_n(k+\ell)\big| \le \Big[\prod_{\substack{0\le\nu\le3 \\ \ell_\nu\ne0}} |k_\nu|\Big] \prod_{\nu=0}^3 \frac{24}{|\ell_\nu|+\pi}$ *if* $0 \ne \ell \in \hat{\mathcal{B}}_n$*.*

(c) $\big|u_n(k) - 1\big| \le 4^3|k|^2$*.*

(d) If $\ell \in \hat{\mathcal{B}}_n$ *and* $\ell_{\tilde\nu} \ne 0$ *for some* $0 \le \tilde\nu \le 3$*, then* $u_n(k+\ell) = \sin\big(\frac12 k_{\tilde\nu}\big) v_{n,\tilde\nu}(k+\ell)$ *with* $\big|v_{n,\tilde\nu}(k+\ell)\big| \le \prod_{\nu=0}^3 \frac{24}{|\ell_\nu|+\pi}$*.*

(e) For all $\ell \in \hat{\mathcal{B}}_n$*,*

$$\big|\mathrm{Im}\, u_n(k+\ell)\big| \le 12\ |\mathrm{Im}\, k| \prod_{\nu=0}^{3} \tfrac{24}{|\ell_\nu|+\pi}$$
$$\big|\mathrm{Im}\, u_n(k+\ell)^{\mathfrak{q}}\big| \le 12\,\mathfrak{q}\ |\mathrm{Im}\, k|\ \Big[\prod_{\nu=0}^{3} \tfrac{24}{|\ell_\nu|+\pi}\Big]^{\mathfrak{q}}$$

(f) Recall that $|\mathrm{Re}\, k_\nu| \le \pi$, $|\mathrm{Im}\, k_\nu| \le 2$ *for each* $0 \le \nu \le 3$*. We have*

$$\tfrac{1}{4\pi^4} \le |u_n(k)| \le 4\pi^4$$

If, in addition, k *is real*

$$\big(\tfrac{2}{\pi}\big)^4 \le u_n(k) \le \big(\tfrac{\pi}{2}\big)^4$$

Proof. Set $s(x) = \frac{\sin x}{x}$. By the definitions of $u_n(p)$ in Remark 2.1.b, $\sigma(k)$ in (2.3) and $\varepsilon_{n,\nu}$ in (2.6),

$$u_n(p) = \frac{\sin \frac{1}{2} p_0}{\frac{1}{\varepsilon_n^2} \sin \frac{1}{2}\varepsilon_n^2 p_0} \prod_{\nu=1}^{3} \frac{\sin \frac{1}{2}\mathbf{p}_\nu}{\frac{1}{\varepsilon_n} \sin \frac{1}{2}\varepsilon_n \mathbf{p}_\nu} = \prod_{\nu=0}^{3} \frac{s(p_\nu/2)}{s(\varepsilon_{n,\nu} p_\nu/2)} \tag{2.7}$$

(a) We may assume without loss of generality that ℓ_ν is bounded, as a real number, by $\frac{\pi}{\varepsilon_{n,\nu}} - \pi$. (Recall that $\frac{1}{\varepsilon_{n,\nu}}$ is an odd natural number.) So $\left|\operatorname{Re} k_\nu + \ell_\nu\right|$ is always bounded by $\frac{\pi}{\varepsilon_{n,\nu}}$. Consequently, the hypotheses of Lemma A.1.c, with $x+iy = k_\nu + \ell_\nu$ and $\varepsilon = \varepsilon_{n,\nu}$, are satisfied and

$$\left| \frac{\sin \frac{1}{2}(k_\nu + \ell_\nu)}{\frac{1}{\varepsilon_{n,\nu}} \sin \frac{1}{2}\varepsilon_{n,\nu}(k_\nu + \ell_\nu)} \right| \le \left\{ \begin{matrix} 4 & \text{if } \ell_\nu = 0 \\ \frac{8}{|\operatorname{Re} k_\nu + \ell_\nu|} & \text{if } |\ell_\nu| \ge 2\pi \end{matrix} \right\} \le \frac{24}{|\ell_\nu| + \pi}$$

since $|\operatorname{Re} k_\nu| \le \pi$ and $\ell_\nu \in 2\pi\mathbb{Z}$.

When $\mathfrak{q} > 1$, $|u_n(k+\ell)|^{\mathfrak{q}}$ is summable in ℓ and the bound on $\|Q_n\|_{m=1}$ follows from [5, Lemma 12.c]. When $\mathfrak{q} = 1$, we use that the action of

$$Q_n : \mathcal{H}_n^{(0)} \to \mathcal{H}_0^{(n)}$$

in position space is

$$(Q_n \phi)(x) = \sum_{v \in \varepsilon_n^2 \mathbb{Z} \times \varepsilon_n \mathbb{Z}^3} u_n(v) \phi(x + [v]) \tag{2.8}$$

where $[v]$ denotes the class of $v \in \varepsilon_n^2 \mathbb{Z} \times \varepsilon_n \mathbb{Z}^3$ in the quotient space

$$\mathcal{X}_n^{(0)} = \left(\varepsilon_n^2 \mathbb{Z} / \varepsilon_n^2 L_{\rm tp} \mathbb{Z}\right) \times \left(\varepsilon_n \mathbb{Z}^3 / \varepsilon_n L_{\rm sp} \mathbb{Z}^3\right)$$

and x runs over

$$\mathcal{X}_0^{(n)} = \left(\mathbb{Z} / \varepsilon_n^2 L_{\rm tp} \mathbb{Z}\right) \times \left(\mathbb{Z}^3 / \varepsilon_n L_{\rm sp} \mathbb{Z}^3\right)$$

When $\mathfrak{q} = 1$, the averaging profile u_n is the characteristic function, $1_\square(v)$ (the dependence on n is suppressed in the notation), of the rectangle

$$\left(\left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^3\right) \cap \left(\varepsilon_n^2 \mathbb{Z} \times \varepsilon_n \mathbb{Z}^3\right)$$

Note that $u_n = 1_\square$ is already normalized to have integral one. The L^1–L^∞ norm of u_n (with mass zero) is exactly one. The L^1–L^∞ norm, with mass $m = 1$, of u_n is bounded by $\exp\left\{\left|\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right|\right\} = e$.

(b) If $\ell_\nu \neq 0$,

$$\tfrac{1}{2}\big|k_\nu + \ell_\nu\big| \geq \tfrac{1}{2}\big|\mathrm{Re}\, k_\nu + \ell_\nu\big| \geq \tfrac{1}{6}\big(\pi + |\ell_\nu|\big)$$

so that, by Lemma A.1.a, the denominator

$$\tfrac{1}{\varepsilon_{n,\nu}}\big|\sin\tfrac{1}{2}\varepsilon_{n,\nu}(k_\nu + \ell_\nu)\big| \geq \tfrac{1}{\varepsilon_{n,\nu}}\tfrac{\sqrt{2}}{\pi}\tfrac{1}{6}\varepsilon_{n,\nu}\big(\pi + |\ell_\nu|\big) = \tfrac{1}{3\sqrt{2}\pi}\big(\pi + |\ell_\nu|\big)$$

On the other hand, the numerator, by Lemma A.1.b,

$$\big|\sin\tfrac{1}{2}(k_\nu + \ell_\nu)\big| = \big|\sin\big(\tfrac{1}{2}k_\nu\big)\big| \leq |k_\nu|$$

As $\ell \neq 0$, there is at least one ν with $\ell_\nu \neq 0$. For each ν with $\ell_\nu \neq 0$ bound the factor

$$\left|\frac{\sin\frac{1}{2}(k_\nu + \ell_\nu)}{\frac{1}{\varepsilon_{n,\nu}}\sin\frac{1}{2}\varepsilon_{n,\nu}(k_\nu + \ell_\nu)}\right| \leq \frac{3\sqrt{2}\pi|k_\nu|}{|\ell_\nu| + \pi} \leq \frac{24|k_\nu|}{|\ell_\nu| + \pi}$$

Bound the remaining factors, with ν having $\ell_\nu = 0$, by $\frac{24}{|\ell_\nu|+\pi}$ as in part (a). All together

$$\prod_{\nu=0}^{3}\left|\frac{\sin\frac{1}{2}(k_\nu + \ell_\nu)}{\frac{1}{\varepsilon_{n,\nu}}\sin\frac{1}{2}\varepsilon_{n,\nu}(k_\nu + \ell_\nu)}\right| \leq \Bigg[\prod_{\substack{0\leq\nu\leq 3\\ \ell_\nu\neq 0}}|k_\nu|\Bigg]\prod_{\nu=0}^{3}\frac{24}{|\ell_\nu| + \pi}$$

(c) For both $z = \frac{1}{2}\varepsilon_{n,\nu}k_\nu$ and $z = \frac{1}{2}k_\nu$, $|z|^2 \leq \frac{\pi^2}{4} + 1 < 4$ so that, by Lemma A.1.e,

$$\big|\tfrac{\sin z}{z} - 1\big| \leq \tfrac{1}{2}|z|^2$$

Using $\frac{a}{b} - 1 = \frac{(a-1)-(b-1)}{b}$ we have, by Lemma A.1.a,

$$\left|\frac{\sin(\frac{1}{2}k_\nu)}{\frac{1}{\varepsilon_{n,\nu}}\sin(\frac{1}{2}\varepsilon_{n,\nu}k_\nu)} - 1\right| \leq \frac{\frac{1}{2}|\frac{1}{2}k_\nu|^2 + \frac{1}{2}|\frac{1}{2}\varepsilon_{n,\nu}k_\nu|^2}{|\sin(\frac{1}{2}\varepsilon_{n,\nu}k_\nu)|/|\frac{1}{2}\varepsilon_{n,\nu}k_\nu|} \leq \tfrac{\pi}{\sqrt{2}}\tfrac{1}{8}(1 + \varepsilon_{n,\nu}^2)|k_\nu|^2 \leq |k_\nu|^2$$

Finally, using

$$\prod_{\nu=0}^{3}A_\nu - 1 = (A_0 - 1)\prod_{\nu=1}^{3}A_\nu + (A_1 - 1)\prod_{\nu=2}^{3}A_\nu + (A_2 - 1)A_3 + (A_3 - 1)$$

we have, by Lemma A.1.c,

$$\big|u_n(k) - 1\big| \leq 4^3|k_0|^2 + 4^{3-1}|k_1|^2 + 4|k_2|^2 + |k_3|^2 \leq 4^3|k|^2$$

(d) The proof is the same as that for part (b), except that the factor $\sin\left(\frac{1}{2}k_{\tilde\nu}\right)$ in the numerator

$$\sin\tfrac{1}{2}(k_{\tilde\nu}+\ell_{\tilde\nu}) = (-1)^{\frac{\ell_{\tilde\nu}}{2\pi}}\sin\left(\tfrac{1}{2}k_{\tilde\nu}\right)$$

is pulled out of u_n, leaving $v_{n,\tilde\nu}$, rather than being bounded by $|k_{\tilde\nu}|$.

(e) By Lemma A.1.c

$$\left|\mathrm{Im}\,\frac{\sin\frac{1}{2}(k_\nu+\ell_\nu)}{\frac{1}{\varepsilon_{n,\nu}}\sin\frac{1}{2}\varepsilon_{n,\nu}(k_\nu+\ell_\nu)}\right| \le 6|\mathrm{Im}\,k_\nu|\frac{24}{|\ell_\nu|+\pi}$$

In general, for any complex numbers $z_j = r_j e^{i\theta_j}$, $1\le j\le J$,

$$\Big|\mathrm{Im}\,\prod_{j=1}^{J} z_j\Big| = \Big|\sin\Big(\sum_{j=1}^{J}\theta_j\Big)\Big|\prod_{j=1}^{J} r_j$$

Repeatedly using

$$\big|\sin(\theta+\theta')\big| = \big|\sin(\theta)\cos(\theta')+\cos(\theta)\sin(\theta')\big| \le |\sin(\theta)|+|\sin(\theta')|$$

we have

$$\Big|\mathrm{Im}\,\prod_{j=1}^{J} z_j\Big| \le \sum_{j=1}^{J}\big|\sin(\theta_j)\big|\prod_{j=1}^{J} r_j = \sum_{j=1}^{J}\big|\mathrm{Im}\,z_j\big|\prod_{j'\ne j}|z_{j'}|$$

So

$$\big|\mathrm{Im}\,u_n(k+\ell)\big| \le \Big(\sum_{\nu=0}^{3}6|\mathrm{Im}\,k_\nu|\Big)\prod_{\nu=0}^{3}\tfrac{24}{|\ell_\nu|+\pi} \le 12\ |\mathrm{Im}\,k|\prod_{\nu=0}^{3}\tfrac{24}{|\ell_\nu|+\pi}$$

and

$$\big|\mathrm{Im}\,u_n(k+\ell)^{\mathfrak{q}}\big| \le 12\,\mathfrak{q}\ |\mathrm{Im}\,k|\ \Big[\prod_{\nu=0}^{3}\tfrac{24}{|\ell_\nu|+\pi}\Big]^{\mathfrak{q}}$$

(f) Just apply Lemma A.1.a,b separately to all of the numerators and denominators in the right hand side of (2.7). When k is real, apply Lemma A.1.d instead.

□

Lemma 2.3. *Let* $\mathfrak{q} \in \mathbb{N}$. *Assume that* $|\mathrm{Re}\, \mathfrak{k}_\nu| \le \frac{\pi}{L_\nu}$ *and* $|\mathrm{Im}\, \mathfrak{k}_\nu| \le \frac{2}{L_\nu}$ *for each* $0 \le \nu \le 3$.

(a) $\big|u_+(\mathfrak{k}+\ell)\big| \le \prod_{\nu=0}^{3} \frac{24}{L_\nu|\ell_\nu|+\pi}$ *for all* $\ell \in \hat{\mathcal{B}}^+$. *We use* $|\ell_\nu|$ *to denote the magnitude of the smallest representative of* ℓ_ν *in its equivalence class, as an element of* $\hat{\mathcal{B}}^+$.

(b) $\big|u_+(\mathfrak{k}+\ell)\big| \le \Big[\prod_{\nu\in\mathcal{I}_+} L_\nu|\mathfrak{k}_\nu|\Big]\Big[\prod_{\nu=0}^{3} \frac{24}{L_\nu|\ell_\nu|+\pi}\Big]$ *for all* $\ell \in \hat{\mathcal{B}}^+$ *with* $\ell \ne 0$. *Here* $\mathcal{I}_+$ *is any subset of* $\big\{\ \nu\ \big|\ 0 \le \nu \le 3,\ \ell_\nu \ne 0\ \big\}$.

(c) $\big|u_+(\mathfrak{k}) - 1\big| \le 4^3 \sum_{\nu=0}^{3} L_\nu^2|\mathfrak{k}_\nu|^2$.

(d) *If* $\ell \in \hat{\mathcal{B}}^+$ *and* $\ell_{\tilde\nu} \ne 0$ *for some* $0 \le \tilde\nu \le 3$, *then* $u_+(\mathfrak{k}+\ell) = \sin\big(\frac{L_{\tilde\nu}}{2}\mathfrak{k}_{\tilde\nu}\big)v_{+,\tilde\nu}(\mathfrak{k}+\ell)$ *with* $\big|v_{+,\tilde\nu}(\mathfrak{k}+\ell)\big| \le \prod_{\nu=0}^{3} \frac{24}{L_\nu|\ell_\nu|+\pi}$.

(e) *For all* $\ell \in \hat{\mathcal{B}}^+$,

$$\big|\mathrm{Im}\, u_+(\mathfrak{k}+\ell)\big| \le 12\ |\mathrm{Im}\, \mathbb{L}\mathfrak{k}| \prod_{\nu=0}^{3} \frac{24}{L_\nu|\ell_\nu|+\pi}$$
$$\big|\mathrm{Im}\, u_+(\mathfrak{k}+\ell)^{\mathfrak{q}}\big| \le 12\,\mathfrak{q}\ |\mathrm{Im}\, \mathbb{L}\mathfrak{k}|\ \Big[\prod_{\nu=0}^{3} \frac{24}{L_\nu|\ell_\nu|+\pi}\Big]^{\mathfrak{q}}$$

Proof. (a) We may assume without loss of generality that ℓ_ν is bounded, as a real number, by $\pi - \frac{\pi}{L_\nu}$. (Recall that L is an odd natural number.) So we will always have $\big|\mathrm{Re}\, \mathfrak{k}_\nu + \ell_\nu\big| \le \pi$. So, by Lemma A.1.c with $\varepsilon = \frac{1}{L_\nu}$ and $x + iy = L_\nu(\mathfrak{k}_\nu + \ell_\nu)$,

$$\left|\frac{\sin \frac{L_\nu}{2}(\mathfrak{k}_\nu+\ell_\nu)}{L_\nu \sin\frac{1}{2}(\mathfrak{k}_\nu+\ell_\nu)}\right| \le \left\{\begin{matrix} 4 & \text{if } \ell_\nu = 0 \\ \frac{8}{L_\nu|\mathrm{Re}\,\mathfrak{k}_\nu+\ell_\nu|} & \text{if } |L_\nu\ell_\nu| \ge 2\pi \end{matrix}\right\} \le \frac{24}{L_\nu|\ell_\nu|+\pi}$$

since $|\mathrm{Re}\, \mathfrak{k}_\nu| \le \frac{\pi}{L_\nu}$, $|\mathrm{Im}\, \mathfrak{k}_\nu| \le \frac{2}{L_\nu}$ and $\ell_\nu \in \frac{2\pi}{L_\nu}\mathbb{Z}$.

(b) If $\ell_\nu \ne 0$,

$$\tfrac{1}{2}\big|\mathfrak{k}_\nu + \ell_\nu\big| \ge \tfrac{1}{2}\big|\mathrm{Re}\, \mathfrak{k}_\nu + \ell_\nu\big| \ge \tfrac{1}{6}\big(\tfrac{\pi}{L_\nu} + |\ell_\nu|\big)$$

and $\frac{1}{2}\big|\mathrm{Re}\, \mathfrak{k}_\nu + \ell_\nu\big| \le \frac{\pi}{2}$ so that, by Lemma A.1.a, the denominator

$$L_\nu\big|\sin\tfrac{1}{2}(\mathfrak{k}_\nu+\ell_\nu)\big| \ge \tfrac{\sqrt{2}}{6\pi}\big(\pi + L_\nu|\ell_\nu|\big)$$

On the other hand, the numerator, by Lemma A.1.b,

$$\left|\sin\tfrac{L_\nu}{2}(\mathfrak{k}_\nu+\ell_\nu)\right| = \left|\sin\left(\tfrac{L_\nu}{2}\mathfrak{k}_\nu\right)\right| \le 2\tfrac{L_\nu}{2}|\mathfrak{k}_\nu| = L_\nu|\mathfrak{k}_\nu|$$

As $\ell \neq 0$, there is at least one ν with $\ell_\nu \neq 0$. Bound each factor with $\nu \in \mathcal{I}_+$ by

$$\left|\frac{\sin\frac{L_\nu}{2}(\mathfrak{k}_\nu+\ell_\nu)}{L_\nu \sin\frac{1}{2}(\mathfrak{k}_\nu+\ell_\nu)}\right| \le \frac{6\pi}{\sqrt{2}}\frac{L_\nu|\mathfrak{k}_\nu|}{L_\nu|\ell_\nu|+\pi}$$

Bound the remaining factors by $\frac{24}{L_\nu|\ell_\nu|+\pi}$.

(c) has the same proof as that of Lemma 2.2.c, with k_ν replaced by $L_\nu\mathfrak{k}_\nu$ and $\frac{1}{\varepsilon_{n,\nu}}$ replaced by L_ν.

(d) The proof is similar to that for part (b), except that the factor $\sin\left(\frac{L_{\tilde\nu}}{2}\mathfrak{k}_{\tilde\nu}\right)$ in the numerator

$$\sin\tfrac{L_{\tilde\nu}}{2}(\mathfrak{k}_{\tilde\nu}+\ell_{\tilde\nu}) = (-1)^{\frac{L_{\tilde\nu}\ell_{\tilde\nu}}{2\pi}}\sin\left(\tfrac{L_{\tilde\nu}}{2}\mathfrak{k}_{\tilde\nu}\right)$$

is pulled out of u_+, leaving $v_{+,\tilde\nu}$, rather than being bounded by $L_{\tilde\nu}|\mathfrak{k}_{\tilde\nu}|$. Also, each $\left|\frac{\sin\frac{L_\nu}{2}(\mathfrak{k}_\nu+\ell_\nu)}{L_\nu\sin\frac{1}{2}(\mathfrak{k}_\nu+\ell_\nu)}\right|$ with $\nu \neq \tilde\nu$ is bounded by $\frac{24}{L_\nu|\ell_\nu|+\pi}$.

(e) By Lemma A.1.c

$$\left|\mathrm{Im}\,\frac{\sin\frac{L_\nu}{2}(\mathfrak{k}_\nu+\ell_\nu)}{L_\nu\sin\frac{1}{2}(\mathfrak{k}_\nu+\ell_\nu)}\right| \le 8|\mathrm{Im}\,L_\nu\mathfrak{k}_\nu|\frac{24}{L_\nu|\ell_\nu|+\pi}$$

The proof now continues as in Lemma 2.2.e, just by substituting $k_\nu \to L_\nu\mathfrak{k}_\nu$, $\ell_\nu \to L_\nu\ell_\nu$ and $\varepsilon_{n,\nu} = \frac{1}{L_\nu}$. □

Proposition 2.4. *Let $\mathfrak{q} \in \mathbb{N}$. There are constants Γ_2, depending only on $\mathfrak{q}$, and Γ_3, depending only on a, such that the following hold for all $L > \Gamma_2$.*

(a) On the domain $\left\{\, k \in \mathbb{C}\times\mathbb{C}^3 \;\middle|\; |\mathrm{Im}\,k_\nu| < 2 \text{ for each } 0 \le \nu \le 3 \,\right\}$ $\hat{\mathfrak{Q}}_n(k)$ is analytic, and invariant under $k_\nu \to -k_\nu$ for each $0 \le \nu \le 3$ and under $k_\nu \leftrightarrow k_{\nu'}$ for all $1 \le \nu, \nu' \le 3$, and obeys

$$\tfrac{5}{6}a \le |\hat{\mathfrak{Q}}_n(k)| \le \tfrac{5}{4}a \qquad \mathrm{Re}\,\hat{\mathfrak{Q}}_n(k) \ge \tfrac{a}{2}$$

If k is real $\frac{5}{6}a \le \hat{\mathfrak{Q}}_n(k) \le a$.

(b) If $|\mathrm{Re}\,k_\nu| \le \pi$ and $|\mathrm{Im}\,k_\nu| \le 2$ for each $0 \le \nu \le 3$, then

$$\left|\hat{\mathfrak{Q}}_n(k) - a_n\right| \le \tfrac{a}{3003}|k|^2 \qquad \text{where} \qquad a_n = a\tfrac{1-L^{-2}}{1-L^{-2n}}$$

(c) $\|\mathfrak{Q}_n\|_{m=1} \le \Gamma_3$

Proof. (a) Recall, from Remark 2.1.e, that $\hat{\mathfrak{Q}}_n(k) = a\Big[1+\sum\limits_{j=1}^{n-1}\sum\limits_{\ell_j\in\hat{\mathcal{B}}_j}\tfrac{1}{L^{2j}}u_j(k+\ell_j)^{2\mathfrak{q}}\Big]^{-1}$.
By Lemma 2.2.a,

$$\sum_{j=1}^{n-1}\sum_{\ell_j\in\hat{\mathcal{B}}_j}\tfrac{1}{L^{2j}}|u_j(k+\ell_j)|^{2\mathfrak{q}} \le \sum_{j=1}^{\infty}\tfrac{1}{L^{2j}}\sum_{\ell\in 2\pi\mathbb{Z}\times 2\pi\mathbb{Z}^3}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_\nu|+\pi}\big)^{2\mathfrak{q}} = \tfrac{c_{\mathfrak{q}}}{L^2-1}$$

where $c_{\mathfrak{q}} = \Big[\sum_{j\in\mathbb{Z}}\big(\tfrac{24/\pi}{[2|j|+1)}\big)^{2\mathfrak{q}}\Big]^4$. Just pick L large enough that $\tfrac{c_{\mathfrak{q}}}{L^2-1} < \tfrac{1}{5}$ and use that, if $|z| \le \tfrac{1}{5}$

$$\mathrm{Re}\,\tfrac{1}{1+z} = \tfrac{\mathrm{Re}\,(1+\bar{z})}{|1+z|^2} \ge \tfrac{4/5}{(6/5)^2} = \tfrac{20}{36}$$

(b) Using $O(|k|^2)$ to denote any function that is bounded by a constant, depending only on $\mathfrak{q}$,

$$\begin{aligned}
1+\sum_{j=1}^{n-1}\sum_{\ell_j\in\hat{\mathcal{B}}_j}\tfrac{1}{L^{2j}}u_j(k+\ell_j)^{2\mathfrak{q}} &= \sum_{j=0}^{n-1}\tfrac{1}{L^{2j}} + \sum_{j=1}^{n-1}\tfrac{1}{L^{2j}}[u_j(k)^{2\mathfrak{q}}-1] \\
&\qquad + \sum_{j=1}^{n-1}\sum_{\substack{\ell_j\in\hat{\mathcal{B}}_j\\ \ell\ne 0}}\tfrac{1}{L^{2j}}u_j(k+\ell_j)^{2\mathfrak{q}} \\
&\le \tfrac{1-L^{-2n}}{1-L^{-2}} + \sum_{j=1}^{n-1}\tfrac{1}{L^{2j}}O\big(|k|^2\big) \qquad \text{by Lemma 2.2.b,c} \\
&\le \tfrac{1-L^{-2n}}{1-L^{-2}} + \tfrac{1}{L^2}O\big(|k|^2\big)
\end{aligned}$$

So

$$\hat{\mathfrak{Q}}_n(k) = a\big[\tfrac{1-L^{-2n}}{1-L^{-2}} + \tfrac{1}{L^2}O\big(|k|^2\big)\big]^{-1} = a_n\big[1+\tfrac{1}{L^2}\tfrac{1-L^{-2}}{1-L^{-2n}}O\big(|k|^2\big)\big]^{-1}$$

and it suffices to choose Γ_2 large enough that

$$\tfrac{9}{10} \le \tfrac{1-L^{-2}}{1-L^{-2n}} \le \tfrac{11}{10} \qquad \Big|\tfrac{1}{L^2}\tfrac{1-L^{-2}}{1-L^{-2n}}O\big(|k|^2\big)\Big| \le \tfrac{10}{22}\tfrac{1}{3003}|k|^2 \qquad \Big|\tfrac{1}{L^2}\tfrac{1-L^{-2}}{1-L^{-2n}}O\big(|k|^2\big)\Big| \le \tfrac{1}{2}$$

for all allowed k's and L's.

(c) follows immediately from part (a) and [5, Lemma 12.b] with $\mathcal{X}_{\mathrm{fin}} = \mathcal{X}_{\mathrm{crs}} = \mathcal{X}_0^{(n)}$. □

We now define operators $Q^{(\pm)}_{n,\nu}$ and $Q^{(\pm)}_{+,\nu}$ so that the next remark holds.

Remark 2.5. Let $0 \le \nu \le 3$. We have

$$\partial_\nu Q_n^* = Q^{(+)}_{n,\nu}\partial_\nu \qquad \partial_\nu Q_n = Q^{(-)}_{n,\nu}\partial_\nu \qquad \partial_\nu Q^* = Q^{(+)}_{+,\nu}\partial_\nu \qquad \partial_\nu Q = Q^{(-)}_{+,\nu}\partial_\nu$$

If $S : \mathcal{H}_n \to \mathcal{H}_n$ and $T : \mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$ are linear operators that are translation invariant with respect to $\mathcal{X}_n$ and $\mathcal{X}_0^{(n)}$, respectively, then

$$Q^{(-)}_{n,\nu} S Q^{(+)}_{n,\nu} = Q_n S Q_n^* \qquad Q^{(-)}_{+,\nu} T Q^{(+)}_{+,\nu} = Q T Q^*$$

To prepare for the definitions, recall that the forward derivatives of $\alpha \in \mathcal{H}_j^{(n)}$ are defined by

$$(\partial_\nu \alpha)(x) = \tfrac{1}{\varepsilon_{j,\nu}}\big[\alpha(x + \varepsilon_{j,\nu} e_\nu) - \alpha(x)\big] \tag{2.9}$$

where e_ν is a unit vector in the ν^{th} direction. The Fourier transforms

$$\begin{aligned}
\big(\widehat{\partial_\nu \phi}\big)(p) &= 2ie^{i\varepsilon_{n,\nu}p_\nu/2}\ \tfrac{\sin(\varepsilon_{n,\nu}p_\nu/2)}{\varepsilon_{n,\nu}}\hat\phi(p) &&\text{for all } \phi \in \mathcal{H}_n &&\text{and } p \in \hat{\mathcal{X}}_n\\
\big(\widehat{\partial_\nu \psi}\big)(k) &= 2ie^{ik_\nu/2}\ \sin(k_\nu/2)\hat\psi(k) &&\text{for all } \psi \in \mathcal{H}_0^{(n)} &&\text{and } k \in \hat{\mathcal{X}}_0^{(n)}\\
\big(\widehat{\partial_\nu \theta}\big)(\mathfrak{k}) &= 2ie^{iL_\nu \mathfrak{k}_\nu/2}\ \tfrac{\sin(L_\nu \mathfrak{k}_\nu/2)}{L_\nu}\hat\theta(\mathfrak{k}) &&\text{for all } \theta \in \mathcal{H}_{-1}^{(n+1)} &&\text{and } \mathfrak{k} \in \hat{\mathcal{X}}_{-1}^{(n+1)}
\end{aligned} \tag{2.10}$$

Set

$$u^{(+)}_{n,\nu}(p) = \prod_{\substack{0\le\nu'\le 3\\ \nu'\ne\nu}} \frac{\sin\frac12 p_{\nu'}}{\frac{1}{\varepsilon_{n,\nu'}}\sin\frac12\varepsilon_{n,\nu'}p_{\nu'}} \qquad u^{(+)}_{+,\nu}(k) = \prod_{\substack{0\le\nu'\le 3\\ \nu'\ne\nu}} \frac{\sin\frac12 L_{\nu'}k_{\nu'}}{L_{\nu'}\sin\frac12 k_{\nu'}}$$

$$u^{(-)}_{n,\nu}(p) = \frac{\sin\frac12 p_\nu}{\frac{1}{\varepsilon_{n,\nu}}\sin\frac12\varepsilon_{n,\nu}p_\nu}\prod_{\nu'=0}^{3} \frac{\sin\frac12 p_{\nu'}}{\frac{1}{\varepsilon_{n,\nu'}}\sin\frac12\varepsilon_{n,\nu'}p_{\nu'}} \qquad u^{(-)}_{+,\nu}(k) = \frac{\sin\frac12 L_\nu k_\nu}{L_\nu \sin\frac12 k_\nu}\prod_{\nu'=0}^{3} \frac{\sin\frac12 L_{\nu'}k_{\nu'}}{L_{\nu'}\sin\frac12 k_{\nu'}}$$

and

$$\begin{aligned}
\zeta^{(+)}_{n,\nu}(k,\ell_n) &= e^{i\varepsilon_{n,\nu}(k+\ell_n)_\nu/2}e^{-ik_\nu/2}\cos\tfrac12\ell_{n,\nu} \qquad & \zeta^{(+)}_{+,\nu}(\mathfrak{k},\ell) &= e^{i(\mathfrak{k}+\ell)_\nu/2}e^{-iL_\nu\mathfrak{k}_\nu/2}\cos\tfrac12 L_\nu\ell_\nu\\
\zeta^{(-)}_{n,\nu}(k,\ell_n) &= e^{ik_\nu/2}e^{-i\varepsilon_{n,\nu}(k+\ell_n)_\nu/2}\cos\tfrac12\ell_{n,\nu} \qquad & \zeta^{(-)}_{+,\nu}(\mathfrak{k},\ell) &= e^{iL_\nu\mathfrak{k}_\nu/2}e^{-i(\mathfrak{k}+\ell)_\nu/2}\cos\tfrac12 L_\nu\ell_\nu
\end{aligned}$$

Define the operators $Q^{(+)}_{n,\nu} : \mathcal{H}_0^{(n)} \to \mathcal{H}_n$ and $Q^{(-)}_{n,\nu} : \mathcal{H}_n \to \mathcal{H}_0^{(n)}$ by

$$\begin{aligned}
\big(\widehat{Q^{(+)}_{n,\nu}\psi}\big)(k+\ell_n) &= \zeta^{(+)}_{n,\nu}(k,\ell_n)u^{(+)}_{n,\nu}(k+\ell_n)u_n(k+\ell_n)^{\mathfrak{q}-1}\hat\psi(k)\\
\big(\widehat{Q^{(-)}_{n,\nu}\phi}\big)(k) &= \sum_{\ell_n\in\hat{\mathcal{B}}_n}\zeta^{(-)}_{n,\nu}(k,\ell_n)u^{(-)}_{n,\nu}(k+\ell_n)u_n(k+\ell_n)^{\mathfrak{q}-1}\hat\phi(k+\ell_n)
\end{aligned} \tag{2.11}$$

and the operators $Q^{(+)}_{+,\nu} : \mathcal{H}^{(n+1)}_{-1} \to \mathcal{H}^{(n)}_0$ and $Q^{(-)}_{+,\nu} : \mathcal{H}^{(n)}_0 \to \mathcal{H}^{(n+1)}_{-1}$ by

$$\begin{aligned} \big(\widehat{Q^{(+)}_{+,\nu}\theta}\big)(\mathfrak{k}+\ell) &= \zeta^{(+)}_{+,\nu}(\mathfrak{k},\ell)u^{(+)}_{+,\nu}(\mathfrak{k}+\ell)u_+(\mathfrak{k}+\ell)^{\mathfrak{q}-1}\hat\theta(\mathfrak{k}) \\ \big(\widehat{Q^{(-)}_{+,\nu}\psi}\big)(\mathfrak{k}) &= \sum_{\ell\in\hat{\mathcal{B}}^+} \zeta^{(-)}_{+,\nu}(\mathfrak{k},\ell)u^{(-)}_{+,\nu}(\mathfrak{k}+\ell)u_+(\mathfrak{k}+\ell)^{\mathfrak{q}-1}\hat\psi(\mathfrak{k}+\ell) \end{aligned} \tag{2.12}$$

Proof of Remark 2.5. For the "Q_n^*" and "Q_n" cases, it suffices to observe that

$$\big(2ie^{i\varepsilon_{n,\nu}(k+\ell)_\nu/2}\ \tfrac{\sin(\varepsilon_{n,\nu}(k+\ell)_\nu/2)}{\varepsilon_{n,\nu}}\big)u_n(k+\ell) = \zeta^{(+)}_{n,\nu}(k,\ell)u^{(+)}_{n,\nu}(k+\ell)\big(2ie^{ik_\nu/2}\ \sin(k_\nu/2)\big)$$

and

$$\big(2ie^{ik_\nu/2}\ \sin(k_\nu/2)\big)u_n(k+\ell) = \zeta^{(-)}_{n,\nu}(k,\ell)u^{(-)}_{n,\nu}(k+\ell)\big(2ie^{i\varepsilon_{n,\nu}(k+\ell)_\nu/2}\ \tfrac{\sin(\varepsilon_{n,\nu}(k+\ell)_\nu/2)}{\varepsilon_{n,\nu}}\big)$$

and

$$\zeta^{(-)}_{n,\nu}(k,\ell)u^{(-)}_{n,\nu}(k+\ell)\zeta^{(+)}_{n,\nu}(k,\ell)u^{(+)}_{n,\nu}(k+\ell) = u_n(k+\ell)^2$$

for all k, ℓ, ν. We remark that "$Q^{(-)}_{n,\nu}SQ^{(+)}_{n,\nu} = Q_nSQ_n^*$" should not be surprising since $\partial_\nu Q_nSQ_n^* = Q^{(-)}_{n,\nu}SQ^{(+)}_{n,\nu}\partial_\nu$ and $Q_nSQ_n^*$ is translation invariant on the unit scale and so commutes with ∂_ν. The proof for the "Q^*" and "Q" cases are virtually identical. ☐

Lemma 2.6. *Let $0\le\nu\le 3$, $\ell\in\hat{\mathcal{B}}^+$ and $\ell_n\in\hat{\mathcal{B}}_n$.*

(a) $\zeta^{(+)}_{n,\nu}(k,\ell_n)u^{(+)}_{n,\nu}(k+\ell_n)$ *and* $\zeta^{(-)}_{n,\nu}(k,\ell_n)u^{(-)}_{n,\nu}(k+\ell_n)$ *are entire in k and* $\zeta^{(+)}_{+,\nu}(\mathfrak{k},\ell)u^{(+)}_{+,\nu}(\mathfrak{k}+\ell)$ *and* $\zeta^{(-)}_{+,\nu}(\mathfrak{k},\ell)u^{(-)}_{+,\nu}(k+\ell)$ *are entire in $\mathfrak{k}$*

(b) Assume that $|\mathrm{Re}\,k_{\nu'}|\le\pi$, $|\mathrm{Im}\,k_{\nu'}|\le 2$, $|\mathrm{Re}\,\mathfrak{k}_{\nu'}|\le\frac{\pi}{L_{\nu'}}$ *and* $|\mathrm{Im}\,\mathfrak{k}_{\nu'}|\le\frac{2}{L_{\nu'}}$ *for each* $0\le\nu'\le 3$. *Then*

$$\begin{aligned} \big|\zeta^{(+)}_{n,\nu}(k,\ell_n)u^{(+)}_{n,\nu}(k+\ell_n)\big| &\le e\prod_{0\le\nu'\le 3\ \nu'\ne\nu}\tfrac{24}{|\ell_{n,\nu'}|+\pi} \\ \big|\zeta^{(+)}_{+,\nu}(\mathfrak{k},\ell)u^{(+)}_{+,\nu}(\mathfrak{k}+\ell)\big| &\le e\prod_{0\le\nu'\le 3\ \nu'\ne\nu}\tfrac{24}{L_\nu|\ell_{\nu'}|+\pi} \\ \big|\zeta^{(-)}_{n,\nu}(k,\ell_n)u^{(-)}_{n,\nu}(k+\ell_n)\big| &\le \tfrac{24e}{|\ell_{n,\nu}|+\pi}\prod_{\nu'=0}^{3}\tfrac{24}{|\ell_{n,\nu'}|+\pi} \\ \big|\zeta^{(-)}_{+,\nu}(\mathfrak{k},\ell)u^{(-)}_{+,\nu}(\mathfrak{k}+\ell)\big| &\le \tfrac{24e}{L_\nu|\ell_\nu|+\pi}\prod_{\nu'=0}^{3}\tfrac{24}{L_{\nu'}|\ell_{\nu'}|+\pi} \end{aligned}$$

(c) There is a constant Γ_4, depending only on $\mathfrak{q}$, such that $\left\| Q_{n,\nu}^{(\pm)} \right\|_{m=1} \le \Gamma_4$.

Proof. (a) The proof is virtually identical to that of Remark 2.1.d.

(b) The proof is virtually identical to that of Lemmas 2.2.a and 2.3.a.

(c) By (2.11), the Fourier transform of $Q_{n,\nu}^{(\pm)}$ is $\zeta_{n,\nu}^{(\pm)}(k,\ell_n) u_{n,\nu}^{(\pm)}(k+\ell_n) u_n(k+\ell_n)^{\mathfrak{q}-1}$, which by part (b) and Lemma 2.2.a, is bounded in magnitude by

$$e \prod_{0 \le \nu' \le 3 \atop \nu' \ne \nu} \tfrac{24}{|\ell_{n,\nu'}|+\pi} \prod_{\nu=0}^{3} \left(\tfrac{24}{|\ell_\nu|+\pi}\right)^{\mathfrak{q}-1}$$

As $\mathfrak{q} > 2$, the claim now follows by [5, Lemma 12.c]. ☐

Remark 2.7. The principle obstruction to allowing $\mathfrak{q} = 1$ arises when a differential operator ∂_ν is intertwined with the block spin averaging operator Q_n, as happens in Remark 2.5. See, for example, the proof of Lemma 2.6.c. We use the condition $\mathfrak{q} > 1$ starting at Lemma 4.3 in §4 and in §5, 6. (See Lemma 5.5.) We use the condition $\mathfrak{q} > 2$ in Proposition 6.1 and Lemma 2.6.c.

3 Differential Operators

In [7, Definition 1.5.a] we associated to an operator h_0 on $L^2\big(\mathbb{Z}^3/L_{\rm sp}\mathbb{Z}^3\big)$ the operators

$$D_n = L^{2n}\,\mathbb{L}_*^{-n}\big(\mathbb{1} - e^{-h_0} - e^{-h_0}\partial_0\big)\mathbb{L}_*^n \tag{3.1}$$

Here ∂_0 is the forward time derivative of (2.9). In this chapter we assume that h_0 is the periodization (see [5, §3]) of a translation invariant operator $\mathbf{h}_0$ on $L^2\big(\mathbb{Z}^3\big)$ whose Fourier transform $\hat{\mathbf{h}}_0(\mathbf{p})$

- is entire in $\mathbf{p}$ and invariant under $\mathbf{p}_\nu \to -\mathbf{p}_\nu$ for each $1 \le \nu \le 3$
- is nonnegative when $\mathbf{p}$ is real and is strictly positive when $\mathbf{p} \in \mathbb{R}^3 \setminus 2\pi\mathbb{Z}^3$
- obeys $\hat{\mathbf{h}}_0(\mathbf{0}) = \frac{\partial\,\hat{\mathbf{h}}_0}{\partial \mathbf{p}_\nu}(\mathbf{0}) = 0$ for $1 \le \nu \le 3$ and has strictly positive Jacobian matrix $H = \Big[\frac{\partial^2\,\hat{\mathbf{h}}_0}{\partial\mathbf{p}_\mu\partial\mathbf{p}_\nu}(\mathbf{0})\Big]_{1\le\mu,\nu\le3}$.

Remark 3.1.

(a) The operator D_n is the periodization of a translation invariant operator $\mathbf{D}_n$, acting on $L^2\big(\varepsilon_n\mathbb{Z} \times \varepsilon_n^2\mathbb{Z}^3\big)$, whose Fourier transform is

$$\hat{\mathbf{D}}_n(p) = \tfrac{1}{2}\varepsilon_n^2 p_0^2 e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})}\left[\frac{\sin\frac{1}{2}\varepsilon_n^2 p_0}{\frac{1}{2}\varepsilon_n^2 p_0}\right]^2 + \mathbf{p}^2\frac{1-e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})}}{\varepsilon_n^2\mathbf{p}^2} - ip_0\,e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})}\frac{\sin\varepsilon_n^2 p_0}{\varepsilon_n^2 p_0}$$

with $p = (p_0, \mathbf{p}) \in \mathbb{C} \times \mathbb{C}^3$.

(b) $\hat{\mathbf{D}}_n(p)$ is entire in p and invariant under $\mathbf{p}_\nu \to -\mathbf{p}_\nu$ for each $1 \le \nu \le 3$.

(c) $\hat{\mathbf{D}}_n(p)$ has nonnegative real part when p is real.

Proof. (a) follows from (1.3) and the observation, by (2.10), that the Fourier transform of ∂_0, on $\mathbb{Z}$, is

$$2ie^{ik_0/2}\ \sin(k_0/2) = -2\sin^2(k_0/2) + i\sin(k_0)$$

(b) and (c) are obvious. □

Lemma 3.2. *There are constants γ_1, Γ_5 and a function $\bar{m}(c) > 0$ that depend only on $\hat{\mathbf{h}}_0$ and in particular are independent of n and L, such that the following hold.*

(a) For all $p \in \mathbb{R} \times \mathbb{R}^3$,

$$\big|\hat{\mathbf{D}}_n(p)\big| \geq \gamma_1\big(|p_0| + \sum_{\nu=1}^{3} |\mathbf{p}_\nu|^2\big)$$

We use $|p_0|$, $|\mathbf{p}_\nu|$ *and* $|\mathbf{p}|$ *to refer to the magnitudes of the smallest representatives of* $p_0 \in \mathbb{C}$, $\mathbf{p}_\nu \in \mathbb{C}$ *and* $\mathbf{p} \in \mathbb{C}^3$ *in* $\mathbb{C}/\frac{2\pi}{\varepsilon_n^2}\mathbb{Z}$, $\mathbb{C}/\frac{2\pi}{\varepsilon_n}\mathbb{Z}$ *and* $\mathbb{C}^3/\frac{2\pi}{\varepsilon_n}\mathbb{Z}^3$, *respectively.*

(b) For all $p \in \mathbb{C} \times \mathbb{C}^3$ *with* $\varepsilon_n^2 p_0, \varepsilon_n \mathbf{p}$ *having modulus less than one,*

$$\hat{\mathbf{D}}_n(p) = -ip_0 + \tfrac{1}{2}\varepsilon_n^2 p_0^2 + \tfrac{1}{2}\sum_{\nu,\nu'=1}^{3} H_{\nu,\nu'}\mathbf{p}_\nu \mathbf{p}_{\nu'} + O\Big(\varepsilon_n|\mathbf{p}|^3 + \varepsilon_n^4|p_0|^3\Big)$$

The higher order part $O(\,\cdot\,)$ *is uniform in* n *and* L.

(c) We have, for all $p \in \mathbb{C} \times \mathbb{C}^3$ *with* $\varepsilon_n^2|\mathrm{Im}\, p_0| \leq 1$ *and* $\varepsilon_n|\mathrm{Im}\,\mathbf{p}| \leq 1$,

$$\big|\hat{\mathbf{D}}_n(p)\big| \leq \Gamma_5\big(|p_0| + \sum_{\nu=1}^{3} |\mathbf{p}_\nu|^2\big)$$

and

$$\big|\tfrac{\partial^{\ell_i}}{\partial p_\nu^{\ell_i}}\hat{\mathbf{D}}_n(p)\big| \leq \Gamma_5 \begin{cases} \frac{1}{[1+|p_0|+|\mathbf{p}|^2]^{\ell_i-1}} & \text{if } \nu = 0,\ \ell_i = 1,2 \\ \frac{1}{[1+|p_0|+|\mathbf{p}|^2]^{\ell_i/2-1}} & \text{if } 1 \leq \nu \leq 3,\ 1 \leq \ell_i \leq 4 \end{cases}$$

(d) For all $c > 0$ *and* $p \in \mathbb{C} \times \mathbb{C}^3$, *with* $|p_0| + |\mathbf{p}| \geq c$ *and* $|\mathrm{Im}\, p| \leq \bar{m}(c)$,

$$\big|\hat{\mathbf{D}}_n(p)\big| \geq \gamma_1\left(|p_0| + \sum_{\nu=1}^{3} |\mathbf{p}_\nu|^2\right)$$

(e) For all $c > 0$ *and all* p *in the set*

$$\left\{\ p \in \mathbb{C} \times \mathbb{C}^3 \ \middle|\ |\mathrm{Im}\, p| \leq \bar{m}(c),\ |\mathbf{p}| \geq c\ \right\} \cup \left\{\ p \in \mathbb{C} \times \mathbb{C}^3 \ \middle|\ |\mathrm{Im}\, p| \leq \bar{m}(c),\ |\varepsilon_n p_0| \geq c\ \right\}$$

we have

$$\mathrm{Re}\,\hat{\mathbf{D}}_n(p) \geq \gamma_1\left(\varepsilon_n^2|p_0|^2 + \sum_{\nu=1}^{3} |\mathbf{p}_\nu|^2\right)$$

Proof. (a) By the hypotheses on $\hat{h}$,

$$\frac{1-e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})}}{\varepsilon_n^2\mathbf{p}^2} = \frac{\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})+O(\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})^2)}{\varepsilon_n^2\mathbf{p}^2} = \frac{\frac{1}{2}\varepsilon_n^2\mathbf{p}\cdot H\mathbf{p}+O(|\varepsilon_n\mathbf{p}|^3)}{\varepsilon_n^2\mathbf{p}^2}$$

for $\varepsilon_n\mathbf{p}$ is a real neighbourhood of $\mathbf{0}$. This is strictly positive and bounded away from 0 on some real neighbourhood of 0, uniformly in ε_n. Since $\hat{h}$ is continuous and strictly positive on $\mathbb{R}^3\setminus 2\pi\mathbb{Z}^3$, there is a constant $\gamma_1'>0$, independent of ε_n, such that

$$\frac{1-e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})}}{\varepsilon_n^2|\mathbf{p}|^2}\ge\gamma_1' \qquad\text{and}\qquad e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})}\ge\gamma_1'$$

for all $\mathbf{p}\in\mathbb{R}^3$. The claim now follows from Lemma A.1.a.

(b) Expanding $\frac{\sin z}{z}=1+O(|z|^2)$, for $|z|\le 1$, and

$$e^{-\hat{h}(\varepsilon_n\mathbf{p})}=1-\tfrac{1}{2}\varepsilon_n^2\sum_{\nu,\nu'=1}^{3}H_{\nu,\nu'}\mathbf{p}_\nu\mathbf{p}_{\nu'}+O\Big(\big(\varepsilon_n|\mathbf{p}|\big)^3\Big) \qquad\qquad \text{for}\quad \varepsilon_n|\mathbf{p}|\le 1$$

gives, using Remark 3.1.a,

$$\begin{aligned}\hat{\mathbf{D}}_n(p) &= \Big[-ip_0+\tfrac{1}{2}\varepsilon_n^2p_0^2\Big]\Big[1+O\Big(\big(\varepsilon_n|\mathbf{p}|\big)^2+\big(\varepsilon_n^2p_0\big)^2\Big)\Big]+\tfrac{1}{2}\sum_{\nu,\nu'=1}^{3}H_{\nu,\nu'}\mathbf{p}_\nu\mathbf{p}_{\nu'}+O\Big(\varepsilon_n|\mathbf{p}|^3\Big)\\ &= -ip_0+\tfrac{1}{2}\varepsilon_n^2p_0^2+\tfrac{1}{2}\sum_{\nu,\nu'=1}^{3}H_{\nu,\nu'}\mathbf{p}_\nu\mathbf{p}_{\nu'}+O\Big(\varepsilon_n|\mathbf{p}|^3+\varepsilon_n^2|p_0||\mathbf{p}|^2+\varepsilon_n^4|p_0|^3\Big)\end{aligned}$$

The claim now follows from

$$\varepsilon_n^2|p_0||\mathbf{p}|^2\le\big(\varepsilon_n^4|p_0|^3\big)^{1/3}\big(\varepsilon_n|\mathbf{p}|^3\big)^{2/3}$$

(c) Since $\hat{\mathbf{D}}_n(p)$ is periodic with respect to $\frac{2\pi}{\varepsilon_n^2}\mathbb{Z}\times\frac{2\pi}{\varepsilon_n}\mathbb{Z}^3$, we may assume that $\varepsilon_n^2\mathrm{Re}\,p_0$ and each $\varepsilon_n\mathrm{Re}\,\mathbf{p}_\nu$, $1\le\nu\le 3$ is bounded in magnitude by π. By Lemma A.1.b, $\left|\frac{\sin z}{z}\right|\le 2$ for all $z\in\mathbb{C}$ with $|\mathrm{Im}\,z|\le 1$. Since $\varepsilon_n\mathbf{p}$ runs over a compact set (independently of n and L), $e^{-\hat{h}(\varepsilon_n\mathbf{p})}$ is bounded. So

$$\left|\tfrac{1}{2}\varepsilon_n^2p_0^2e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})}\left[\frac{\sin\frac{1}{2}\varepsilon_n^2p_0}{\frac{1}{2}\varepsilon_n^2p_0}\right]^2\right|,\ \left|ip_0\,e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})}\frac{\sin\varepsilon_n^2p_0}{\varepsilon_n^2p_0}\right|\ \le\ \mathrm{const}|p_0|$$

and

$$\left|\frac{1-e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{p})}}{\varepsilon_n^2}\right|\le\mathrm{const}\tfrac{|\varepsilon_n\mathbf{p}|^2}{\varepsilon_n^2}\le\mathrm{const}|\mathbf{p}|^2$$

This gives the bound on $\left|\hat{\mathbf{D}}_n(p)\right|$.

The bounds

$$
\begin{aligned}
&\tfrac{\partial}{\partial p_0}\varepsilon_n^2\left[\tfrac{\sin(\frac{1}{2}\varepsilon_n^2 p_0)}{\frac{1}{2}\varepsilon_n^2}\right]^2 = 2\sin(\varepsilon_n^2 p_0) = O(1) && \tfrac{\partial}{\partial p_0}\left[\tfrac{\sin(\varepsilon_n^2 p_0)}{\varepsilon_n^2}\right] = \cos(\varepsilon_n^2 p_0) = O(1)\\
&\tfrac{\partial^2}{\partial p_0^2}\varepsilon_n^2\left[\tfrac{\sin(\frac{1}{2}\varepsilon_n^2 p_0)}{\frac{1}{2}\varepsilon_n^2}\right]^2 = 2\varepsilon_n^2\cos(\varepsilon_n^2 p_0) = O(\varepsilon_n^2) && \tfrac{\partial^2}{\partial p_0^2}\left[\tfrac{\sin(\varepsilon_n^2 p_0)}{\varepsilon_n^2}\right] = -\varepsilon_n^2\sin(\varepsilon_n^2 p_0) = O(\varepsilon_n^2)\\
&\tfrac{\partial}{\partial \mathbf{p}_\nu}\left[\tfrac{1}{\varepsilon_n^2}e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{p})}\right] = O(|\mathbf{p}|) && \tfrac{\partial^2}{\partial \mathbf{p}_\nu^2}\left[\tfrac{1}{\varepsilon_n^2}e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{p})}\right] = O(1+\varepsilon_n^2|\mathbf{p}|^2) = O(1)\\
&\tfrac{\partial^3}{\partial \mathbf{p}_\nu^3}\left[\tfrac{1}{\varepsilon_n^2}e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{p})}\right] = O(\varepsilon_n^2|\mathbf{p}|+\varepsilon_n^4|\mathbf{p}|^3) && \tfrac{\partial^4}{\partial \mathbf{p}_\nu^4}\left[\tfrac{1}{\varepsilon_n^2}e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{p})}\right] = O(\varepsilon_n^2+\varepsilon_n^4|\mathbf{p}|^2+\varepsilon_n^6|\mathbf{p}|^4)\\
&\qquad = O(\varepsilon_n) && \qquad = O(\varepsilon_n^2)
\end{aligned}
$$

together with

$$\varepsilon_n^2 \le \tfrac{\mathrm{const}}{1+|p_0|+|\mathbf{p}|^2} \qquad \varepsilon_n^2|p_0| \le \mathrm{const}$$

yield the bounds on the derivatives.

(d) Write $p = P + iQ$ with $P = (P_0, \mathbf{P}),\ Q = (Q_0, \mathbf{Q}) \in \mathbb{R}\times\mathbb{R}^3$. We may choose $\bar{m}(c)$ sufficiently small that, if $|P+iQ| \ge c$ and $|Q| \le \bar{m}(c)$, then

$$
\begin{aligned}
|P_0 + iQ_0| + \sum_{\nu=1}^3 |\mathbf{P}_\nu + i\mathbf{Q}_\nu|^2 &\le \Big(|P_0| + \sum_{\nu=1}^3 |\mathbf{P}_\nu|^2\Big) + \Big(|Q_0| + \sum_{\nu=1}^3 |\mathbf{Q}_\nu|^2\Big)\\
&\le 2\Big(|P_0| + \sum_{\nu=1}^3 |\mathbf{P}_\nu|^2\Big)
\end{aligned}
$$

If $\gamma_1 > 0$ is chosen small enough and Γ_5 is chosen large enough, then, for all such P, Q, we have, by parts (a) and (c),

$$
\begin{aligned}
\left|\hat{\mathbf{D}}_n(P)\right| &\ge 4\gamma_1\big(|P_0| + \sum_{\nu=1}^3 |\mathbf{P}_\nu|^2\big)\\
&\ge 2\gamma_1\big(|P_0+iQ_0| + \sum_{\nu=1}^3 |\mathbf{P}_\nu + i\mathbf{Q}_\nu|^2\big)\\
\left|\hat{\mathbf{D}}_n(P+iQ) - \hat{\mathbf{D}}_n(P)\right| &\le 2\,\Gamma_5\Big(1 + |P_0+iQ_0| + \sum_{\nu=1}^3 |\mathbf{P}_\nu + i\mathbf{Q}_\nu|^2\Big)|Q|
\end{aligned}
$$

Recalling that $|P_0+iQ_0| + \sum_{\nu=1}^3 |\mathbf{P}_\nu + i\mathbf{Q}_\nu|^2 \ge \min\left\{\frac{c}{2}, \frac{c^2}{4}\right\}$, it now suffices to choose $\bar{m}(c)$ small enough that

$$2\,\Gamma_5\bar{m}(c) \le \tfrac{1}{2}\gamma_1\min\left\{\tfrac{c}{2}, \tfrac{c^2}{4}\right\} \qquad \text{and} \qquad 2\,\Gamma_5\bar{m}(c) \le \tfrac{1}{2}\gamma_1$$

(e) Again write $p = P + iQ$ with $P = (P_0, \mathbf{P})$, $Q = (Q_0, \mathbf{Q}) \in \mathbb{R} \times \mathbb{R}^3$. Since $\hat{\mathbf{D}}_n(P + iQ)$ is periodic with respect to $P \in \frac{2\pi}{\varepsilon_n^2}\mathbb{Z} \times \frac{2\pi}{\varepsilon_n}\mathbb{Z}^3$, we may assume that $\left|\varepsilon_n^2 P_0\right| \le \pi$ and $\left|\varepsilon_n \mathbf{P}_\nu\right| \le \pi$, for each $1 \le \nu \le 3$. If the constant γ_1 was chosen small enough, then, as in part (a),

$$\operatorname{Re}\hat{\mathbf{D}}_n(P) = \tfrac{1}{2}\varepsilon_n^2 P_0^2 e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{P})}\left[\frac{\sin\frac{1}{2}\varepsilon_n^2 P_0}{\frac{1}{2}\varepsilon_n^2 P_0}\right]^2 + \mathbf{P}^2\frac{1 - e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{P})}}{\varepsilon_n^2 \mathbf{P}^2} \ge 2\gamma_1\big(\varepsilon_n^2 P_0^2 + |\mathbf{P}|^2\big)$$

Hence it suffices to prove that it is possible to choose $\bar{m} = \bar{m}(c)$ so that

$$\left|\operatorname{Re}\hat{\mathbf{D}}_n(P + iQ) - \operatorname{Re}\hat{\mathbf{D}}_n(P)\right| \le \gamma_1 |\mathbf{P}|^2 \qquad \text{when } |\mathbf{P}| \ge c \text{ and } |Q| \le \bar{m} \tag{3.2}$$

and that

$$\left|\operatorname{Re}\hat{\mathbf{D}}_n(P + iQ) - \operatorname{Re}\hat{\mathbf{D}}_n(P)\right| \le \gamma_1 c^2 \qquad \text{when } |\mathbf{P}| \le c,\ |\varepsilon_n P_0| \ge c \text{ and } |Q| \le \bar{m} \tag{3.3}$$

This is a consequence of the following bounds on the derivatives of the real parts of the three terms making up $\hat{\mathbf{D}}_n(P + iQ)$ in Remark 3.1.a. For the first term,

$$\begin{aligned}
&\left|\tfrac{d}{dt}\varepsilon_n^2(P_0 + itQ_0)^2 e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{P})}\left[\frac{\sin\frac{1}{2}\varepsilon_n^2(P_0 + itQ_0)}{\frac{1}{2}\varepsilon_n^2(P_0 + itQ_0)}\right]^2\right| \\
&\quad \le \text{const}\left[\varepsilon_n^2 |P_0 + iQ_0|\,|Q_0| + \varepsilon_n^2|P_0 + iQ_0|^2\ \varepsilon_n^2|Q_0|\right] \\
&\quad \le \text{const}\,\bar{m}
\end{aligned}$$

$$\tfrac{d}{dt}\operatorname{Re}\varepsilon_n^2(P_0 + itQ_0)^2\left[e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{P} + it\varepsilon_n \mathbf{Q})} - e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{P})}\right]\left[\frac{\sin\frac{1}{2}\varepsilon_n^2(P_0 + itQ_0)}{\frac{1}{2}\varepsilon_n^2(P_0 + itQ_0)}\right]^2\Bigg|_{t=0} = 0$$

$$\begin{aligned}
&\left|\tfrac{d^2}{dt^2}\varepsilon_n^2(P_0 + itQ_0)^2\left[e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{P} + it\varepsilon_n \mathbf{Q})} - e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{P})}\right]\left[\frac{\sin\frac{1}{2}\varepsilon_n^2(P_0 + itQ_0)}{\frac{1}{2}\varepsilon_n^2(P_0 + itQ_0)}\right]^2\right| \\
&\quad \le \text{const}\left[\varepsilon_n^2|Q_0|^2 + \varepsilon_n^2|P_0 + iQ_0|\,|Q_0|\big(\varepsilon_n|\mathbf{Q}| + \varepsilon_n^2|Q_0|\big) + \varepsilon_n^2|P_0 + iQ_0|^2\ \big(\varepsilon_n|\mathbf{Q}| + \varepsilon_n^2|Q_0|\big)^2\right] \\
&\quad \le \text{const}\,\bar{m}^2
\end{aligned}$$

For the second term,

$$\tfrac{d}{dt}\operatorname{Re}\tfrac{1 - e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{P} + it\varepsilon_n \mathbf{Q})}}{\varepsilon_n^2}\Bigg|_{t=0} = 0$$

$$\left|\tfrac{d^2}{dt^2}\tfrac{1 - e^{-\hat{\mathbf{h}}_0(\varepsilon_n \mathbf{P} + it\varepsilon_n \mathbf{Q})}}{\varepsilon_n^2}\right| \le \text{const}\tfrac{1}{\varepsilon_n^2}\big(\varepsilon_n|\mathbf{Q}|\big)^2 \le \text{const}\,\bar{m}^2$$

For the third term,

$$\left|\tfrac{d}{dt}\mathrm{Re}\, i(P_0+itQ_0)\, e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{P})}\frac{\sin\varepsilon_n^2(P_0+itQ_0)}{\varepsilon_n^2(P_0+itQ_0)}\right|$$
$$\leq \mathrm{const}|Q_0| + \mathrm{const}|P_0+iQ_0|\ \varepsilon_n^2|Q_0|$$
$$\leq \mathrm{const}\,\bar{m}$$
$$\left|\tfrac{d}{dt}\mathrm{Re}\, i(P_0+itQ_0)\left[e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{P}+it\varepsilon_n\mathbf{Q})} - e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{P})}\right]\frac{\sin\varepsilon_n^2(P_0+itQ_0)}{\varepsilon_n^2(P_0+itQ_0)}\bigg|_{t=0}\right|$$
$$= \left|\mathrm{Re}\, iP_0\left[\tfrac{d}{dt}\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{P}+it\varepsilon_n\mathbf{Q})\right]_{t=0} e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{P})}\frac{\sin\varepsilon_n^2(P_0)}{\varepsilon_n^2P_0}\right|$$
$$= \left|\varepsilon_nP_0\ \big(\mathbf{Q}\cdot\hat{\mathbf{h}}_0'(\varepsilon_n\mathbf{P})\big)\ e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{P})}\frac{\sin\varepsilon_n^2(P_0)}{\varepsilon_n^2P_0}\right|$$
$$\leq \mathrm{const}\varepsilon_n^2|P_0|\,|\mathbf{P}|\,\bar{m}$$
$$\leq \mathrm{const}|\mathbf{P}|\,\bar{m}$$
$$\left|\tfrac{d^2}{dt^2} i(P_0+itQ_0)\left[e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{P}+it\varepsilon_n\mathbf{Q})} - e^{-\hat{\mathbf{h}}_0(\varepsilon_n\mathbf{P})}\right]\frac{\sin\varepsilon_n^2(P_0+itQ_0)}{\varepsilon_n^2(P_0+itQ_0)}\right|$$
$$\leq \mathrm{const}\left[|Q_0|\big(\varepsilon_n|\mathbf{Q}|+\varepsilon_n^2|Q_0|\big) + |P_0+iQ_0|\ \big(\varepsilon_n|\mathbf{Q}|+\varepsilon_n^2|Q_0|\big)^2\right]$$
$$\leq \mathrm{const}\,\bar{m}^2$$

Now choose $\bar{m} = \bar{m}(c)$ small enough that (3.2) and (3.3) are satisfied. □

4 The Covariance

The covariance for the fluctuation integral in [8] is

$$C^{(n)} = (\tfrac{a}{L^2}Q^*Q + \Delta^{(n)})^{-1}$$

where

$$\Delta^{(n)} = \begin{Bmatrix} \big(\mathbb{1} + \mathfrak{Q}_n Q_n D_n^{-1} Q_n^*\big)^{-1}\mathfrak{Q}_n & \text{if } n \ge 1 \\ D_0 & \text{if } n = 0 \end{Bmatrix} : \mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$$

See [7, (1.15) and (1.14)]. In Lemma 4.2 we study the properties of $\Delta^{(n)}$ and in Corollary 4.5 we study the properties of $C^{(n)}$ and its square root.

Remark 4.1. Let $n \ge 1$.

(a) The operator $\Delta^{(n)}$ is the periodization of a translation invariant operator $\boldsymbol{\Delta}^{(n)}$, acting on $L^2\big(\mathbb{Z}\times\mathbb{Z}^3\big)$, whose Fourier transform is

$$\begin{aligned}\hat{\boldsymbol{\Delta}}^{(n)}(k) &= \hat{\mathfrak{Q}}_n(k)\Big(1 + \hat{\mathfrak{Q}}_n(k)\sum_{\ell\in\hat{\mathcal{B}}_n} u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\Big)^{-1} \\ &= \frac{\hat{\mathfrak{Q}}_n(k)\ \hat{\mathbf{D}}_n(k)}{\hat{\mathbf{D}}_n(k) + \hat{\mathfrak{Q}}_n(k)\sum_\ell u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\hat{\mathbf{D}}_n(k)}\end{aligned}$$

with $k \in \mathbb{C}\times\mathbb{C}^3$, where $u_n(p)$ and $\hat{\mathcal{B}}_n$ were defined in parts (b) and (e) of Remark 2.1, respectively.

(b) $\hat{\boldsymbol{\Delta}}^{(n)}(k)$ is invariant under $\mathbf{k}_\nu \to -\mathbf{k}_\nu$ for each $1 \le \nu \le 3$.

(c) $\hat{\boldsymbol{\Delta}}^{(n)}(k)$ has nonnegative real part when k is real.

Lemma 4.2. *There are constants $m_1 > 0$, γ_2, Γ_6 and Γ_7, such that, for $L > \Gamma_2$, the following hold.*

(a) $\hat{\boldsymbol{\Delta}}^{(n)}(k)$ *is analytic on* $|\mathrm{Im}\, k| < 3m_1$.

(b) For all $k \in \mathbb{C}\times\mathbb{C}^3$ with $|\mathrm{Im}\, k| \le 3m_1$.

$$\hat{\boldsymbol{\Delta}}^{(n)}(k) = -ik_0 + \big(\tfrac{1}{a_n} + \tfrac{\varepsilon_n^2}{2}\big)k_0^2 + \tfrac{1}{2}\sum_{\nu,\nu'=1}^{3} H_{\nu,\nu'}\mathbf{k}_\nu\mathbf{k}_{\nu'} + O\big(|k|^3\big)$$

$$\hat{\boldsymbol{\Delta}}^{(n)}(k)\,\hat{\mathbf{D}}_n^{-1}(k) = 1 + \tfrac{ik_0}{a_n} + O\big(|k|^2\big)$$

The higher order part $O(\,\cdot\,)$ is uniform in n and L.

(c) $\big|\hat{\mathbf{\Delta}}^{(n)}(k)\big| \le 2a$ *and* $\big|\frac{\partial}{\partial k_\nu}\hat{\mathbf{\Delta}}^{(n)}(k)\big|, \big|\frac{\partial^2}{\partial k_\nu \partial k_{\nu'}}\hat{\mathbf{\Delta}}^{(n)}(k)\big| \le \Gamma_7$ *for all* $0 \le \nu, \nu' \le 3$ *and* $k \in \mathbb{C} \times \mathbb{C}^3$ *with* $|\mathrm{Im}\, k| < 3m_1$.

(d) There is a function $\rho(c) > 0$*, which is defined for all* $c > 0$ *and which depends only on* m_1, $\mathfrak{q}$, $\hat{\mathbf{h}}_0$ *and* a *and, in particular, is independent of* n *and* L*, such that*

$$\mathrm{Re}\,\hat{\mathbf{\Delta}}^{(n)}(k) \ge \rho(c)$$

for all $k \in \mathbb{C} \times \mathbb{C}^3$ *with* $|k| \ge c$ *and* $|\mathrm{Im}\, k| \le 3m_1$.

(e) For all $k \in \mathbb{C} \times \mathbb{C}^3$ *with* $|\mathrm{Im}\, k| \le 3m_1$ *and* $|\mathrm{Re}\, k_\nu| \le \pi$ *for all* $0 \le \nu \le 3$,

$$\big|\hat{\mathbf{\Delta}}^{(n)}(k)\big| \ge \gamma_2 \big|\hat{\mathbf{D}}_n(k)\big|$$

(f) $\hat{\mathbf{D}}_n^{-1}(p)\,\hat{\mathbf{\Delta}}^{(n)}(p)$ *is analytic on* $|\mathrm{Im}\, p| \le 3m_1$*. Furthermore, for all* $p \in \mathbb{C} \times \mathbb{C}^3$ *with* $|\mathrm{Im}\, p| \le 3m_1$,

$$\big|\hat{\mathbf{D}}_n^{-1}(p)\,\hat{\mathbf{\Delta}}^{(n)}(p)\big| \le \frac{\Gamma_6}{1 + |p_0| + \sum_{\nu=1}^3 |\mathbf{p}_\nu|^2}$$

Here, as usual, $|p_0|$ *and* $|\mathbf{p}_\nu|$ *refer to the magnitudes of the smallest representatives of* $p_0 \in \mathbb{C}$ *and* $\mathbf{p}_\nu \in \mathbb{C}$ *in* $\mathbb{C}/\frac{2\pi}{\varepsilon_n^2}\mathbb{Z}$ *and* $\mathbb{C}/\frac{2\pi}{\varepsilon_n}\mathbb{Z}$ *respectively.*

(g) $\big|\frac{\partial}{\partial \mathbf{k}_\nu}\hat{\mathbf{\Delta}}^{(n)}(k)\big| \le \Gamma_7 |\mathbf{k}_\nu|$ *for all* $1 \le \nu \le 3$ *and* $k \in \mathbb{C} \times \mathbb{C}^3$ *with* $|\mathrm{Im}\, k| < 3m_1$.

Proof. We first prove part (b). Using that

- $u_n(k)^{2\mathfrak{q}} = 1 + O\big(|k|^2\big)$ by Lemma 2.2.b
- $\hat{\mathfrak{Q}}_n(k) = a_n + O\big(|k|^2\big)$ by Proposition 2.4.b
- $|\hat{\mathbf{D}}_n(k)| \le \mathrm{const}\big(|k_0| + |\mathbf{k}|^2\big)$ by Lemma 3.2.c

and that, for $\ell \ne 0$,

- $|u_n(k+\ell)|^{2\mathfrak{q}} \le \mathrm{const}|k|^{2\mathfrak{q}} \prod_{\nu=0}^3 \frac{1}{(|\ell|_\nu + \pi)^{2\mathfrak{q}}}$ by Lemma 2.2.a
- $|\hat{\mathbf{D}}_n^{-1}(k+\ell)| \le \mathrm{const}$ by Lemma 3.2.d

we obtain, by Remark 4.1.a and Lemma 3.2.b,

$$\begin{aligned}
\hat{\mathbf{\Delta}}^{(n)}(k) &= \frac{\hat{\mathfrak{Q}}_n(k)\hat{\mathbf{D}}_n(k)}{\hat{\mathfrak{Q}}_n(k)u_n(k)^{2\mathfrak{q}} + \hat{\mathbf{D}}_n(k) + O\big(|k|^3\big)} \\
&= \hat{\mathbf{D}}_n(k)\frac{a_n + O\big(|k|^2\big)}{a_n + \hat{\mathbf{D}}_n(k) + O\big(|k|^2\big)}
\end{aligned}$$

$$\begin{aligned}
&= \hat{\mathbf{D}}_n(k)\Big\{1 - \tfrac{1}{a_n}\hat{\mathbf{D}}_n(k) + O\big(|k|^2\big)\Big\} \\
&= -ik_0 + \tfrac{1}{2}\varepsilon_n^2 k_0^2 + \tfrac{1}{2}\sum_{\nu,\nu'=1}^{3} H_{\nu,\nu'}\mathbf{k}_\nu \mathbf{k}_{\nu'} - \tfrac{1}{a_n}\hat{\mathbf{D}}_n(k)^2 + O\big(|k|^3\big) \\
&= -ik_0 + \big(\tfrac{1}{a_n} + \tfrac{\varepsilon_n^2}{2}\big)k_0^2 + \tfrac{1}{2}\sum_{\nu,\nu'=1}^{3} H_{\nu,\nu'}\mathbf{k}_\nu \mathbf{k}_{\nu'} + O\big(|k|^3\big)
\end{aligned}$$

This also shows that, in a neighbourhood of the origin, $\hat{\mathbf{\Delta}}^{(n)}(k)$ is analytic and bounded in magnitude by $2a$. For the second expansion,

$$\begin{aligned}
\hat{\mathbf{\Delta}}^{(n)}(k)\,\hat{\mathbf{D}}_n^{-1}(k) &= \frac{a_n + O\big(|k|^2\big)}{a_n + \hat{\mathbf{D}}_n(k) + O\big(|k|^2\big)} = 1 - \tfrac{1}{a_n}\hat{\mathbf{D}}_n(k) + O\big(|k|^2\big) \\
&= 1 + \tfrac{ik_0}{a_n} + O\big(|k|^2\big)
\end{aligned}$$

(a), (c) We first prove the analyticity and the bound $\big|\hat{\mathbf{\Delta}}^{(n)}(k)\big| \le 2a$ of part (c). We have already done so for a neighbourhood of the origin. So it suffices to consider $|k| > c_0$, for some suitably small c_0. Since $\hat{\mathbf{\Delta}}^{(n)}$ is periodic with respect to $2\pi\mathbb{Z}\times 2\pi\mathbb{Z}^3$, it suffices to consider k in the set

$$M(m_1) = \Big\{\ k \in \mathbb{C}\times\mathbb{C}^3\ \Big|\ |\mathrm{Re}\, k_\nu| \le \pi \text{ for all } 0 \le \nu \le 3, |\mathrm{Im}\, k| \le 3m_1,\ |k| > c_0\ \Big\}$$

Recall, from Remarks 2.1.d and 3.1.b, that $u_n(p)$ and $\hat{\mathbf{D}}_n(p)$ are entire. If m_1 is small enough, then, by Lemma 3.2.d, the functions $k \mapsto \hat{\mathbf{D}}_n(k+\ell)$, $\ell \in \hat{\mathcal{B}}_n$, $n \ge 1$, have no zeroes in $M(m_1)$. Hence each term in the infinite sum

$$1 + \hat{\mathfrak{Q}}_n(k)\sum_{\ell\in\hat{\mathcal{B}}_n} u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)$$

is analytic and it suffices to prove that, for $k \in M(m_1)$, the sum converges uniformly and $\big|1 + \hat{\mathfrak{Q}}_n(k)\sum_{\ell\in\hat{\mathcal{B}}_n} u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\big| \ge \frac{6}{10}$.

By Proposition 2.4.a, Remark 2.1.d and Lemma 3.2.d, there is an $l > 0$ such that

$$|\hat{\mathfrak{Q}}_n(k)| \sum_{\substack{\ell\in\hat{\mathcal{B}}_n\\ |\ell|\ge l}} \big|u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\big| \le \tfrac{6}{5}a \sum_{\substack{\ell\in\hat{\mathcal{B}}_n\\ |\ell|\ge l}} \tfrac{1}{\gamma_1\pi}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_\nu|+\pi}\big)^{2\mathfrak{q}} \le \tfrac{\mathrm{const}}{l} \le \tfrac{2}{10} \tag{4.1}$$

Hence we have uniform convergence and

$$\Big|1 + \hat{\mathfrak{Q}}_n(k)\sum_{\ell\in\hat{\mathcal{B}}_n} u_n(k+\ell)^2\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\Big| \ge \Big|1 + \hat{\mathfrak{Q}}_n(k)\sum_{\substack{\ell\in\hat{\mathcal{B}}_n\\ |\ell|< l}} u_n(k+\ell)^2\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\Big| - \tfrac{2}{10}$$

For real k and every $\ell \in \hat{\mathcal{B}}_n$, the real part of $\hat{\mathfrak{Q}}_n(k)u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)$ is nonnegative. Consequently, if m_1 is small enough and $|\mathrm{Im}\, k| \le 3m_1$

$$\mathrm{Re}\,\hat{\mathfrak{Q}}_n(k) \sum_{\ell\in\hat{\mathcal{B}}_n\,|\ell|<l} u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell) \ge -\tfrac{2}{10} \tag{4.2}$$

since, for all $k \in M(m_1)$ and $\ell \in \hat{\mathcal{B}}_n$ with $|\ell| < l$,

- $\hat{\mathbf{D}}_n(k+\ell)$ is bounded away from zero (by Lemma 3.2.d) and has bounded first derivative (by Lemma 3.2.c) and
- $u_n(k+\ell)$ and $\hat{\mathfrak{Q}}_n(k)$ are bounded with bounded first derivatives (by analyticity, Remark 2.1.d and Proposition 2.4.a).

So, by (4.2),

$$\Big|1 + \hat{\mathfrak{Q}}_n(k) \sum_{\ell\in\hat{\mathcal{B}}_n\,|\ell|<l} u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\Big| \ge \tfrac{8}{10}$$

All of this is uniform in n and L.

Shrinking m_1 by a factor of 2, the bounds $\big|\frac{\partial}{\partial k_\nu}\hat{\mathbf{\Delta}}^{(n)}(k)\big|, \big|\frac{\partial^2}{\partial k_\nu \partial k_{\nu'}}\hat{\mathbf{\Delta}}^{(n)}(k)\big| \le \Gamma_7$, with Γ_7 being the maximum of $4a$ divided by the original $3m_1$ (for first order derivatives) and $16a$ divided by the square of the original $3m_1$ (for second order derivatives), follow by the Cauchy integral formula.

(d) Denote the real and imaginary parts of the numerator and denominator by

$$R_n = \mathrm{Re}\,\hat{\mathfrak{Q}}_n(k) R_d = \mathrm{Re}\Big[1 + \hat{\mathfrak{Q}}_n(k) \sum_{\ell\in\hat{\mathcal{B}}_n} u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\Big]$$
$$I_n = \mathrm{Im}\,\hat{\mathfrak{Q}}_n(k)\ I_d = \mathrm{Im}\Big[1 + \hat{\mathfrak{Q}}_n(k) \sum_{\ell\in\hat{\mathcal{B}}_n} u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\Big]$$

By Proposition 2.4.a and (4.1), (4.2),

$$R_n \ge \tfrac{a}{2} \qquad R_d \ge \tfrac{6}{10}$$

By Proposition 2.4.a, Remark 2.1.d and Lemma 3.2.d,

$$I_d \le \Big|\hat{\mathfrak{Q}}_n(k) \sum_{\ell\in\hat{\mathcal{B}}_n} u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\Big| \le a\tilde{I}_d(c)$$

where $\tilde{I}_d(c) = \frac{6}{5}a\sum_{\ell\in\hat{\mathcal{B}}_n} \frac{1}{\gamma_1 \min\{c/2, c^2/4\}} \prod_{\nu=0}^{3} \frac{24^{2\mathfrak{q}}}{(|\ell_\nu|+\pi)^{2\mathfrak{q}}}$. When k is real, $\hat{\mathfrak{Q}}_n(k)$ is real. By analyticity and Proposition 2.4.a, the first order derivatives of $\hat{\mathfrak{Q}}_n(k)$ are bounded.

So if m_1 is small enough, $|I_n| \le \frac{2}{10\tilde{I}_d(c)}$. So

$$\mathrm{Re}\,\hat{\mathbf{\Delta}}^{(n)}(k) = \tfrac{R_nR_d+I_nI_d}{R_d^2+I_d^2} \ge \tfrac{a(3/10)-a(2/10)}{(1+a\tilde{I}_d(c))^2} = \tfrac{a}{10(1+a\tilde{I}_d(c))^2}$$

(e) By Remark 2.1.d, Proposition 2.4.a and Lemma 3.2.d,

$$\begin{aligned}
&\Big|1+\hat{\mathfrak{Q}}_n(k)\sum_{\ell\in\hat{\mathcal{B}}_n} u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\Big| \\
&\qquad \le \big|\hat{\mathfrak{Q}}_n(k)\ u_n(k)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k)\big| + 1 + \sum_{\mathbf{0}\ne\ell\in\hat{\mathcal{B}}_n}\big|\hat{\mathfrak{Q}}_n(k)\ u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\big| \\
&\qquad \le \tfrac{6}{5}a\ \big(\tfrac{24}{\pi}\big)^{8\mathfrak{q}}\ \big|\hat{\mathbf{D}}_n^{-1}(k)\big| + 1 + \tfrac{6}{5}a\sum_{\mathbf{0}\ne\ell\in\hat{\mathcal{B}}_n}\tfrac{1}{\gamma_1\pi}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_\nu|+\pi}\big)^{2\mathfrak{q}}
\end{aligned}$$

and so, by Lemma 3.2.c,

$$\begin{aligned}
&\big|\hat{\mathbf{D}}_n(k)\big|\Big|1+\hat{\mathfrak{Q}}_n(k)\sum_{\ell\in\hat{\mathcal{B}}_n} u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\Big| \\
&\qquad \le \tfrac{6}{5}a\big(\tfrac{24}{\pi}\big)^{8\mathfrak{q}} + \Gamma_5\ \big(\pi+1+3(\pi+1)^2\big)\Big[1+\tfrac{6}{5}a\sum_{\mathbf{0}\ne\ell\in\hat{\mathcal{B}}_n}\tfrac{1}{\gamma_1\pi}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_\nu|+\pi}\big)^{2\mathfrak{q}}\Big]
\end{aligned}$$

(f) By part (c) of this lemma, Remark 3.1.b and Lemma 3.2.d, it suffices to consider $p=k$ with $|\mathrm{Im}\,k| \le 3m_1$ and $|k| < c_0 < \pi$ where c_0 is sufficiently small. Now

$$\begin{aligned}
&\hat{\mathbf{D}}_n^{-1}(k)\hat{\mathbf{\Delta}}^{(n)}(k) \\
&\qquad = \frac{\hat{\mathfrak{Q}}_n(k)}{\hat{\mathfrak{Q}}_n(k)u_n(k)^{2\mathfrak{q}} + \hat{\mathbf{D}}_n(k)\big[1+\hat{\mathfrak{Q}}_n(k)\sum_{\mathbf{0}\ne\ell\in\hat{\mathcal{B}}_n} u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\big]}
\end{aligned}$$

By Proposition 2.4.a, Remark 2.1.d, Lemma 2.2.b and parts (c) and (d) of Lemma 3.2, we may choose $c_0 > 0$ so that

$$\Big|\hat{\mathbf{D}}_n(k)\big[1+\hat{\mathfrak{Q}}_n(k)\sum_{\mathbf{0}\ne\ell\in\hat{\mathcal{B}}_n} u_n(k+\ell)^{2\mathfrak{q}}\ \hat{\mathbf{D}}_n^{-1}(k+\ell)\big]\Big| < \tfrac{a}{5}$$

and

$$\big|\hat{\mathfrak{Q}}_n(k)\big|\ \big|u_n(k)^{2\mathfrak{q}}-1\big| < \tfrac{a}{5}$$

for all $|k| < c_0$ with $|\mathrm{Im}\, k| \le 3m_1$. That does it.

(g) Fix any $1 \le \nu \le 3$. Observe that

$$\tfrac{\partial \hat{\mathbf{\Delta}}^{(n)}}{\partial k_\nu}(k) = \mathbf{k}_\nu \int_0^1 \tfrac{\partial^2 \hat{\mathbf{\Delta}}^{(n)}}{\partial k_\nu^2}\big(k(t)\big)\ dt \qquad \text{with} \quad k(t)_{\nu'} = \begin{cases} k_{\nu'} & \text{if } \nu' \ne \nu \\ tk_\nu, & \text{if } \nu = \nu' \end{cases}$$

since $\tfrac{\partial \hat{\mathbf{\Delta}}^{(n)}}{\partial k_\nu}\big(k(0)\big) = 0$, by Remark 4.1.b. Now apply the bound on the second derivative from part (c). $\square$

We next consider the "resolvents"

$$R_\zeta^{(n)} = \big(\zeta \mathbb{1} - \tfrac{a}{L^2} Q^* Q - \Delta^{(n)}\big)^{-1}$$

in preparation for studying C_n and $\sqrt{C_n}$. We shall use [5, Lemma 14], with

$$\varepsilon_T = 1 \qquad \varepsilon_X = 1 \qquad L_T = L^2 \qquad L_X = L \qquad \mathcal{L}_T = L_{\mathrm{tp}} \qquad \mathcal{L}_X = L_{\mathrm{sp}}$$

and

$$\mathcal{X}_{\mathrm{fin}} = \mathcal{X}_0^{(n)} \quad \mathcal{Z}_{\mathrm{fin}} = \mathbb{Z} \times \mathbb{Z}^3 \quad \mathcal{X}_{\mathrm{crs}} = \mathcal{X}_{-1}^{(n+1)} \quad \mathcal{Z}_{\mathrm{crs}} = L^2\mathbb{Z} \times L\mathbb{Z}^3 \quad \mathcal{B} = \mathcal{B}^+$$

We wish to apply [5, Lemma 14], with $A \to D_n = \frac{a}{L^2} Q^* Q + \Delta^{(n)}$ (scaled). By [5, Lemmas 9.b and 5.b], for each $\ell, \ell' \in \hat{\mathcal{B}} = \hat{\mathcal{B}}^+$,

$$a_{\mathfrak{k}}(\ell, \ell') \to d_{n,\mathfrak{k}}(\ell, \ell') = \tfrac{a}{L^2} u_+(\mathfrak{k} + \ell)^{\mathfrak{q}} u_+(\mathfrak{k} + \ell')^{\mathfrak{q}} + \delta_{\ell,\ell'} \hat{\Delta}^{(n)}(\mathfrak{k} + \ell) \tag{4.3}$$

Here we are denoting

- momenta dual to the L–lattice by $\mathfrak{k} \in \big(\mathbb{R}/\frac{2\pi}{L^2}\mathbb{Z}\big) \times \big(\mathbb{R}^3/\frac{2\pi}{L}\mathbb{Z}^3\big)$ and
- momenta dual to the unit lattice $\mathbb{Z} \times \mathbb{Z}^3$ by $k \in \big(\mathbb{R}/2\pi\mathbb{Z}\big) \times \big(\mathbb{R}^3/2\pi\mathbb{Z}^3\big)$ and decompose $k = \mathfrak{k} + \ell$ or $k = \mathfrak{k} + \ell'$ with $\mathfrak{k}$ in a fundamental cell for $\big(\mathbb{R}/\frac{2\pi}{L^2}\mathbb{Z}\big) \times \big(\mathbb{R}^3/\frac{2\pi}{L}\mathbb{Z}^3\big)$ and $\ell, \ell' \in \big(\frac{2\pi}{L^2}\mathbb{Z}/2\pi\mathbb{Z}\big) \times \big(\frac{2\pi}{L}\mathbb{Z}^3/2\pi\mathbb{Z}^3\big) = \hat{\mathcal{B}}^+$.

We also use $d_{n,\mathfrak{k}}$ to denote the $\hat{\mathcal{B}}^+ \times \hat{\mathcal{B}}^+$ matrix $\big[d_{n,\mathfrak{k}}(\ell, \ell')\big]_{\ell,\ell' \in \hat{\mathcal{B}}^+}$. Observe, by Remark 2.1.d and Lemma 4.2.a, that $d_{n,\mathfrak{k}}\big(\ell, \ell'\big)$ is analytic in the strip $|\mathrm{Im}\, \mathfrak{k}| < 3m_1$.

Let $\big[v_\ell\big]_{\ell \in \hat{\mathcal{B}}^+}$ and $\big[w_\ell\big]_{\ell \in \hat{\mathcal{B}}^+}$ be any vectors in $L^2(\hat{\mathcal{B}}^+)$. Then, if $\mathfrak{k} = \mathbb{L}^{-1}(k)$,

$$\begin{aligned} &\big\langle \bar{v}, d_{n,\mathfrak{k}} w \big\rangle \\ &= \tfrac{a}{L^2} \Big[\sum_{\ell \in \hat{\mathcal{B}}^+} u_+(\mathbb{L}^{-1}(k) + \ell)^{\mathfrak{q}} \overline{v_\ell} \Big] \Big[\sum_{\ell \in \hat{\mathcal{B}}^+} u_+(\mathbb{L}^{-1}(k) + \ell)^{\mathfrak{q}} w_\ell \Big] + \sum_{\ell \in \hat{\mathcal{B}}^+} \hat{\Delta}^{(n)}(\mathbb{L}^{-1}(k) + \ell) \overline{v_\ell} w_\ell \\ &= \sum_{\ell, \ell' \in \hat{\mathcal{B}}_1} \overline{v_{\mathbb{L}(\ell)}}\ d_{n,k}^{(s)}(\ell, \ell')\ w_{\mathbb{L}(\ell')} \end{aligned}$$

where the "scaled" matrix

$$d^{(s)}_{n,k}(\ell,\ell') = \tfrac{a}{L^2} u_+\big(\mathbb{L}^{-1}(k+\ell)\big)^{\mathfrak{q}} u_+\big(\mathbb{L}^{-1}(k+\ell')\big)^{\mathfrak{q}} + \delta_{\ell,\ell'}\hat\Delta^{(n)}\big(\mathbb{L}^{-1}(k+\ell)\big)$$

Lemma 4.3. *There are constants $m_2, \lambda_0, \Gamma_8 > 0$, such that, for all $L > \Gamma_2$ and $k \in \mathbb{C}\times\mathbb{C}^3$ with $|\operatorname{Im} k| < 3m_2$, the following hold.*

(a) Write $\mathfrak{k} = \mathbb{L}^{-1}(k)$. For both the operator and (matrix) ℓ^1–ℓ^∞ norms

$$\|d_{n,\mathfrak{k}}\| \le \Gamma_8 \qquad \|d^{(s)}_{n,k}\| \le \Gamma_8$$

(b) Let $\lambda \in \mathbb{C}$ be within a distance $\frac{\lambda_0}{L^2}$ of the negative real axis. Then the resolvent

$$\big\|\big(\lambda\mathbb{1} - d^{(s)}_{n,k}\big)^{-1}\big\| \le \Gamma_8 L^2$$

This is true for both the operator and (matrix) ℓ^1–ℓ^∞ norms.

Proof. Since $d_{n,\mathfrak{k}+p}(\ell,\ell') = d_{n,\mathfrak{k}}(\ell+p,\ell'+p)$, for all $p,\ell,\ell' \in \hat{\mathcal{B}}^+$, we may always assume that $|\operatorname{Re}\mathfrak{k}_0| \le \frac{\pi}{L^2}$ and $|\operatorname{Re}\mathfrak{k}_\nu| \le \frac{\pi}{L}$ for $1 \le \nu \le 3$.

(a) By Lemmas 2.3.a and 4.2.c, the matrices

$$\big[\tfrac{a}{L^2}u_+(\mathfrak{k}+\ell)^{\mathfrak{q}}u_+(\mathfrak{k}+\ell')^{\mathfrak{q}}\big]_{\ell,\ell'\in\hat{\mathcal{B}}^+} \quad\text{and}\quad \big[\tfrac{a}{L^2}u_+\big(\mathbb{L}^{-1}(k+\ell)\big)^{\mathfrak{q}}u_+\big(\mathbb{L}^{-1}(k+\ell')\big)^{\mathfrak{q}}\big]_{\ell,\ell'\in\hat{\mathcal{B}}_1}$$

have finite L^1–L^∞ norms and the matrix elements of

$$\big[\delta_{\ell,\ell'}\hat\Delta^{(n)}(\mathfrak{k}+\ell)\big]_{\ell,\ell'\in\hat{\mathcal{B}}^+} \quad\text{and}\quad \big[\delta_{\ell,\ell'}\hat\Delta^{(n)}\big(\mathbb{L}^{-1}(k+\ell)\big)\big]_{\ell,\ell'\in\hat{\mathcal{B}}^1}$$

are all bounded.

(b) *Case $|k| \le c_0$, with c_0 being chosen later in this case:* We first consider those diagonal matrix elements of $d^{(s)}_{n,k}(\ell,\ell')$ having k,ℓ such that $|\mathbb{L}^{-1}(k+\ell)| < \tilde c_0$, where $\tilde c_0 > 0$ is a small number to be chosen shortly. This, and all other constants chosen in the course of this argument are to be independent of L. Then, by Lemma 4.2.b, we have the following.

- If at least one of ℓ_ν, $1 \le \nu \le 3$ is nonzero, then $\operatorname{Re}\hat\Delta^{(n)}\big(\mathbb{L}^{-1}(k+\ell)\big) \ge \frac{c_1}{L^2}$, provided m_2 and $\tilde c_0$ are chosen small enough. Here c_1 and the constraints on m_2 and $\tilde c_0$ depend only on the largest and smallest eigenvalues of $\big[H_{\nu,\nu'}\big]$, a_n and the $O(|k|^3)$. To see this, denote by $\mathbf{k}$ and $\boldsymbol{\ell}$ the spatial parts of k and ℓ and observe that
  - ∘ $\big|\operatorname{Re}\mathbb{L}^{-1}(\mathbf{k}+\boldsymbol{\ell})\big| \ge \frac{1}{L}\max\big\{\pi, \frac{1}{2}|\boldsymbol{\ell}|\big\}$,

- ◦ $\left|\mathrm{Im}\,\mathbb{L}^{-1}(\mathbf{k}+\boldsymbol{\ell})\right| = \frac{1}{L}\left|\mathrm{Im}\,\mathbf{k}\right| \le \frac{3m_2}{L}$ and
- ◦ $\left|\mathrm{Im}\,\mathbb{L}^{-1}(k+\ell)_0\right| = \frac{1}{L^2}\left|\mathrm{Im}\,k_0\right| \le \frac{3m_2}{L^2}$.

  In controlling the contribution from $O\Big(\left|\mathbb{L}^{-1}(k+\ell)\right|^3\Big)$ when $\frac{|\ell_0|}{L^2}$ is larger than $\frac{|\boldsymbol{\ell}|}{L}$, we have to use that, in this case, the real part of $\big(\frac{1}{a_n}+\frac{\varepsilon_n^2}{2}\big)\big(\mathbb{L}^{-1}(k+\ell)\big)_0^2$ is at least a strictly positive constant times $\frac{\ell_0^2}{L^4}$.

- If $\ell_\nu = 0$ for all $1 \le \nu \le 3$ but $\ell_0 \ne 0$, then $-\mathrm{sgn}\,\ell_0\ \mathrm{Im}\,\hat{\Delta}^{(n)}\big(\mathbb{L}^{-1}(k+\ell)\big) \ge \frac{\pi}{2L^2}$, provided m_2 and $\tilde{c}_0$ is chosen small enough. To see this observe that
  - ◦ $\left|\mathrm{Re}\,\mathbb{L}^{-1}(\mathbf{k}+\boldsymbol{\ell})\right| = \frac{1}{L}|\mathrm{Re}\,\mathbf{k}| \le \sqrt{3}\frac{\pi}{L}$,
  - ◦ $\left|\mathrm{Im}\,\mathbb{L}^{-1}(\mathbf{k}+\boldsymbol{\ell})\right| = \frac{1}{L}\left|\mathrm{Im}\,\mathbf{k}\right| \le \frac{3m_2}{L}$,
  - ◦ $\mathrm{sgn}\,\ell_0\ \mathrm{Re}\,\mathbb{L}^{-1}(k+\ell)_0 \ge \frac{1}{L^2}\max\big\{\pi, \frac{1}{2}|\ell_0|\big\}$,
  - ◦ $\left|\mathrm{Re}\,\mathbb{L}^{-1}(k+\ell)_0\right| \le \frac{3}{2L^2}|\ell_0|$ and
  - ◦ $\left|\mathrm{Im}\,\mathbb{L}^{-1}(k+\ell)_0\right| = \frac{1}{L^2}\left|\mathrm{Im}\,k_0\right| \le \frac{3m_2}{L^2}$.
- If $\ell \ne 0$, then, by parts (a) and (b) of Lemma 2.3,

$$\tfrac{a}{L^2}\left|u_+\big(\mathbb{L}^{-1}(k+\ell)\big)^{2\mathfrak{q}}\right| \le \tfrac{a}{L^2}|k|\Big[\prod_{\nu=0}^{3}\tfrac{24}{|\ell_\nu|+\pi}\Big]^{2q}$$

- If $\ell = 0$ then $\mathrm{Re}\,\big\{\frac{a}{L^2}u_+\big(\mathbb{L}^{-1}(k)\big)^{2q} + \hat{\Delta}^{(n)}\big(\mathbb{L}^{-1}(k)\big)\big\} \ge \frac{a}{2L^2}$, provided m_2 and $\tilde{c}_0$ are chosen small enough. To see this observe that
  - ◦ $\left|u_+\big(\mathbb{L}^{-1}(k)\big)-1\right| \le 4^3|k|^2$, by Lemma 2.3.c and $\left|u_+\big(\mathbb{L}^{-1}(k)\big)\right| \le \big(\frac{24}{\pi}\big)^4$ by Lemma 2.3.a.
  - ◦ $\mathrm{Re}\,\hat{\Delta}^{(n)}\big(\mathbb{L}^{-1}(k)\big) \ge -c_2\big(\frac{m_2}{L^2}+\tilde{c}_0\big(\frac{|k|}{L}\big)^2\big)$ where c_2 depends only on a_n, the largest eigenvalue of $\big[H_{\nu,\nu'}\big]$, and the $O(|k|^3)$.

Note that we have now fixed $\tilde{c}_0$. Now we consider the remaining matrix elements.

- For the remaining diagonal matrix elements we have $|\mathbb{L}^{-1}(k+\ell)| \ge \tilde{c}_0$ and then, by Lemma 4.2.d, $\mathrm{Re}\,\hat{\Delta}^{(n)}\big(\mathbb{L}^{-1}(k+\ell)\big) \ge \rho(\tilde{c}_0)$
- Finally, the off–diagonal matrix elements of $d_{n,k}^{(s)}$ obey, by parts (a) and (b) of Lemma 2.3,

$$\tfrac{a}{L^2}\left|u_+\big(\mathbb{L}^{-1}(k+\ell)\big)^{\mathfrak{q}}u_+\big(\mathbb{L}^{-1}(k+\ell')\big)^{\mathfrak{q}}\right| \le \tfrac{a}{L^2}|k|\Big[\prod_{\nu=0}^{3}\tfrac{24}{|\ell_\nu|+\pi}\Big]^{q}\Big[\prod_{\nu=0}^{3}\tfrac{24}{|\ell'_\nu|+\pi}\Big]^{q}$$

  Hence the off–diagonal part of $d_{n,k}^{(s)}$ has Hilbert-Schmidt, matrix and, as $\mathfrak{q} > 1$, L^1–L^∞ norms all bounded by a universal constant times $\frac{a}{L^2}|k|$.

Thus $\lambda\mathbb{1} - d_{n,k}^{(s)}$ has diagonal matrix elements of magnitude at least

$$\min\big\{\tfrac{c_1}{L^2}, \tfrac{\pi}{2L^2}, \tfrac{a}{2L^2}, \rho(\tilde{c}_0)\big\} - \tfrac{a}{L^2}|k|\Big[\prod_{\nu=0}^{3}\tfrac{24}{|\ell_\nu|+\pi}\Big]^{2q} - \tfrac{\lambda_0}{L^2}$$

and off diagonal part with L^1–L^∞ norm bounded by a universal constant times $\frac{a}{L^2}|k|$. It now suffices to choose c_0 and λ_0 small enough that every diagonal matrix element has magnitude at least $\frac{1}{2L^2}\min\big\{c_1, \frac{\pi}{2}, \frac{a}{2}, \rho(\tilde c_0)\big\}$ and the off diagonal part has L^1–L^∞ norm bounded by $\frac{1}{4L^2}\min\big\{c_1, \frac{\pi}{2}, \frac{a}{2}, \rho(\tilde c_0)\big\}$ and then do a Neumann expansion.

(b) *Case* $|k| \ge c_0$*, with the* c_0 *just chosen:* We may assume that $|\mathrm{Re}\, k_\nu| \le \pi$ for each $0 \le \nu \le 3$.

- If $|\mathbb{L}^{-1}(k+\ell)| < \tilde c_0$ and $\ell_\nu \ne 0$ for at least one $1 \le \nu \le 3$ or if $|\mathbb{L}^{-1}(k+\ell)| \ge \tilde c_0$, then $\mathrm{Re}\,\hat\Delta^{(n)}\big(\mathbb{L}^{-1}(k+\ell)\big) \ge \min\big\{\rho(\tilde c_0), \frac{c_1}{L^2}\big\}$. The proof of this given in the case $|k| \le c_0$ applies now too.
- If $|\mathbb{L}^{-1}(k+\ell)| < \tilde c_0$, $\ell_\nu = 0$ for all $\nu \ge 1$ then

$$\begin{aligned} \mathrm{Re}\,\hat\Delta^{(n)}\big(\mathbb{L}^{-1}(\mathrm{Re}\,k+\ell)\big) &\ge 0 \\ \big|\hat\Delta^{(n)}\big(\mathbb{L}^{-1}(k+\ell)\big) - \hat\Delta^{(n)}\big(\mathbb{L}^{-1}(\mathrm{Re}\,k+\ell)\big)\big| &\le 4\pi\Gamma_7\tfrac{1}{L^2}|\mathrm{Im}\,k| \end{aligned}$$

  The first bound follows immediately from Remark 4.1.c. The second bound follows from Lemma 4.2.c (for the $\frac{1}{L^2}\mathrm{Im}\,k_0 = \mathrm{Im}\,(\mathbb{L}^{-1}(k+\ell)_0$ contribution) and Lemma 4.2.g (for the $\frac{1}{L}\mathrm{Im}\,\mathbf{k}_\nu$ contribution, with $1 \le \nu \le 3$, — note that on the line segment from $\mathbb{L}^{-1}(\mathrm{Re}\,k+\ell)$ to $\mathbb{L}^{-1}(k+\ell)$, $\big|\frac{\partial\hat\Delta^{(n)}}{\partial k_\nu}\big|$ is bounded by $\Gamma_7|\mathbb{L}^{-1}(k+\ell)_\nu| = \frac{\Gamma_7}{L}|\mathbf{k}_\nu| \le \Gamma_7\frac{\pi+3m_2}{L}$). Furthermore

$$\begin{aligned} -\mathrm{sgn}\,\ell_0\ \mathrm{Im}\,\hat\Delta^{(n)}\big(\mathbb{L}^{-1}(\mathrm{Re}\,k+\ell)\big) &\ge \tfrac{\pi}{2L^2} && \text{if } \ell_0 \ne 0 \\ \big|\hat\Delta^{(n)}\big(\mathbb{L}^{-1}(\mathrm{Re}\,k)\big)\big| &\ge \gamma_1\gamma_2\min\big\{\tfrac{c_0}{\sqrt2}\ ,\ \tfrac{c_0^2}{2}\big\}\ \tfrac{1}{L^2} && \text{if } \ell_0 = 0 \end{aligned}$$

  In the case $\ell_0 \ne 0$, the proof of the bound given in the case $|k| \le c_0$ applies now too. (Just apply it to $\mathrm{Re}\,k$.) The bound for the case $\ell_0 = 0$ follows from Lemma 4.2.e and Lemma 3.2.a.
- For all ℓ, by parts (a) and (e) of Lemma 2.3,

$$\begin{aligned} \big|u_+\big(\mathbb{L}^{-1}(k+\ell)\big)^{\mathfrak q}\big| &\le \Big[\prod_{\nu=0}^{3}\tfrac{24}{|\ell_\nu|+\pi}\Big]^{\mathfrak q} \\ \big|\mathrm{Im}\,u_+\big(\mathbb{L}^{-1}(k+\ell)\big)^{\mathfrak q}\big| &\le 16\mathfrak{q}\ |\mathrm{Im}\,k|\ \Big[\prod_{\nu=0}^{3}\tfrac{24}{|\ell_\nu|+\pi}\Big]^{\mathfrak q} \end{aligned}$$

We split $\lambda\mathbb{1} - d^{(s)}_{n,k}$ into three pieces

$$\big(\lambda\mathbb{1} - d^{(s)}_{n,k}\big)(\ell,\ell') = D(\ell,\ell') - P(\ell,\ell') + I(\ell,\ell')$$

where

$$D(\ell,\ell') = \delta_{\ell,\ell'} d_\ell \qquad \text{with } d_\ell = \lambda_- - \begin{cases} \hat\Delta^{(n)}\big(\mathbb{L}^{-1}(k+\ell)\big) & \ell_\nu \ne 0 \text{ for some } \nu \ge 1 \\ \hat\Delta^{(n)}\big(\mathbb{L}^{-1}(k+\ell)\big) & |\mathbb{L}^{-1}(k+\ell)| \ge \tilde c_0 \\ \hat\Delta^{(n)}\big(\mathbb{L}^{-1}(\mathrm{Re}\, k+\ell)\big) & \text{otherwise} \end{cases}$$

$$\lambda_- = \min\{\mathrm{Re}\,\lambda, 0\} + i\mathrm{Im}\,\lambda$$

$$P(\ell,\ell') = \tfrac{a}{L^2} v(\ell) v(\ell') \text{ with } v(\ell) = \mathrm{Re}\, u_+\big(\mathbb{L}^{-1}(k+\ell)\big)^{\mathfrak{q}}$$

We have chosen λ_- so that $\mathrm{Re}\,\lambda_- \le 0$ and $|\lambda - \lambda_-| \le \frac{\lambda_0}{L^2}$. As P is a rank one operator

$$(D-P)^{-1}(\ell,\ell') = \tfrac{1}{d_\ell}\delta_{\ell,\ell'} + \tfrac{1}{1-\kappa}\tfrac{a}{L^2}\tfrac{v(\ell)}{d_\ell}\tfrac{v(\ell')}{d_{\ell'}}$$

with

$$\kappa = \sum_{\ell''} \tfrac{a}{L^2}\tfrac{1}{d_{\ell''}} v(\ell'')^2$$

We have shown above that

$$\begin{aligned} |d_\ell| &\ge \tfrac{\lambda_1}{L^2} \\ \mathrm{Re}\, d_\ell &\le \begin{cases} 0 & \text{if } |\mathbb{L}^{-1}(k+\ell)| < \tilde c_0,\ \ell_\nu = 0 \text{ for all } \nu \ge 1 \\ -\frac{\lambda_1}{L^2} & \text{otherwise} \end{cases} \\ |v(\ell)| &\le \Big[\prod_{\nu=0}^{3} \tfrac{24}{|\ell_\nu|+\pi}\Big]^q \\ |I(\ell,\ell')| &\le \big(\tfrac{\lambda_0}{L^2} + 4\pi\Gamma_7 \tfrac{1}{L^2}|\mathrm{Im}\,k|\big)\ \delta_{\ell,\ell'} + 32\mathfrak{q}\ |\mathrm{Im}\,k|\ \tfrac{a}{L^2}\Big[\prod_{\nu=0}^{3}\tfrac{24}{|\ell_\nu|+\pi}\Big]^{\mathfrak{q}}\Big[\prod_{\nu=0}^{3}\tfrac{24}{|\ell'_\nu|+\pi}\Big]^{\mathfrak{q}} \end{aligned}$$

provided we choose $0 < \lambda_1 \le \min\big\{c_1, \rho(\tilde c_0), \frac{\pi}{2}, \gamma_1\gamma_2\frac{c_0}{\sqrt{2}}, \gamma_1\gamma_2\frac{c_0^2}{2}\big\}$.

Since $\mathrm{Re}\, z \le 0 \Rightarrow \mathrm{Re}\,\frac{1}{z} \le 0$, and $v(\ell) \in \mathbb{R}$ for all ℓ, we have that $\mathrm{Re}\,\kappa \le 0$ so that $\big|\frac{1}{1-\kappa}\big| \le 1$ and

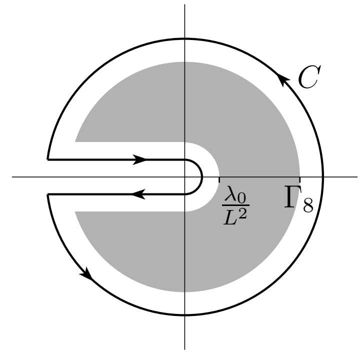
$$\|(D-P)^{-1}\|_{\ell^1-\ell^\infty} \le \tfrac{L^2}{\lambda_2} \qquad \|I\|_{\ell^1-\ell^\infty} \le \tfrac{\Gamma'_5}{L^2}\big(\lambda_0 + |\mathrm{Im}\,k|\big)$$

with the constant $\lambda_2 > 0$ depending only on a and λ_1 and the constant Γ'_5 depending only on Γ_7, a and $\mathfrak{q}$. It now suffices to choose λ_0 and m_2 smaller than $\frac{\lambda_2}{12\Gamma'_5}$ and use a Neumann expansion to give

$$\big\|\big(\lambda\mathbb{1} - d^{(s)}_{n,k}\big)^{-1}\big\|_{\ell^1-\ell^\infty} \le 2\tfrac{L^2}{\lambda_2}$$

□

Proposition 4.4. *Let m_2, λ_0, Γ_8 be as in Lemma 4.3 and use $\mathbb{R}_-$ to denote the negative real axis in $\mathbb{C}$. Set*



$$\mathcal{O}_C = \left\{ z \in \mathbb{C} \;\middle|\; \mathrm{dist}(z, \mathbb{R}_-) > \tfrac{\lambda_0}{2L^2},\ |z| < \Gamma_8 + 1 \right\}$$
$$\mathcal{O} = \left\{ z \in \mathbb{C} \;\middle|\; \mathrm{dist}(z, \mathbb{R}_-) > \tfrac{\lambda_0}{3L^2},\ |z| < \Gamma_8 + 2 \right\}$$

and let

- $C = \partial\mathcal{O}_C$*, oriented counterclockwise*
- $f : \mathcal{O} \to \mathbb{C}$ *be analytic.*

Then $f\big((\frac{a}{L^2}Q^*Q + \Delta^{(n)})^{(s)}\big)$, defined by [5, (12) and Lemma 15.a], exists and there is a constant[3] *Γ_9 such that*

$$\|f\big((\tfrac{a}{L^2}Q^*Q + \Delta^{(n)})^{(s)}\big)\|_{m_2} \le \Gamma_9 L^7 \sup_{\zeta \in C} |f(\zeta)|$$

Proof. Apply [5, Lemma 14] with $\hat{a}_k(\ell, \ell') = d^{(s)}_{n,k}(\ell, \ell')$ and

$$\mathcal{X}_{\rm fin} = \mathcal{X}_1^{(n)} \quad \mathcal{Z}_{\rm fin} = \varepsilon_1^2\mathbb{Z} \times \varepsilon_1\mathbb{Z}^3 \quad \mathcal{X}_{\rm crs} = \mathcal{X}_0^{(n+1)} \quad \mathcal{Z}_{\rm crs} = \mathbb{Z} \times \mathbb{Z}^3 \quad \mathcal{B} = \mathcal{B}_1$$

and $m = 3m_2$, $m' = 2m_2$ and $m'' = m_2$. Then $\mathrm{vol}_c = 1$, $\#\hat{\mathcal{B}}_1 = L^5$. Observe in particular that, by Lemma 4.3, the spectrum of $d^{(s)}_{n,k}$ is contained in

$$\left\{ z \in \mathbb{C} \;\middle|\; \mathrm{dist}(z, \mathbb{R}_-) > \tfrac{\lambda_0}{L^2},\ |z| \le \Gamma_8 \right\}$$

which is the shaded region in the figure above. So the lemma gives

$$\begin{aligned}
&\big\|f\big((\tfrac{a}{L^2}Q^*Q + \Delta^{(n)})^{(s)}\big)\big\|_{m''} \\
&\quad\le \tfrac{C_{m'-m''}}{2\pi\ \mathrm{vol}_c}|C| \sup_{\zeta\in C}|f(\zeta)| \sup_{\substack{|\mathrm{Im}\,\mathrm{k}|=m' \\ \zeta\in C}} \sum_{\ell,\ell'\in\hat{\mathcal{B}}_1} \big|(\zeta\mathbb{1} - \hat{d}^{(s)}_{n,\mathrm{k}})^{-1}(\ell,\ell')\big| \\
&\quad\le \tfrac{C_{m'-m''}}{2\pi}(\Gamma_8+1)(3\pi+2) \sup_{\zeta\in C}|f(\zeta)|\ L^5 \sup_{\substack{|\mathrm{Im}\,\mathrm{k}|=m' \\ \zeta\in C}} \sup_{\ell\in\hat{\mathcal{B}}_1} \sum_{\ell'\in\hat{\mathcal{B}}_1} \big|(\zeta\mathbb{1} - \hat{d}^{(s)}_{n,\mathrm{k}})^{-1}(\ell,\ell')\big| \\
&\quad\le \tfrac{C_{m'-m''}}{2\pi}(\Gamma_8+1)(3\pi+2) \sup_{\zeta\in C}|f(\zeta)| L^5\ \Gamma_8 L^2
\end{aligned}$$

by Lemma 4.3.b.

□

[3] Recall Convention 1.2.

Applying Proposition 4.4 with $f(z)=\frac{1}{z}$, $f(z)=\frac{1}{\sqrt{z}}$ and $f(z)=\sqrt{z}$, where $\sqrt{z}$ is the principal value of the square root gives

Corollary 4.5. *The operators $C^{(n)}$, $\sqrt{C^{(n)}}$ and $\left(\sqrt{C^{(n)}}\right)^{-1}$ all exist. There is a constant Γ_{10} such that*

$$\left\|\mathbb{L}_*^{-1}C^{(n)}\mathbb{L}_*\right\|_{m_2}\ ,\ \left\|\sqrt{\mathbb{L}_*^{-1}C^{(n)}\mathbb{L}_*}\,\right\|_{m_2}\ ,\ \left\|\left(\sqrt{\mathbb{L}_*^{-1}C^{(n)}\mathbb{L}_*}\,\right)^{-1}\right\|_{m_2}\le\Gamma_{10}L^9$$

5 The Green's Functions

In this chapter, we discuss the inverses of the operators

$$D_n + Q_n^* \mathfrak{Q}_n Q_n$$

These inverses, and variations thereof, are constituents of the leading part of the power series expansion of the background fields of [7, 8, 9]. See [7, Proposition 1.14] and [9, Proposition 2.1]. In Proposition 5.1, below, we show that for sufficiently small μ, the operators $D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu$ are invertible, and we estimate the decay of the kernels $S_n(\mu)(x,y)$ of their inverses

$$S_n(\mu) = \big[D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu\big]^{-1}$$

By Remark 2.5

$$\partial_\nu \big(S_n(\mu)^*\big)^{-1} = \big(S_{n,\nu}^{(+)}(\mu)\big)^{-1}\partial_\nu \qquad \partial_\nu S_n(\mu)^{-1} = \big(S_{n,\nu}^{(-)}(\mu)\big)^{-1}\partial_\nu \tag{5.1}$$

where

$$S_{n,\nu}^{(+)}(\mu) = \big[D_n^* + Q_{n,\nu}^{(+)} \mathfrak{Q}_n Q_{n,\nu}^{(-)} - \mu\big]^{-1} \qquad S_{n,\nu}^{(-)}(\mu) = \big[D_n + Q_{n,\nu}^{(+)} \mathfrak{Q}_n Q_{n,\nu}^{(-)} - \mu\big]^{-1} \tag{5.2}$$

and $Q_{n,\nu}^{(+)}, Q_{n,\nu}^{(-)}$ were defined in (2.11). We shall write

$$S_n = S_n(0) \qquad\qquad S_{n,\nu}^{(\pm)} = S_{n,\nu}^{(\pm)}(0) \tag{5.3}$$

The main result extends the statement of [7, Theorem 1.13]. It is

Proposition 5.1. *There are constants $\mu_{\rm up}, m_3 > 0$ and Γ_{11}, depending only on $\mathfrak{q}$, $\mathbf{h}_0$ and a, and in particular independent of n and $L > \Gamma_2$, such that, for $|\mu| \le \mu_{\rm up}$, the operators $D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu$ and $D_n^* + Q_{n,\nu}^{(+)} \mathfrak{Q}_n Q_{n,\nu}^{(-)} - \mu$, $D_n + Q_{n,\nu}^{(+)} \mathfrak{Q}_n Q_{n,\nu}^{(-)} - \mu$ are invertible, and their inverses $S_n(\mu)$ and $S_{n,\nu}^{(+)}(\mu)$, $S_{n,\nu}^{(-)}(\mu)$, respectively, fulfill*

$$\begin{gathered} \|\, S_n(\mu)\|_{m_3}, \|\, S_{n,\nu}^{(\pm)}(\mu)\|_{m_3} \le \Gamma_{11} \\ \|\, S_n(\mu) - S_n\|_{m_3}, \|\, S_{n,\nu}^{(\pm)}(\mu) - S_{n,\nu}^{(\pm)}\|_{m_3} \le |\mu|\, \Gamma_{11} \end{gathered}$$

This Proposition is proven following the proof of Lemma 5.5.

Example 5.2. As a model computation, we evaluate the inverse transform

$$s(x) = \int_{\mathbb{R}\times\mathbb{R}^3} \tfrac{e^{ip\cdot x}}{-ip_0 + m^2 + \mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{p}_3^2} \tfrac{dp_0\, d^3\mathbf{p}}{(2\pi)^4}$$

of $\hat{s}(p) = \frac{1}{-ip_0+m^2+\mathbf{p}^2}$. It is designed to mimic the behaviour of S_n in the limit $n \to \infty$. Write $x = (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3$. We first compute the p_0 integral. Observe that the integrand has exactly one pole, which is at $p_0 = -i(m^2 + \mathbf{p}^2)$, and that the $e^{ip_0 t}$ in the integrand forces us to close the contour in the upper half plane when $t > 0$ and in the lower half plane when $t < 0$. Thus

$$\int_{-\infty}^{\infty} \frac{e^{ip_0 t + i\mathbf{p}\cdot\mathbf{x}}}{-ip_0+m^2+\mathbf{p}^2} \frac{dp_0}{(2\pi)^4} = \begin{cases} 0 & \text{if } t > 0 \\ e^{(m+\mathbf{p}^2)t+i\mathbf{p}\cdot\mathbf{x}} \frac{1}{(2\pi)^3} & \text{if } t < 0 \end{cases}$$

Hence $s(x) = 0$ for $t > 0$ and, for $t < 0$,

$$\begin{aligned} s(x) &= \int_{\mathbb{R}^3} e^{-(m^2+\mathbf{p}^2)|t|} e^{i\mathbf{p}\cdot\mathbf{x}} \frac{d^3\mathbf{p}}{(2\pi)^3} = e^{-m^2|t|} \prod_{j=1}^{3} \int_{-\infty}^{\infty} e^{-|t|\mathbf{p}_j^2} e^{i\mathbf{p}_j \mathbf{x}_j} \frac{d\mathbf{p}_j}{2\pi} \\ &= \frac{1}{2^3(\pi|t|)^{3/2}} e^{-m^2|t|} e^{-\frac{\mathbf{x}^2}{4|t|}} \end{aligned}$$

Here are some observations about $s(x)$.

- Since $m^2|t| + \frac{\mathbf{x}^2}{4|t|} \geq m|\mathbf{x}|$ (the minimum is at $|t| = \frac{|\mathbf{x}|}{2m}$), $s(x)$ decays exponentially for large $|x|$ in all directions.
- For $\mathbf{x} \neq 0$, $\lim_{t \nearrow 0} e^{-\frac{\mathbf{x}^2}{4|t|}} = 0$, so $s(x)$ is continuous everywhere except at $x = 0$.
- $s(x)$ has an integrable singularity at $x = 0$. There are a number of ways to see this. For example, the inequality $e^{-\frac{\mathbf{x}^2}{8|t|}} \leq \text{const}\big(\frac{|t|}{|\mathbf{x}|^2}\big)^{\kappa/2}$ implies that $|s(x)|$ is bounded near $x = 0$ by a constant times $\frac{1}{|t|^{(3-\kappa)/2}} \frac{1}{|\mathbf{x}|^\kappa}$. This is integrable if $1 < \kappa < 3$.
- If we send $x \to 0$ along a curve with $\mathbf{x}^2 = -4\gamma|t| \ln|t|$, $s(x) \approx \text{const}\frac{1}{|t|^{3/2-\gamma}}$.
- We see, using $\mathbf{x}_j^4 e^{-\frac{\mathbf{x}_j^2}{8|t|}} \leq \text{const}|t|^2$, that $\big(t^2 + \sum_{j=1}^3 \mathbf{x}_j^4\big)|s(x)|$ is bounded and exponentially decaying. Note that $\frac{1}{t^2+\Sigma_j \mathbf{x}_j^4}$ has an integrable singularity at the origin, since $\int_{-\infty}^{\infty} \frac{1}{t^2+\Sigma_j \mathbf{x}_j^4} dt = \frac{\text{const}}{\sqrt{\Sigma_j \mathbf{x}_j^4}}$.

As preparation for and in addition to the position space estimates of Proposition 5.1, we also derive bounds on the Fourier transforms of these and related operators. To convert bounds in momentum space into bounds in position space, we shall use [5, Lemma 12], with

$$\mathcal{X}_{\text{fin}} = \mathcal{X}_n \qquad \mathcal{Z}_{\text{fin}} = \varepsilon_n^2\mathbb{Z} \times \varepsilon_n\mathbb{Z}^3 \qquad \mathcal{X}_{\text{crs}} = \mathcal{X}_0^{(n)} \qquad \mathcal{Z}_{\text{crs}} = \mathbb{Z} \times \mathbb{Z}^3 \qquad \mathcal{B} = \mathcal{B}_n \tag{5.4}$$

We shall routinely use $|p_0|$, $|\mathbf{p}_\nu|$ and $|\mathbf{p}|$ to refer to the magnitudes of the smallest representatives of $p_0 \in \mathbb{C}$, $\mathbf{p}_\nu \in \mathbb{C}$ and $\mathbf{p} \in \mathbb{C}^3$ in $\mathbb{C}/\frac{2\pi}{\varepsilon_n^2}\mathbb{Z}$, $\mathbb{C}/\frac{2\pi}{\varepsilon_n}\mathbb{Z}$ and $\mathbb{C}^3/\frac{2\pi}{\varepsilon_n}\mathbb{Z}^3$, respectively.

The operators $S_n(\mu)$ act on functions on the lattice $\mathcal{X}_n$, but they are only translation invariant with respect to the sublattice $\mathcal{X}_0^{(n)}$. An exponentially decaying operator which is fully translation invariant, and has the same local singularity as S_n, is the operator

$$\begin{aligned} S'_n &= \big[D_n + a_n \exp\{-\Delta_n\}\big]^{-1} \qquad \text{where} \\ \Delta_n &= \partial_0^*\partial_0 + \big(\partial_1^*\partial_1 + \partial_2^*\partial_2 + \partial_3^*\partial_3\big) \end{aligned} \tag{5.5}$$

and $a_n = a\frac{1-L^{-2}}{1-L^{-2n}}$ as in Proposition 2.4.b, and the forward derivatives ∂_ν are defined in (2.9). Obviously S'_n has Fourier transform

$$\widehat{S}'_n(p) = \big[\hat{\mathbf{D}}_n(p) + a_n \exp\{-\Delta_n(p)\}\big]^{-1} \quad \text{where } \Delta_n(p) = \Big[\tfrac{\sin\frac{1}{2}\varepsilon_n^2 p_0}{\frac{1}{2}\varepsilon_n^2}\Big]^2 + \sum_{\nu=1}^{3}\Big[\tfrac{\sin\frac{1}{2}\varepsilon_n \mathbf{p}_\nu}{\frac{1}{2}\varepsilon_n}\Big]^2$$

Before we discuss the properties of S'_n and of the difference $\delta S = S_n - S'_n$ we note

Remark 5.3. The Fourier transform $\Delta_n(p)$ of the four dimensional Laplacian Δ_n is entire. For $p \in \mathbb{R}\times\mathbb{R}^3$,

$$\Delta_n(p) \ge \tfrac{2}{\pi^2}\big[|p_0|^2 + |\mathbf{p}|^2\big]$$

For $p \in \mathbb{C}\times\mathbb{C}^3$ with $\varepsilon_n^2|\mathrm{Im}\, p_0| \le 1$ and $\varepsilon_n|\mathrm{Im}\,\mathbf{p}| \le 1$

$$|\Delta_n(p)| \le 4\big[|p_0|^2 + |\mathbf{p}|^2\big] \qquad |\tfrac{\partial}{\partial p_\nu}\Delta_n(p)| \le 4|p_\nu| \qquad |\tfrac{\partial^\ell}{\partial p_\nu^\ell}\Delta_n(p)| \le 4\varepsilon_{n,\nu}^{\ell-2} \text{ if } \ell \ge 2$$

with the $\varepsilon_{n,\nu}$ of (2.6). For $p \in \mathbb{C}\times\mathbb{C}^3$ with $|\mathrm{Im}\, p| \le 1$,

$$\mathrm{Re}\,\Delta_n(p) \ge -5\pi^2 + \tfrac{1}{\pi^2}\big[|p_0|^2 + |\mathbf{p}|^2\big]$$

Proof. For the first two claims, just apply parts (a) and (b) of Lemma A.1. For the derivatives, use

$$\tfrac{d}{d\theta}\Big[\tfrac{\sin(\eta\theta)}{\eta}\Big]^2 = \tfrac{\sin(2\eta\theta)}{\eta} \qquad \Longrightarrow \qquad \tfrac{d^\ell}{d\theta^\ell}\Big[\tfrac{\sin(\eta\theta)}{\eta}\Big]^2 = \pm(2\eta)^{\ell-1}\tfrac{1}{\eta}\begin{cases}\sin(2\eta\theta) & \text{for } \ell \text{ odd}\\ \cos(2\eta\theta) & \text{for } \ell \text{ even}\end{cases}$$

For the final claim, write $p = P + iQ$ with $P, Q \in \mathbb{R}\times\mathbb{R}^3$. Then

$$\begin{aligned} \mathrm{Re}\,\Delta_n(P+iQ) &\ge \Delta_n(P) - \big|\mathrm{Re}\,\Delta_n(P+iQ) - \Delta_n(P)\big| \\ &\ge \tfrac{2}{\pi^2}\big[|P_0|^2 + |\mathbf{P}|^2\big] - 4|Q||P+iQ| \\ &\ge \tfrac{2}{\pi^2}|P+iQ|^2 - 2\big(2+\tfrac{2}{\pi^2}\big)\,|Q|\,|P+iQ| - \tfrac{2}{\pi^2}|Q|^2 \\ &\ge \tfrac{1}{\pi^2}|P+iQ|^2 - \big\{\pi^2\big(2+\tfrac{2}{\pi^2}\big)^2 + \tfrac{2}{\pi^2}\big\}|Q|^2 \end{aligned}$$

□

By the resolvent identity

$$\delta S = S_n - S'_n = -S'_n\big[Q_n^*\mathfrak{Q}_n Q_n - a_n \exp\{-\Delta_n\}\big]S_n$$

S_n and δS are translation invariant with respect to the sublattice $\mathcal{X}_0^{(n)}$ of $\mathcal{X}_n$. By "Floquet theory" (see [5, Lemma 1]), their Fourier transforms $\widehat{S}_n(p,p')$, $\widehat{\delta S}(p,p')$, $p,p' \in \hat{\mathcal{X}}_n$ vanish unless $\pi_n^{(n,0)}(p) = \pi_n^{(n,0)}(p')$, i.e. unless there are $k \in \hat{\mathcal{X}}_0^{(n)}$ and $\ell,\ell' \in \hat{\mathcal{B}}_n$ such that $p = k+\ell$, $p' = k+\ell'$. The blocks $\widehat{S}_{n,k}^{-1}(\ell,\ell') = \widehat{S}_n^{-1}(k+\ell,k+\ell')$ and $\widehat{\delta S}_k(\ell,\ell') = \widehat{\delta S}(k+\ell,k+\ell')$ are given by

$$\begin{aligned}
\widehat{S}_{n,k}^{-1}(\ell,\ell') &= \hat{\mathbf{D}}_n(k+\ell)\,\delta_{\ell,\ell'} + u_n(k+\ell)^{\mathfrak{q}}\hat{\mathfrak{Q}}_n(k)u_n(k+\ell')^{\mathfrak{q}} \\
\widehat{\delta S}_k(\ell,\ell') &= -\sum_{\ell''\in\hat{\mathcal{B}}_n} \widehat{S}'_n(k+\ell)\big[u_n(k+\ell)^{\mathfrak{q}}\hat{\mathfrak{Q}}_n(k)u_n(k+\ell'')^{\mathfrak{q}} - a_n e^{-\Delta_n(k+\ell)}\delta_{\ell,\ell''}\big]\widehat{S}_{n,k}(\ell'',\ell')
\end{aligned} \tag{5.6}$$

where u_n and $\hat{\mathfrak{Q}}_n$ are given in parts (b) and (e) of Remark 2.1.

Lemma 5.4. *There are constants*[4] $m_4 > 0$ *and* Γ_{12}, *such that the following hold for all* $L > \Gamma_2$.

(a) $\widehat{S}'_n(p)$ *is analytic in* $|\mathrm{Im}\,p| < 3m_4$ *and obeys*

$$\big|\widehat{S}'_n(p)\big| \le \tfrac{\Gamma_{12}}{1+|p_0|+|\mathbf{p}|^2} \quad \textit{and} \quad \big|\tfrac{\partial^2}{\partial p_0^2}\widehat{S}'_n(p)\big|, \big|\tfrac{\partial^4}{\partial \mathbf{p}_\nu^4}\widehat{S}'_n(p)\big| \le \tfrac{\Gamma_{12}}{(1+|p_0|+|\mathbf{p}|^2)^3}$$

for $1 \le \nu \le 3$ *there.*

(b) For all $\ell,\ell' \in \hat{\mathcal{B}}_n$, $\widehat{S}_{n,k}(\ell,\ell')$ *is analytic in* $|\mathrm{Im}\,k| < 3m_4$ *and obeys*

$$\big|\widehat{S}_{n,k}(\ell,\ell')\big| \le \tfrac{\Gamma_{12}}{1+|\ell_0|+\Sigma_{\nu=1}^3|\ell_\nu|^2}\Big\{\delta_{\ell,\ell'} + \tfrac{1}{1+|\ell'_0|+\Sigma_{\nu=1}^3|\ell'_\nu|^2}\prod_{\nu=0}^{3}\tfrac{1}{(|\ell_\nu|+1)^{\mathfrak{q}}}\prod_{\nu=0}^{3}\tfrac{1}{(|\ell'_\nu|+1)^{\mathfrak{q}}}\Big\}$$

there.

(c) For all $\ell,\ell' \in \hat{\mathcal{B}}_n$, $\widehat{\delta S}_k(\ell,\ell')$ *is analytic in* $|\mathrm{Im}\,k| < 3m_4$ *and obeys*

$$\begin{aligned}
\big|\widehat{\delta S}_k(\ell,\ell')\big| &\le \Gamma_{12}\exp\big\{-\tfrac{1}{40}\Sigma_{\nu=0}^3|\ell_\nu|^2\big\}\ \delta_{\ell,\ell'} \\
&\quad + \tfrac{\Gamma_{12}}{1+|\ell_0|+\Sigma_{\nu=1}^3|\ell_\nu|^2}\ \Big\{\prod_{\nu=0}^{3}\tfrac{1}{(|\ell_\nu|+1)^{\mathfrak{q}}}\prod_{\nu=0}^{3}\tfrac{1}{(|\ell'_\nu|+1)^{\mathfrak{q}}}\Big\}\ \tfrac{1}{1+|\ell'_0|+\Sigma_{\nu=1}^3|\ell'_\nu|^2}
\end{aligned}$$

there.

[4] Recall Convention 1.2.

(d) For all $u, u' \in \mathcal{X}_n$,

$$\big|S_n(u,u') - S'_n(u,u')\big| \le \Gamma_{12} e^{-2m_4|u-u'|}$$
$$\big|S'_n(u,u')\big| \le \Gamma_{12} \min\Big\{ \tfrac{e^{-2m_4|u-u'|}}{|u_0-u'_0|^2+|\mathbf{u}-\mathbf{u}'|^4}\ ,\ L^{5n}\Big\}$$

Proof. (a) Obviously $\widehat{S}'_n(p)^{-1}$ is entire. For real p

$$\big|\widehat{S}'_n(p)^{-1}\big| \ge a_n \exp\{-\Delta_n(p)\} + \mathrm{const}\big\{|p_0| + |\mathbf{p}|^2\big\} \ge \mathrm{const}\big\{1 + |p_0| + |\mathbf{p}|^2\big\} \tag{5.7}$$

by Remark 5.3, Lemma 3.2.a, and the fact that $\Delta_n(p), \mathrm{Re}\,\hat{\mathbf{D}}_n(p) \ge 0$ for real p. The bound on $\big|\frac{\partial}{\partial p_\nu}\hat{\mathbf{D}}_n(p)\big|$ of Lemma 3.2.c and Remark 5.3 shows that (5.7) is valid for all $|\mathrm{Im}\, p| < 3m_4$, if m_4 is chosen sufficiently small.

We now bound the derivatives. For any $0 \le \nu \le 3$ and $\ell \in \mathbb{N}$, $\frac{\partial^\ell}{\partial p_\nu^\ell}\widehat{S'_n}(p)$ is a finite linear combination of terms of the form

$$\widehat{S'_n}(p)^{1+j} \prod_{i=1}^{j} \tfrac{\partial^{\ell_i}}{\partial p_\nu^{\ell_i}}\big[\hat{\mathbf{D}}_n(p) + a_n \exp\{-\Delta_n(p)\}\big]$$

with each $\ell_i \ge 1$ and $\sum_{i=1}^j \ell_i = \ell$. By Remark 5.3, all derivatives of $\exp\{-\Delta_n(p)\}$ of order $\ell_i \in \mathbb{N}$ are bounded by $\frac{\mathrm{const}_{\ell_i}}{[1+|p_0|+|\mathbf{p}|^2]^{\ell_i}}$. Hence, by Lemma 3.2.c,

$$\big|\tfrac{\partial^{\ell_i}}{\partial p_\nu^{\ell_i}}\big[\hat{\mathbf{D}}_n(p) + a_n \exp\{-\Delta_n(p)\}\big]\big| \le \mathrm{const}\begin{cases} \frac{1}{[1+|p_0|+|\mathbf{p}|^2]^{\ell_i-1}} & \text{if } \nu=0,\ \ell_i = 1,2 \\ \frac{1}{[1+|p_0|+|\mathbf{p}|^2]^{\ell_i/2-1}} & \text{if } \nu \ge 1,\ 1 \le \ell_i \le 4 \end{cases}$$

As $\big|\widehat{S}'_n(p)\big| \le \frac{\mathrm{const}}{1+|p_0|+|\mathbf{p}|^2}$,

$$\Big|\widehat{S'_n}(p)^{1+j} \prod_{i=1}^{j} \tfrac{\partial^{\ell_i}}{\partial p_\nu^{\ell_i}}\big[\hat{\mathbf{D}}_n(p) + a_n \exp\{-\Delta_n(p)\}\big]\Big|$$
$$\le \tfrac{\mathrm{const}}{1+|p_0|+|\mathbf{p}|^2} \prod_{i=1}^{j} \begin{cases} \frac{1}{[1+|p_0|+|\mathbf{p}|^2]^{\ell_i}} & \text{if } \nu=0,\ \ell_i = 1,2 \\ \frac{1}{[1+|p_0|+|\mathbf{p}|^2]^{\ell_i/2}} & \text{if } \nu \ge 1,\ 1 \le \ell_i \le 4 \end{cases}$$

and the claim follows.

(b) For any $c''_0 > 0$ we have, for $|k| \ge c''_0$ and $|\mathrm{Im}\, k_0| < 3m_4$, analyticity and the bound

$$\big|\widehat{S}_{n,k}(\ell,\ell')\big| \le \tfrac{\Gamma'_{10}}{1+|\ell_0|+\Sigma_{\nu=1}^3|\ell_\nu|^2}\delta_{\ell,\ell'} + \tfrac{\Gamma'_{10}}{1+|\ell_0|+\Sigma_{\nu=1}^3|\ell_\nu|^2} \prod_{\nu=0}^{3} \tfrac{1}{(|\ell_\nu|+1)^{\mathfrak{q}}} \prod_{\nu=0}^{3} \tfrac{1}{(|\ell'_\nu|+1)^{\mathfrak{q}}} \quad \tfrac{1}{1+|\ell'_0|+\Sigma_{\nu=1}^3|\ell'_\nu|^2}$$

(with Γ'_{10} depending on c''_0) which follows from the representation

$$S_n = D_n^{-1} - D_n^{-1} Q_n^* \Delta^{(n)} Q_n D_n^{-1}$$

(see [6, Remark 10.b]) and Lemmas 3.2.d, 2.2.a and 4.2.c.

For $|k| < c'_0$, with c'_0 to be shortly chosen sufficiently small, and $|\operatorname{Im} k| < 3m_4$ we use the representation

$$\hat{S}_{n,k}^{-1}(\ell,\ell') = \hat{\mathbf{D}}_n(k+\ell)\,\delta_{\ell,\ell'} + u_n(k+\ell)^{\mathfrak{q}} \hat{\mathfrak{Q}}_n(k) u_n(k+\ell')^{\mathfrak{q}} = D_{\ell,\ell'} + B_{\ell,\ell'} \tag{5.8}$$

with

$$D_{\ell,\ell'} = \hat{\mathbf{D}}_n(k+\ell)\,\delta_{\ell,\ell'} + \begin{cases} a_n & \text{if } \ell,\ell' = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$B_{\ell,\ell'} = \begin{cases} \hat{\mathfrak{Q}}_n(k) u_n(k)^{2\mathfrak{q}} - a_n & \text{if } \ell = \ell' = 0 \\ u_n(k+\ell)^{\mathfrak{q}} \hat{\mathfrak{Q}}_n(k) u_n(k+\ell')^{\mathfrak{q}} & \text{otherwise} \end{cases}$$

By parts (c) and (d) of Lemma 3.2, assuming that $|k| < c'_0$ with c'_0 small enough, D is invertible and the inverse is a diagonal matrix with every diagonal matrix element obeying

$$\big|D_{\ell,\ell}^{-1}\big| \le \tfrac{\Gamma''_{10}}{1+|\ell_0|+\Sigma_{\nu=1}^3 |\ell_\nu|^2}$$

for some Γ''_{10} which is independent of c'_0. By parts (b) and (c) of Lemma 2.2 and parts (a) and (b) of Proposition 2.4,

$$\big|B_{\ell,\ell'}\big| \le \mathrm{const}_{a,\mathfrak{q}}\ \ |k| \prod_{\nu=0}^{3} \big(\tfrac{24}{|\ell_\nu|+\pi}\big)^{\mathfrak{q}} \prod_{\nu=0}^{3} \big(\tfrac{24}{|\ell'_\nu|+\pi}\big)^{\mathfrak{q}}$$

So if $|k| < c'_0$ with c'_0 small enough, $D+B$ is invertible with the inverse given by the Neumann expansion $D^{-1} + \sum_{p=1}^{\infty} (-1)^p D^{-1}\big(BD^{-1}\big)^p$. Since D and B are both analytic on $|\operatorname{Im} k| < 2$ and

$$\sum_{\ell\in 2\pi\mathbb{Z}^4} \mathrm{const}_{a,\mathfrak{q}}\ \ |k| \Big(\prod_{\nu=0}^{3} \tfrac{24}{|\ell_\nu|+\pi}\Big)^{\mathfrak{q}} \tfrac{\Gamma''_{10}}{1+|\ell_0|+\Sigma_{\nu=1}^3 |\ell_\nu|^2} \Big(\prod_{\nu=0}^{3} \tfrac{24}{|\ell_\nu|+\pi}\Big)^{\mathfrak{q}} < \tfrac{1}{2}$$

if c'_0 is small enough, we again get the desired analyticity and bound on $\big|\widehat{S}_{n,k}(\ell,\ell')\big|$.

(c) Just apply Remark 5.3 and parts (a) and (b) of this lemma, Lemma 2.2.a and Proposition 2.4.a and the fact that

$$\sum_{\ell\in 2\pi\mathbb{Z}^4} \Big(\prod_{\nu=0}^{3} \tfrac{1}{|\ell_\nu|+\pi}\Big)^{\mathfrak{q}} \tfrac{1}{1+|\ell_0|+\Sigma_{\nu=1}^3 |\ell_\nu|^2} \Big(\prod_{\nu=0}^{3} \tfrac{1}{|\ell_\nu|+\pi}\Big)^{\mathfrak{q}}$$

is bounded uniformly in n and L to (5.6).

(d) The bound on $\big|S_n(u,u') - S'_n(u,u')\big|$ follows from part (c) by [5, Lemma 12.b] with the replacements (5.4). The bound on $\big|S'_n(u,u')\big|$ follows from part (a), noting in particular that $\frac{\Gamma_{12}}{(1+|p_0|+|\mathbf{p}|^2)^3} \in L^1(\hat{\mathcal{Z}}_{\rm fin})$, and

$$|u_\nu - u'_\nu|^j S'_n(u,u') = \int_{\hat{\mathcal{Z}}_{\rm fin}} \tfrac{\partial^j \widehat{S}'_n}{\partial p_\nu^j}(p)\ e^{-ip\cdot(u-u')}\tfrac{d^4p}{(2\pi)^4}$$

and

$$\big|S'_n(u,u')\big| \le \bigg|\int_{\hat{\mathcal{Z}}_{\rm fin}} \widehat{S}'_n(p)\ e^{-ip\cdot(u-u')}\tfrac{d^4p}{(2\pi)^4}\bigg| \le \Gamma_{12}\int_{\hat{\mathcal{Z}}_{\rm fin}} \tfrac{d^4p}{(2\pi)^4} = \Gamma_{12}\,L^{5n}$$

□

We now prove the analog of Lemma 5.4 for the operators $S^{(+)}_{n,\nu}$ and $S^{(-)}_{n,\nu}$ of (5.3). As in (5.5)–(5.6), we decompose

$$S^{(+)}_{n,\nu} = S'^{\,*}_n + \delta S^{(+)}_\nu \qquad S^{(-)}_{n,\nu} = S'_n + \delta S^{(-)}_\nu$$

with

$$\begin{aligned}\delta S^{(+)}_\nu &= S^{(+)}_{n,\nu} - S'^{\,*}_n = -S'^{\,*}_n\big[Q^{(+)}_{n,\nu}\mathfrak{Q}_n Q^{(-)}_{n,\nu} - a_n\exp\{-\Delta_n\}\big]S^{(+)}_{n,\nu}\\ \delta S^{(-)}_\nu &= S^{(-)}_{n,\nu} - S'_n = -S'_n\big[Q^{(+)}_{n,\nu}\mathfrak{Q}_n Q^{(-)}_{n,\nu} - a_n\exp\{-\Delta_n\}\big]S^{(-)}_{n,\nu}\end{aligned}$$

They have Fourier representations

$$\begin{aligned}\widehat{\delta S^{(+)}_{\nu,k}}(\ell,\ell') &= -\sum_{\ell''\in\hat{\mathcal{B}}_n}\overline{\widehat{S}'_n(k+\ell)}\big[U^{(+)}_{n,\nu}(k,\ell)\hat{\mathfrak{Q}}_n(k)U^{(-)}_{n,\nu}(k,\ell'') - a_n e^{-\Delta_n(k+\ell)}\delta_{\ell,\ell''}\big]\widehat{S}^{(+)}_{n,\nu,k}(\ell'',\ell')\\ \widehat{\delta S^{(-)}_{\nu,k}}(\ell,\ell') &= -\sum_{\ell''\in\hat{\mathcal{B}}_n}\widehat{S}'_n(k+\ell)\big[U^{(+)}_{n,\nu}(k,\ell)\hat{\mathfrak{Q}}_n(k)U^{(-)}_{n,\nu}(k,\ell'') - a_n e^{-\Delta_n(k+\ell)}\delta_{\ell,\ell''}\big]\widehat{S}^{(-)}_{n,\nu,k}(\ell'',\ell')\end{aligned} \tag{5.9}$$

where, by (2.11),

$$\begin{aligned}U^{(+)}_{n,\nu}(k,\ell) &= \zeta^{(+)}_{n,\nu}(k,\ell)u^{(+)}_{n,\nu}(k+\ell)u_n(k+\ell)^{\mathfrak{q}-1}\\ U^{(-)}_{n,\nu}(k,\ell'') &= \zeta^{(-)}_{n,\nu}(k,\ell'')u^{(-)}_{n,\nu}(k+\ell'')u_n(k+\ell'')^{\mathfrak{q}-1}\end{aligned}$$

Lemma 5.5. *There are constants $m_5 > 0$ and Γ_{13} such that the following hold for all $L > \Gamma_2$.*

(a) For all $\ell, \ell' \in \hat{\mathcal{B}}_n$, $\widehat{S}^{(\pm)}_{n,\nu,k}(\ell,\ell')$ is analytic in $|\mathrm{Im}\, k| < 3m_5$ and obeys

$$\big|\widehat{S}^{(\pm)}_{n,\nu,k}(\ell,\ell')\big| \le \tfrac{\Gamma_{13}}{1+|\ell_0|+\Sigma_{\nu=1}^3|\ell_\nu|^2}\Big\{\delta_{\ell,\ell'} + \tfrac{1}{1+|\ell'_0|+\Sigma_{\nu=1}^3|\ell'_\nu|^2}\prod_{\nu=0}^{3}\tfrac{1}{(|\ell_\nu|+1)^{\mathfrak{q}-1}}\prod_{\nu=0}^{3}\tfrac{1}{(|\ell'_\nu|+1)^{\mathfrak{q}}}\Big\}$$

there.

(b) For all $\ell, \ell' \in \hat{\mathcal{B}}_n$, $\widehat{\delta S}_k(\ell,\ell')$ is analytic in $|\mathrm{Im}\, k| < 3m_5$ and obeys

$$\begin{aligned}\big|\widehat{\delta S^{(\pm)}_{\nu,k}}(\ell,\ell')\big| &\le \Gamma_{13}\exp\big\{-\tfrac{1}{40}\Sigma_{\nu=0}^3|\ell_\nu|^2\big\}\ \delta_{\ell,\ell'}\\ &\quad + \tfrac{\Gamma_{13}}{1+|\ell_0|+\Sigma_{\nu=1}^3|\ell_\nu|^2}\ \prod_{\nu=0}^{3}\tfrac{1}{(|\ell_\nu|+1)^{\mathfrak{q}-1}}\prod_{\nu=0}^{3}\tfrac{1}{(|\ell'_\nu|+1)^{\mathfrak{q}}}\ \ \tfrac{1}{1+|\ell'_0|+\Sigma_{\nu=1}^3|\ell'_\nu|^2}\end{aligned}$$

there.

(c) For all $u, u' \in \mathcal{X}_n$, $\big|S^{(\pm)}_{n,\nu}(u,u') - S'_n(u,u')\big| \le \Gamma_{13}e^{-2m_5|u-u'|}$.

Proof. (a) For any $c''_0 > 0$ we have, for $|k| \ge c''_0$ and $|\mathrm{Im}\, k_0| < 3m_5$, analyticity and the desired bound follows from the representations (apply [6, Remark 10.b] with $R = Q^{(-)}_{n,\nu}$, $R_* = Q^{(+)}_{n,\nu}$ and use Remark 2.5 to give $R\,D^{-1}R_* = Q_-D^{-1}Q_-^*$)

$$S^{(+)}_{n,\nu} = D_n^{*\,-1} - D_n^{*\,-1}\,Q^{(+)}_{n,\nu}\Delta^{(n)}Q^{(-)}_{n,\nu}D_n^{*\,-1} \qquad S^{(-)}_{n,\nu} = D_n^{-1} - D_n^{-1}\,Q^{(+)}_{n,\nu}\Delta^{(n)}Q^{(-)}_{n,\nu}D_n^{-1}$$

and Lemmas 3.2.d, 2.2.a, 2.6.b and 4.2.c.

For $|k| < c'_0$, with c'_0 to be shortly chosen sufficiently small, and $|\mathrm{Im}\, k| < 3m_5$ we use the representation

$$\big(\hat{S}^{(-)}_{n,\nu,k}\big)^{-1}(\ell,\ell') = \hat{\mathbf{D}}_n(k+\ell)\,\delta_{\ell,\ell'} + U^{(+)}_{n,\nu}(k,\ell)\hat{\mathfrak{Q}}_n(k)U^{(-)}_{n,\nu}(k,\ell') = D_{\ell,\ell'} + B_{\ell,\ell'} \quad (5.10)$$

with

$$D_{\ell,\ell'} = \hat{\mathbf{D}}_n(k+\ell)\,\delta_{\ell,\ell'} + \begin{cases} a_n & \text{if } \ell,\ell' = 0\\ 0 & \text{otherwise}\end{cases}$$

$$B_{\ell,\ell'} = \begin{cases} \hat{\mathfrak{Q}}_n(k)u_n(k)^{2\mathfrak{q}} - a_n & \text{if } \ell = \ell' = 0\\ U^{(+)}_{n,\nu}(k,\ell)\hat{\mathfrak{Q}}_n(k)U^{(-)}_{n,\nu}(k,\ell') & \text{otherwise}\end{cases}$$

and the obvious analog for $\big(\hat{S}^{(+)}_{n,\nu,k}\big)^{-1}$ — just replace $\hat{\mathbf{D}}_n(k+\ell)$ with its complex conjugate. As in the proof of Lemma 5.4.b, D is invertible and the inverse is a diagonal matrix with every diagonal matrix element obeying

$$\big|D^{-1}_{\ell,\ell}\big| \le \tfrac{\Gamma'_{11}}{1+|\ell_0|+\Sigma_{\nu=1}^3|\ell_\nu|^2}$$

for some Γ'_{11} which is independent of c'_0. By parts (b) and (c) of Lemma 2.2, parts (a) and (b) of Proposition 2.4, and part (b) of Lemma 2.6,

$$\big|B_{\ell,\ell'}\big| \le \mathrm{const}_{a,\mathfrak{q}}\ \ |k| \prod_{\nu=0}^{3} \big(\tfrac{24}{|\ell_\nu|+\pi}\big)^{\mathfrak{q}-1} \prod_{\nu=0}^{3} \big(\tfrac{24}{|\ell'_\nu|+\pi}\big)^{\mathfrak{q}}$$

So if $|k| < c'_0$ with c'_0 small enough, $D + B$ is invertible with the inverse given by the Neumann expansion $D^{-1} + \sum_{p=1}^{\infty}(-1)^p D^{-1}\big(BD^{-1}\big)^p$. As $\mathfrak{q} > 1$, the desired analyticity and the desired bound on $\big|\widehat{S}^{(\pm)}_{n,\nu,k}(\ell,\ell')\big|$ follow as in the proof of Lemma 5.4.b

(b) is proven just as Lemma 5.4.c.

(c) follows from part (b) by [5, Lemma 12.b]. □

Proof of Proposition 5.1. Set $m_3 = \min\{m_4, m_5\}$. As $\frac{1}{|u_0|^2+|\mathbf{u}|^4}$ is locally integrable in $\mathbb{R}^4$, the pointwise bounds on $\big|S_n(u,u') - S'_n(u,u')\big|$ and $\big|S'_n(u,u')\big|$, given in Lemma 5.4.d, and on $\big|S^{(\pm)}_{n,\nu}(u,u') - S'_n(u,u')\big|$, given in Lemma 5.5.c, imply

$$\|\ S_n\|_{m_3}, \|\ S^{(\pm)}_{n,\nu}\|_{m_3} \le \tilde{\Gamma}_{11}$$

with $\tilde{\Gamma}_{11}$ a constant, depending only on m_3, times $\max\{\Gamma_{12}, \Gamma_{13}\}$. Setting $\mu_{\rm up} = \frac{1}{2\tilde{\Gamma}_{11}}$ and Γ_{11} to be the maximum of $2\tilde{\Gamma}_{11}$ (for $\|\ S_n(\mu)\|_{m_3}$ and$\|\ S^{(\pm)}_{n,\nu}(\mu)\|_{m_3}$) and $2\tilde{\Gamma}^2_{11}$ (for $\|\ S_n(\mu) - S_n\|_{m_3}$ and$\|\ S^{(\pm)}_{n,\nu}(\mu) - S^{(\pm)}_{n,\nu}\|_{m_3}$) a Neumann expansion gives the specified bounds.

□

We now formulate and prove two more technical lemmas that will be used elsewhere.

Lemma 5.6. *There are constants $m_6 > 0$ and Γ_{14} such that, for all $L > \Gamma_2$,*

$$\big|\big(S_n Q^*_n\big)(y,x)\big| \le \Gamma_{14} e^{-2m_6|x-y|} \qquad \|S_n Q^*_n\|_{m_6} \le \Gamma_{14}$$

Proof. From the definitions of S_n, in (5.3), and $\Delta^{(n)}$, at the beginning of §4, one sees directly that

$$S^{(*)}_n Q^*_n = {D^{(*)}_n}^{-1} Q^*_n \Delta^{(n)(*)} \mathfrak{Q}_n^{-1} : L^2(\mathcal{X}_{\rm crs}) \to L^2(\mathcal{X}_{\rm fin}) \tag{5.11}$$

The Fourier transform of the kernel, $b(y,x)$, of the operator $S_n Q^*_n$ is

$$\hat{b}_{\rm k}(\ell) = \hat{\mathbf{D}}_n^{-1}({\rm k}+\ell) u_n({\rm k}+\ell)^{\mathfrak{q}} \hat{\Delta}^{(n)}({\rm k}) \tfrac{1}{\hat{\mathfrak{Q}}_n({\rm k})}$$

By Lemma 4.2.a,c,f, Remark 3.1.b and Lemma 3.2.d, Proposition 2.4.a, Remark 2.1.d and Lemma 2.2.a, $\hat{b}_{\mathrm{k}}(\ell)$ is analytic in $|\mathrm{Im}\,\mathrm{k}| < 3m_6$ and

$$\big|\hat{b}_{\mathrm{k}}(\ell)\big| \le \frac{\frac{5}{4a}\Gamma_6}{1+|\mathrm{k}_0+\ell_0|+\sum_{\nu=1}^3|\mathrm{k}_\nu+\ell_\nu|^2}\prod_{\nu=0}^3\Big(\frac{24}{|\ell_\nu|+\pi}\Big)^{\mathfrak{q}}$$

The bound is uniform in n and L and is summable in ℓ, so the claims follow from [5, Lemma 12.c]. $\square$

Lemma 5.7. *There are constants $m_7 > 0$ and Γ_{15} such that, for all $L > \Gamma_2$, the operators*

$$\big(S_n(\mu)Q_n^*\mathfrak{Q}_n\big)^*\big(S_n(\mu)Q_n^*\mathfrak{Q}_n\big),\ \big(S_n(\mu)^*Q_n^*\mathfrak{Q}_n\big)^*\big(S_n(\mu)^*Q_n^*\mathfrak{Q}_n\big)$$
$$\big(S_{n,\nu}^{(\pm)}(\mu)Q_{n,\nu}^{(+)}\mathfrak{Q}_n\big)^*\big(S_{n,\nu}^{(\pm)}(\mu)Q_{n,\nu}^{(+)}\mathfrak{Q}_n\big)$$

all have bounded inverses. The $\|\cdot\|_{m_7}$ norms of the inverses are all bounded by Γ_{15}.

Proof. We first consider the case that $\mu = 0$. By (5.11), the operator

$$S_nQ_n^*\mathfrak{Q}_n = D_n^{-1}Q_n^*\Delta^{(n)} : L^2(\mathcal{X}_{\mathrm{crs}}) \to L^2(\mathcal{X}_{\mathrm{fin}})$$

has Fourier transform

$$\tilde{b}_k(\ell) = \hat{\mathbf{D}}_n^{-1}(k+\ell)u_n(k+\ell)^{\mathfrak{q}}\hat{\Delta}^{(n)}(k)$$

The operator $(S_nQ_n^*\mathfrak{Q}_n)^*(S_nQ_n^*\mathfrak{Q}_n)$ maps $L^2(\mathcal{X}_{\mathrm{crs}})$ to $L^2(\mathcal{X}_{\mathrm{crs}})$ and has Fourier transform

$$\sum_{\ell\in\hat{\mathcal{B}}}\tilde{b}_{-k}(-\ell)\tilde{b}_k(\ell) = \sum_{\ell\in\hat{\mathcal{B}}}\hat{\mathbf{D}}_n^{-1}(-k-\ell)\hat{\mathbf{D}}_n^{-1}(k+\ell)u_n(k+\ell)^{2\mathfrak{q}}\hat{\Delta}^{(n)}(-k)\hat{\Delta}^{(n)}(k)$$

For k real,

$$\begin{aligned}\sum_{\ell\in\hat{\mathcal{B}}}\tilde{b}_{-k}(-\ell)\tilde{b}_k(\ell) &= \sum_{\ell\in\hat{\mathcal{B}}}\big|\hat{\mathbf{D}}_n^{-1}(k+\ell)u_n(k+\ell)^{\mathfrak{q}}\hat{\Delta}^{(n)}(k)\big|^2\\ &\ge \big|\hat{\mathbf{D}}_n^{-1}(k)u_n(k)^{\mathfrak{q}}\hat{\Delta}^{(n)}(k)\big|^2\\ &\ge \gamma_2^2\inf_{|k_\nu|\le\pi}|u_n(k)|^{2\mathfrak{q}}\\ &\ge \gamma_2^2\big(\tfrac{2}{\pi}\big)^{8\mathfrak{q}}\end{aligned}$$

by Lemmas 4.2.e and 2.2.f. To show that half this the lower bound extends into a strip along the real axis that has width independent of n and L, we observe that

- all first order derivatives of $u_n(k+\ell)$ are uniformly bounded by $2\prod_{\nu=0}^{3}\frac{24}{|\ell_\nu|+\pi}$ on such a strip by the Cauchy integral formula and Remark 2.1.d and Lemma 2.2.a and
- $u_n(k+\ell)$ itself is uniformly bounded by $\prod_{\nu=0}^{3}\frac{24}{|\ell_\nu|+\pi}$ on such a strip by Lemma 2.2.a and
- all first order derivatives of $\hat\Delta^{(n)}(k)$ are uniformly bounded on such a strip by Lemma 4.2.c and
- for $\ell\neq 0$, all first order derivatives of $\hat{\mathbf{D}}_n^{-1}(k+\ell)$ are uniformly bounded on such a strip by parts (c) and (d) of Lemma 3.2 and
- for $\ell=0$, all first order derivatives of $\hat{\mathbf{D}}_n^{-1}(k)\hat\Delta^{(n)}(k)$ are uniformly bounded on such a strip by Lemma 4.2.f.

The operator $S_n^*Q_n^*\mathfrak{Q}_n = D_n^{-1*}Q_n^*\Delta^{(n)*}$ has Fourier transform $\tilde b_{-k}(-\ell)$. So

$$(S_nQ_n^*\mathfrak{Q}_n)^*(S_nQ_n^*\mathfrak{Q}_n) = (S_n^*Q_n^*\mathfrak{Q}_n)^*(S_n^*Q_n^*\mathfrak{Q}_n)$$

The operators $S_{n,\nu}^{(-)}Q_{n,\nu}^{(+)}\mathfrak{Q}_n = D_n^{-1}Q_{n,\nu}^{(+)}\Delta^{(n)}$ and $S_{n,\nu}^{(+)}Q_{n,\nu}^{(+)}\mathfrak{Q}_n = D_n^{-1*}Q_{n,\nu}^{(+)}\Delta^{(n)*}$ map $L^2(\mathcal{X}_{\rm crs})$ to $L^2(\mathcal{X}_{\rm fin})$ and have Fourier transforms

$$\tilde c_k(\ell) = \hat{\mathbf{D}}_n^{-1}(\pm k\pm\ell)\zeta_{n,\nu}^{(+)}(k,\ell)u_{n,\nu}^{(+)}(k+\ell)u_n(k+\ell)^{\mathfrak{q}-1}\hat\Delta^{(n)}(\pm k)$$

So the operators $(S_{n,\nu}^{(\pm)}Q_{n,\nu}^{(+)}\mathfrak{Q}_n)^*(S_{n,\nu}^{(\pm)}Q_{n,\nu}^{(+)}\mathfrak{Q}_n)$ both map $L^2(\mathcal{X}_{\rm crs})$ to $L^2(\mathcal{X}_{\rm crs})$ and have Fourier transform

$$\begin{aligned}\sum_{\ell\in\hat{\mathcal{B}}}&\tilde c_{-k}(-\ell)\tilde c_k(\ell)\\ &=\sum_{\ell\in\hat{\mathcal{B}}}\hat{\mathbf{D}}_n^{-1}(-k-\ell)\hat{\mathbf{D}}_n^{-1}(k+\ell)u_{n,\nu}^{(+)}(k+\ell)^2u_n(k+\ell)^{2(\mathfrak{q}-1)}\hat\Delta^{(n)}(-k)\hat\Delta^{(n)}(k)\end{aligned}$$

This is bounded just as $\sum_{\ell\in\hat{\mathcal{B}}}\tilde b_{-k}(-\ell)\tilde b_k(\ell)$ was. The specified bounds, in the special case that $\mu=0$ follow.

By Proposition 5.1, a Neumann expansion gives the desired bounds when μ is nonzero.

□

6 The Degree One Part of the Critical Field

In [9, Proposition 5.1 and (5.3)] we derive an expansion for the critical fields of the form

$$\psi_{(*)n}(\theta_*,\theta,\mu,\mathcal{V}) = \tfrac{a}{L^2}C^{(n)}(\mu)^{(*)}Q^*\theta_{(*)} + \psi^{(\ge 3)}_{(*)n}(\theta_*,\theta,\mu,\mathcal{V})$$

with the $C^{(n)}(\mu)$ of [7, Proposition 1.15] and with $\psi^{(\ge 3)}_{(*)n}$ being of degree at least 3 in $\theta_{(*)}$. In this section we derive bounds on a scaled version of $C^{(n)}(\mu)^{(*)}Q^*\theta_{(*)}$ and some related operators. To do so we use the representation

$$\tfrac{a}{L^2}C^{(n)}(\mu)^{(*)}Q^* = \big(\tfrac{a}{L^2}Q^*Q + \mathfrak{Q}_n\big)^{-1}\big\{\tfrac{a}{L^2}Q^* + \mathfrak{Q}_nQ_n\check{S}_{n+1}(\mu)^{(*)}\check{Q}^*_{n+1}\check{\mathfrak{Q}}_{n+1}\big\} \tag{6.1}$$

of [9, (5.2)]. Here, as in [9, Lemma 2.4 and (5.1)],

$$\begin{aligned}
\check{Q}_{n+1} &= \mathbb{L}_*Q_{n+1}\mathbb{L}_*^{-1} &&= QQ_n &&: \mathcal{H}_n \to \mathcal{H}^{(n+1)}_{-1}\\
\check{\mathfrak{Q}}^{-1}_{n+1} &= L^2\,\mathbb{L}_*\mathfrak{Q}^{-1}_{n+1}\mathbb{L}_*^{-1} &&= \tfrac{1}{aL^{-2}}\mathbb{1} + Q\mathfrak{Q}_n^{-1}Q^* &&: \mathcal{H}^{(n+1)}_{-1} \to \mathcal{H}^{(n+1)}_{-1}\\
\check{S}_{n+1}(\mu) &= L^2\,\mathbb{L}_*S_{n+1}(L^2\mu)\mathbb{L}_*^{-1} &&= \big\{D_n - \mu + \check{Q}^*_{n+1}\check{\mathfrak{Q}}_{n+1}\check{Q}_{n+1}\big\}^{-1} &&: \mathcal{H}_n \to \mathcal{H}_n
\end{aligned} \tag{6.2}$$

These operators are all translation invariant with respect to $\mathcal{X}^{(n+1)}_{-1}$. As

$$\check{S}_{n+1}(\mu) = \check{S}_{n+1} + \mu\check{S}_{n+1}\check{S}_{n+1}(\mu) \qquad \text{with } \check{S}_{n+1} = \check{S}_{n+1}(0)$$

we have

$$\tfrac{a}{L^2}C^{(n)}(\mu)^{(*)}Q^* = \tfrac{a}{L^2}C^{(n)\,(*)}Q^* + \mu A_{\psi,\phi}\check{S}^{(*)}_{n+1}\check{S}_{n+1}(\mu)^{(*)}\check{Q}^*_{n+1}\check{\mathfrak{Q}}_{n+1} \tag{6.3}$$

where

$$A_{\psi,\phi} = (aL^{-2}Q^*Q + \mathfrak{Q}_n)^{-1}\mathfrak{Q}_nQ_n : \mathcal{H}_n \to \mathcal{H}^{(n)}_0 \tag{6.4}$$

The operator $A_{\psi,\phi}$ is also used in the course of bounding $\psi^{(\ge 3)}_{(*)n}$ in [9, Proposition 5.1].

The main results of this section are

Proposition 6.1. *There are constants*[5] $m_8 > 0$ *and* Γ_{16}, Γ_{17} *such that the following holds, for each* $L > \Gamma_{17}$ *and each* μ *obeying* $|L^2\mu| \le \mu_{\rm up}$.

(a) $\big\|\mathbb{L}_*^{-1}A_{\psi,\phi}\mathbb{L}_*\big\|_{m=1} \le \Gamma_{16}$ *and* $\big\|\mathbb{L}_*^{-1}\ \tfrac{a}{L^2}C^{(n)}(\mu)^{(*)}Q^*\ \mathbb{L}_*\big\|_{m_8} \le \Gamma_{16}$

[5] Recall Convention 1.2.

(b) Let $0 \le \nu \le 3$. There are operators $A_{\psi,\phi,\nu}$ and $A_{\psi_{(*)}\theta_{(*)}\nu}(\mu)$ such that

$$\partial_\nu A_{\psi,\phi} = A_{\psi,\phi,\nu}\partial_\nu \qquad \partial_\nu \tfrac{a}{L^2} C^{(n)}(\mu)^{(*)} Q^* = A_{\psi_{(*)}\theta_{(*)}\nu}(\mu)\ \partial_\nu$$

and

$$\|\mathbb{L}_*^{-1} A_{\psi,\phi,\nu} \mathbb{L}_*\|_{m=1} \le \Gamma_{16} \qquad \left\|\mathbb{L}_*^{-1} A_{\psi_{(*)}\theta_{(*)}\nu}(\mu) \mathbb{L}_*\right\|_{m_8} \le \Gamma_{16}$$

This proposition is proven at the end of this section, after Lemma 6.6. In this proof we write $\frac{a}{L^2} C^{(n)^{(*)}} Q^* = A_{\psi_{(*)},\theta_{(*)}}$ so that

$$A_{\psi_{(*)},\theta_{(*)}} = (aL^{-2}Q^*Q + \mathfrak{Q}_n)^{-1}\big\{aL^{-2}Q^* + \mathfrak{Q}_n Q_n \check{S}^{(*)}_{n+1} \check{Q}^*_{n+1} \check{\mathfrak{Q}}_{n+1}\big\} : \mathcal{H}^{(n+1)}_{-1} \to \mathcal{H}^{(n)}_0$$

Remark 6.2.

(a) $A_{\psi_{(*)}\theta_{(*)}} = \mathfrak{Q}_n^{-1} Q^* \check{\mathfrak{Q}}_{n+1} + A_{\psi,\phi} D_n^{-1^{(*)}} \check{Q}^*_{n+1}\ \check{\Delta}^{(n+1)(*)}$ with

$$\check{\Delta}^{(n+1)(*)} = \tfrac{1}{L^2}\mathbb{L}_* \Delta^{(n+1)(*)} \mathbb{L}_*^{-1} = \big\{\mathbb{1} + \check{\mathfrak{Q}}_{n+1} \check{Q}_{n+1} D_n^{-1^{(*)}} \check{Q}^*_{n+1}\big\}^{-1} \check{\mathfrak{Q}}_{n+1}$$

being a fully translation invariant operator on $\mathcal{H}^{(n+1)}_{-1}$.

(b) Let $0 \le \nu \le 3$. We have $\partial_\nu A_{\psi,\phi} = A_{\psi,\phi,\nu}\partial_\nu$ and $\partial_\nu A_{\psi_{(*)}\theta_{(*)}}(\mu) = A_{\psi_{(*)}\theta_{(*)}\nu}(\mu)\partial_\nu$ where

$$\begin{aligned}
A_{\psi,\phi,\nu} &= \big[\mathbb{1} - \mathfrak{Q}_n^{-1} Q^{(+)}_{+,\nu} \check{\mathfrak{Q}}_{n+1} Q^{(-)}_{+,\nu}\big] Q^-_{n,\nu} \\
A_{\psi_{(*)}\theta_{(*)}\nu} &= \mathfrak{Q}_n^{-1} Q^{(+)}_{+,\nu} \check{\mathfrak{Q}}_{n+1} + A_{\psi,\phi,\nu} \mathbb{L}_* D^{-1(*)}_{n+1} Q^{(+)}_{n+1,\nu} \Delta^{(n+1)(*)} \mathbb{L}_*^{-1} \\
A_{\psi_*\theta_*\nu}(\mu) &= A_{\psi_*\theta_*\nu} + L^2\mu A_{\psi,\phi,\nu} \mathbb{L}_* S^{(+)}_{n+1,\nu} S^{(+)}_{n+1,\nu}(L^2\mu) Q^{(+)}_{n+1,\nu} \mathfrak{Q}_{n+1} \mathbb{L}_*^{-1} \\
A_{\psi\theta\nu}(\mu) &= A_{\psi\theta\nu} + L^2\mu A_{\psi,\phi,\nu} \mathbb{L}_* S^{(-)}_{n+1,\nu} S^{(-)}_{n+1,\nu}(L^2\mu) Q^{(+)}_{n+1,\nu} \mathfrak{Q}_{n+1} \mathbb{L}_*^{-1}
\end{aligned}$$

Proof. (a) First observe that, by (5.11),

$$Q_n \check{S}^{(*)}_{n+1} \check{Q}^*_{n+1} \check{\mathfrak{Q}}_{n+1} = Q_n \mathbb{L}_* {D^{(*)}_{n+1}}^{-1} Q^*_{n+1} \Delta^{(n+1)(*)} \mathbb{L}_*^{-1} = (Q_n D_n^{-1^{(*)}} Q^*_n)\ Q^*\ \check{\Delta}^{(n+1)(*)} \tag{6.5}$$

Using (6.5), the operator

$$\begin{aligned}
A_{\psi_{(*)}\theta_{(*)}} &= (aL^{-2}Q^*Q + \mathfrak{Q}_n)^{-1}\big\{aL^{-2}Q^* + \mathfrak{Q}_n Q_n D_n^{-1^{(*)}} \check{Q}^*_{n+1}\ \check{\Delta}^{(n+1)(*)}\big\} \\
&= \mathfrak{Q}_n^{-1}(Q^*Q\mathfrak{Q}_n^{-1} + \tfrac{1}{aL^{-2}}\mathbb{1})^{-1} Q^* + A_{\psi,\phi} D_n^{-1^{(*)}} \check{Q}^*_{n+1}\ \check{\Delta}^{(n+1)(*)} \\
&= \mathfrak{Q}_n^{-1} Q^* (Q\mathfrak{Q}_n^{-1}Q^* + \tfrac{1}{aL^{-2}}\mathbb{1})^{-1} + A_{\psi,\phi} D_n^{-1^{(*)}} \check{Q}^*_{n+1}\ \check{\Delta}^{(n+1)(*)} \\
&= \mathfrak{Q}_n^{-1} Q^* \check{\mathfrak{Q}}_{n+1} + A_{\psi,\phi} D_n^{-1^{(*)}} \check{Q}^*_{n+1}\ \check{\Delta}^{(n+1)(*)}
\end{aligned}$$

(b) By Remark 2.5,

$$\begin{aligned}\partial_\nu A_{\psi,\phi} &= \partial_\nu(\mathbb{1} + aL^{-2}\mathfrak{Q}_n^{-1}Q^*Q)^{-1}Q_n \\ &= \partial_\nu Q_n - \partial_\nu aL^{-2}\mathfrak{Q}_n^{-1}Q^*(\mathbb{1} + aL^{-2}Q\mathfrak{Q}_n^{-1}Q^*)^{-1}QQ_n \\ &= Q_{n,\nu}^-\partial_\nu - aL^{-2}\mathfrak{Q}_n^{-1}Q_{+,\nu}^{(+)}(\mathbb{1} + aL^{-2}Q\mathfrak{Q}_n^{-1}Q^*)^{-1}Q_{+,\nu}^{(-)}Q_{n,\nu}^{(-)}\partial_\nu \\ &= \big[\mathbb{1} - \mathfrak{Q}_n^{-1}Q_{+,\nu}^{(+)}\check{\mathfrak{Q}}_{n+1}Q_{+,\nu}^{(-)}\big]Q_{n,\nu}^-\partial_\nu\end{aligned}$$

Therefore by part (a), (3.1), (6.2) and Remark 2.5,

$$\partial_\nu A_{\psi_{(*)}\theta_{(*)}} = \partial_\nu\big[\mathfrak{Q}_n^{-1}Q^*\check{\mathfrak{Q}}_{n+1} + A_{\psi,\phi}\mathbb{L}_* D_{n+1}^{-1(*)}Q_{n+1}^*\Delta^{(n+1)(*)}\mathbb{L}_*^{-1}\big] = A_{\psi_{(*)}\theta_{(*)\nu}}\partial_\nu$$

since $\partial_\nu\mathbb{L}_* = \frac{1}{L_\nu}\mathbb{L}_*\partial_\nu$ by [7, Remark 2.2.a,b]. To get $\partial_\nu A_{\psi_{(*)}\theta_{(*)}}(\mu) = A_{\psi_{(*)}\theta_{(*)}\nu}(\mu)\partial_\nu$ when $\mu \neq 0$, write, using (6.2),

$$\begin{aligned}A_{\psi_{(*)}\theta_{(*)}}(\mu) &= A_{\psi_{(*)}\theta_{(*)}} + \mu A_{\psi,\phi}\check{S}_{n+1}^{(*)}\check{S}_{n+1}(\mu)^{(*)}\check{Q}_{n+1}^*\check{\mathfrak{Q}}_{n+1} \\ &= A_{\psi_{(*)}\theta_{(*)}} + L^2\mu A_{\psi,\phi}\mathbb{L}_* S_{n+1}^{(*)}S_{n+1}(L^2\mu)^{(*)}Q_{n+1}^*\mathfrak{Q}_{n+1}\mathbb{L}_*^{-1}\end{aligned}$$

and use (5.1), Remark 2.5 and the fact that $\mathfrak{Q}_{n+1}$ is fully translation invariant.

□

The operators of principal interest, A_{ψ_*,θ_*} and $A_{\psi,\theta}$, act from $L^2(\mathcal{X}_{\rm crs}) = \mathcal{H}_{-1}^{(n+1)}$ to $L^2(\mathcal{X}_{\rm fin}) = \mathcal{H}_0^{(n)}$ with

$$\mathcal{X}_{\rm fin} = \mathcal{X}_0^{(n)} \qquad \mathcal{X}_{\rm crs} = \mathcal{X}_{-1}^{(n+1)} \qquad \mathcal{B} = \mathcal{B}^+$$

We now give a bunch of Fourier transforms (in the sense of [5, (7) and (8)] – but we shall suppress the ˆ from the notation). All of the operators above are periodized in the sense of [5, Definition 2]. As before we denote

- momenta dual to the L–lattice $L^2\mathbb{Z} \times L\mathbb{Z}^3$ by $\mathfrak{k} \in \big(\mathbb{R}/\frac{2\pi}{L^2}\mathbb{Z}\big) \times \big(\mathbb{R}^3/\frac{2\pi}{L}\mathbb{Z}^3\big)$,
- momenta dual to the unit lattice $\mathbb{Z} \times \mathbb{Z}^3$ by $k \in \big(\mathbb{R}/2\pi\mathbb{Z}\big) \times \big(\mathbb{R}^3/2\pi\mathbb{Z}^3\big)$ and decompose $k = \mathfrak{k} + \ell$ or $k = \mathfrak{k} + \ell'$ with $\mathfrak{k}$ in a fundamental cell for $\big(\mathbb{R}/\frac{2\pi}{L^2}\mathbb{Z}\big) \times \big(\mathbb{R}^3/\frac{2\pi}{L}\mathbb{Z}^3\big)$ and $\ell, \ell' \in \big(\frac{2\pi}{L^2}\mathbb{Z}/2\pi\mathbb{Z}\big) \times \big(\frac{2\pi}{L}\mathbb{Z}^3/2\pi\mathbb{Z}^3\big) = \hat{\mathcal{B}}^+$ and
- momenta dual to the ε_j–lattice $\varepsilon_j^2\mathbb{Z} \times \varepsilon_j\mathbb{Z}^3$ by $p_j \in \big(\mathbb{R}/\frac{2\pi}{\varepsilon_j^2}\mathbb{Z}\big) \times \big(\mathbb{R}^3/\frac{2\pi}{\varepsilon_j}\mathbb{Z}^3\big)$ and decompose $p_j = k + \ell_j$ with k in a fundamental cell for $\big(\mathbb{R}/2\pi\mathbb{Z}\big) \times \big(\mathbb{R}^3/2\pi\mathbb{Z}^3\big)$ and $\ell_j \in \big(2\pi\mathbb{Z}/\frac{2\pi}{\varepsilon_j^2}\mathbb{Z}\big) \times \big(2\pi\mathbb{Z}^3/\frac{2\pi}{\varepsilon_j}\mathbb{Z}^3\big) = \hat{\mathcal{B}}_j$. Here $1 \le j \le n$.

The Fourier transform of $A_{\psi_{(*)}\theta_{(*)}}$ is

$$\big(A_{\psi_{(*)}\theta_{(*)}}\big)_{\mathfrak{k}}(\ell) = \sum_{\ell'\in\hat{\mathcal{B}}^+} \big(\tfrac{a}{L^2}Q^*Q + \mathfrak{Q}_n\big)_{\mathfrak{k}}^{-1}(\ell,\ell') \Big\{\tfrac{a}{L^2}Q^*_{\mathfrak{k}}(\ell') + \mathfrak{Q}_n(\mathfrak{k}+\ell')\,\big(Q_n D_n^{-1^{(*)}}Q_n^*\big)(\mathfrak{k}+\ell')Q^*_{\mathfrak{k}}(\ell')\check{\Delta}^{(n+1)(*)}(\mathfrak{k})\Big\} \tag{6.6}$$

where, by Remark 2.1, Remark 6.2.a and (6.2)

$$\big(aL^{-2}Q^*Q + \mathfrak{Q}_n\big)_{\mathfrak{k}}(\ell,\ell') = \mathfrak{Q}_n(\mathfrak{k}+\ell)\delta_{\ell,\ell'} + aL^{-2}u_+(\mathfrak{k}+\ell)^{\mathfrak{q}}u_+(\mathfrak{k}+\ell')^{\mathfrak{q}}$$

and

$$\begin{aligned}
Q^*_{\mathfrak{k}}(\ell) &= u_+(\mathfrak{k}+\ell)^{\mathfrak{q}}\\
\mathfrak{Q}_n(k) &= a\Big[1+\sum_{j=1}^{n-1}\sum_{\ell_j\in\hat{\mathcal{B}}_j}\tfrac{1}{L^{2j}}u_j(k+\ell_j)^{2\mathfrak{q}}\Big]^{-1}\\
(Q_n\mathbf{D}_n^{-1(*)}Q_n^*)(k) &= \sum_{\ell_n\in\hat{\mathcal{B}}_n} u_n(k+\ell_n)^{2\mathfrak{q}}\mathbf{D}_n^{-1(*)}(k+\ell_n)\\
\check{\Delta}^{(n+1)(*)}(\mathfrak{k}) &= \frac{\check{\mathfrak{Q}}_{n+1}(\mathfrak{k})}{1+\check{\mathfrak{Q}}_{n+1}(\mathfrak{k})\sum\limits_{\substack{\ell_n\in\hat{\mathcal{B}}_n\\ \ell\in\hat{\mathcal{B}}^+}} u_n(\mathfrak{k}+\ell+\ell_n)^{2\mathfrak{q}}\,u_+(\mathfrak{k}+\ell)^{2\mathfrak{q}}\,\mathbf{D}_n^{-1(*)}(\mathfrak{k}+\ell+\ell_n)}\\
\check{\mathfrak{Q}}_{n+1}(\mathfrak{k}) &= \tfrac{a}{L^2}\Big[1+\sum_{\ell\in\hat{\mathcal{B}}^+}\tfrac{a}{L^2}u_+(\mathfrak{k}+\ell)^{2\mathfrak{q}}\mathfrak{Q}_n(\mathfrak{k}+\ell)^{-1}\Big]^{-1}
\end{aligned}$$

Lemma 6.3. *Let $|\mathrm{Im}\,\mathfrak{k}_\nu| \le \frac{2}{L_\nu}$ for each $0\le\nu\le 3$. There is a constant Γ_{17}, depending only on $\mathfrak{q}$, such that the following hold for all $L>\Gamma_{17}$.*

(a) We have $\frac{6}{7}\frac{a}{L^2} \le |\check{\mathfrak{Q}}_{n+1}(\mathfrak{k})| \le \frac{6}{5}\frac{a}{L^2}$ and $\mathrm{Re}\,\check{\mathfrak{Q}}_{n+1}(\mathfrak{k}) \ge \frac{a}{2L^2}$.

(b) We have

$$\Big|\big(aL^{-2}Q^*Q+\mathfrak{Q}_n\big)_{\mathfrak{k}}^{-1}(\ell,\ell') - \mathfrak{Q}_n(\mathfrak{k}+\ell)^{-1}\delta_{\ell,\ell'}\Big| \le \tfrac{2}{aL^2}\prod_{\nu=0}^{3}\big(\tfrac{24}{L_\nu|\ell_\nu|+\pi}\big)^{\mathfrak{q}}\prod_{\nu=0}^{3}\big(\tfrac{24}{L_\nu|\ell'_\nu|+\pi}\big)^{\mathfrak{q}}\begin{cases}1 & \textit{for all } \ell\\ \prod\limits_{\substack{0\le\nu\le3\\ \ell_\nu\ne0}} L_\nu|\mathfrak{k}_\nu| & \textit{if } \ell\ne0\end{cases}$$

(c) Let $0 \le \nu \le 3$. *Then*

$$\big|\big[\mathfrak{Q}_n^{-1} Q_{+,\nu}^{(+)} \check{\mathfrak{Q}}_{n+1} Q_{+,\nu}^{(-)}\big]_{\mathfrak{k}}(\ell,\ell')\big| \le \tfrac{3e^4}{2L^2}\big(\tfrac{24}{\pi}\big)^4 \prod_{\nu=0}^{3}\big(\tfrac{24}{L_\nu|\ell_\nu|+\pi}\big)^{\mathfrak{q}-1} \prod_{\nu=0}^{3}\big(\tfrac{24}{L_\nu|\ell'_\nu|+\pi}\big)^{\mathfrak{q}}$$

Proof. (a) is proven much as Proposition 2.4.a was.

(b) The straight forward Neumann expansion gives

$$\begin{aligned}
&\Big|\big(\tfrac{a}{L^2}Q^*Q + \mathfrak{Q}_n\big)_{\mathfrak{k}}^{-1}(\ell,\ell') - \mathfrak{Q}_n(\mathfrak{k}+\ell)^{-1}\delta_{\ell,\ell'}\Big| \\
&\qquad \le \tfrac{a}{L^2}\Big|\mathfrak{Q}_n(\mathfrak{k}+\ell)^{-1}u_+(\mathfrak{k}+\ell)^{\mathfrak{q}}u_+(\mathfrak{k}+\ell')^{\mathfrak{q}}\mathfrak{Q}_n(\mathfrak{k}+\ell')^{-1}\Big| \sum_{j=0}^{\infty}\Big[\sum_{l\in\hat{\mathcal{B}}^+}\tfrac{a}{L^2}\Big|\tfrac{u_+(\mathfrak{k}+l)^{2\mathfrak{q}}}{\mathfrak{Q}_n(\mathfrak{k}+l)}\Big|\Big]^j \\
&\qquad \le \tfrac{a}{L^2}\big(\tfrac{5}{4a}\big)^2\big|u_+(\mathfrak{k}+\ell)^{\mathfrak{q}}\big| \prod_{\nu=0}^{3}\big(\tfrac{24}{L_\nu|\ell'_\nu|+\pi}\big)^{\mathfrak{q}} \sum_{j=0}^{\infty}\big[\tfrac{5\Gamma'_{15}}{4L^2}\big]^j \\
&\qquad \le \tfrac{2}{aL^2}\prod_{\nu=0}^{3}\big(\tfrac{24}{L_\nu|\ell_\nu|+\pi}\big)^{\mathfrak{q}} \prod_{\nu=0}^{3}\big(\tfrac{24}{L_\nu|\ell'_\nu|+\pi}\big)^{\mathfrak{q}} \begin{cases} 1 & \text{for all } \ell \\ \prod\limits_{\substack{0\le\nu\le3\\ \ell_\nu\ne0}} L_\nu|\mathfrak{k}_\nu| & if \ell \ne 0 \end{cases}
\end{aligned}$$

by part (a), Lemma 2.3.a,b and Proposition 2.4.a, with L satisfying the conditions of part (a).

(c) The specified bound follows from (2.12), part (a), Proposition 2.4.a and Lemmas 2.6.b, 2.3.a. □

Lemma 6.4. *For all*

$$\begin{array}{ll}
\mathfrak{k} = \mathbb{L}^{-1}(k) \in \big(\mathbb{R}/\tfrac{2\pi}{L^2}\mathbb{Z}\big)\times\big(\mathbb{R}^3/\tfrac{2\pi}{L}\mathbb{Z}^3\big) & k \in \big(\mathbb{R}/2\pi\mathbb{Z}\big)\times\big(\mathbb{R}^3/2\pi\mathbb{Z}^3\big) \\
p = \mathbb{L}^{-1}(\mathfrak{p}) \in \big(\mathbb{R}/\tfrac{2\pi}{\varepsilon_n^2}\mathbb{Z}\big)\times\big(\mathbb{R}^3/\tfrac{2\pi}{\varepsilon_n}\mathbb{Z}^3\big) & \mathfrak{p} \in \big(\mathbb{R}/\tfrac{2\pi}{\varepsilon_{n+1}^2}\mathbb{Z}\big)\times\big(\mathbb{R}^3/\tfrac{2\pi}{\varepsilon_{n+1}}\mathbb{Z}^3\big) \\
 & \ell_{n+1} \in \big(2\pi\mathbb{Z}/\tfrac{2\pi}{\varepsilon_{n+1}^2}\mathbb{Z}\big)\times\big(2\pi\mathbb{Z}^3/\tfrac{2\pi}{\varepsilon_{n+1}}\mathbb{Z}^3\big)
\end{array}$$

we have

(a) $u_+\big(\mathbb{L}^{-1}(k)\big) = u_1(k)$ *and* $u_n\big(\mathbb{L}^{-1}(k)\big)u_+\big(\mathbb{L}^{-1}(k)\big) = u_{n+1}(k)$ *for all* $n \in \mathbb{N}$.

(b) $\check{\mathfrak{Q}}_{n+1}\big(\mathbb{L}^{-1}(k)\big) = \frac{1}{L^2}\mathfrak{Q}_{n+1}(k)$

(c) $\mathbf{D}_n^{-1}\big(\mathbb{L}^{-1}(\mathfrak{p})\big) = L^2\mathbf{D}_{n+1}^{-1}(\mathfrak{p})$

(d) $\check{\Delta}^{(n+1)(*)}\big(\mathbb{L}^{-1}(k)\big) = \frac{1}{L^2}\Delta^{(n+1)(*)}(k)$

(e) $\check{S}_{n+1,\mathbb{L}^{-1}(k)}\big(\mathbb{L}^{-1}(\ell_{n+1}),\mathbb{L}^{-1}(\ell'_{n+1})\big) = L^2\,\hat{S}_{n+1,k}(\ell_{n+1},\ell'_{n+1})$

Proof. These all follow from (6.2), Remark 6.2.a and

- $Q_1 = \mathbb{L}_*^{-1} Q \mathbb{L}_*$ by (2.2)
- $\mathbf{D}_{n+1} = L^2\,\mathbb{L}_*^{-1}\mathbf{D}_n\mathbb{L}_*$ by (3.1)

and [5, Lemmas 15.b and 16.b].

□

Corollary 6.5. *There are constants $m_9 > 0$ and Γ_{18}such that, for all $L > \Gamma_{17}$,*

$$\|\mathbb{L}_*^{-1}\check{S}_{n+1}\mathbb{L}_*\|_{m_9} \le L^2\Gamma_{18} \qquad \|\mathbb{L}_*^{-1}\check{S}_{n+1}\check{Q}_{n+1}^*\mathbb{L}_*\|_{m_9} \le L^2\Gamma_{18}$$

Proof. Set $m_9 = \min\{m_3, m_6\}$. Just combine [5, Lemmas 15.b and 16.b] and parts (a) and (e) of Lemma 6.4 to yield

$$\|\mathbb{L}_*^{-1}\check{S}_{n+1}\mathbb{L}_*\|_{m_9} = L^2\|S_{n+1}\|_{m_9} \qquad \|\mathbb{L}_*^{-1}\check{S}_{n+1}\check{Q}_{n+1}^*\mathbb{L}_*\|_{m_9} = L^2\|S_{n+1}Q_{n+1}\|_{m_9}$$

and then apply Proposition 5.1 and Lemma 5.6. □

Lemma 6.6. *Assume that $L > \Gamma_{17}$, the constant of Lemma 6.3. There are constants $m_{10} > 0$ and Γ_{19} such that the following hold for all $k \in \mathbb{C}^4$ with $|\mathrm{Im}\,k| \le m_{10}$.*

(a) $\Big|\big(\mathbb{L}_*^{-1}A_{\psi_{(*)}\theta_{(*)}}\mathbb{L}_*\big)_k(\ell_1)\Big| = \Big|\big(A_{\psi_{(*)}\theta_{(*)}}\big)_{\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1))\Big| \le \Gamma_{19}\prod_{\nu=0}^{3}\big(\frac{24}{|\ell_{1,\nu}|+\pi}\big)^{\mathfrak{q}}$

(b) If $\ell_1 \ne 0$, then $\Big|\big(\mathbb{L}_*^{-1}A_{\psi_{(*)}\theta_{(*)}}\mathbb{L}_*\big)_k(\ell_1)\Big| \le \Gamma_{19}|k|\prod_{\nu=0}^{3}\big(\frac{24}{|\ell_{1,\nu}|+\pi}\big)^{\mathfrak{q}}$

Proof. By (6.6), Lemma 6.4 and Remark 2.1.e,

$$\begin{aligned}
&\big(A_{\psi_{(*)}\theta_{(*)}}\big)_{\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1)) \\
&= \sum_{\ell'_1\in\hat{\mathcal{B}}_1}\big(\tfrac{a}{L^2}Q^*Q+\mathfrak{Q}_n\big)^{-1}_{\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1),\mathbb{L}^{-1}(\ell'_1))\Big\{\tfrac{a}{L^2}Q^*_{\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell'_1)) \\
&\qquad + \mathfrak{Q}_n(\mathbb{L}^{-1}(k+\ell'_1))\,\big(Q_n\mathbf{D}_n^{-1(*)}Q_n^*\big)(\mathbb{L}^{-1}(k+\ell'_1)) \\
&\qquad\qquad Q^*_{\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell'_1))\check{\Delta}^{(n+1)(*)}(\mathbb{L}^{-1}(k))\Big\} \\
&= \sum_{\ell'_1\in\hat{\mathcal{B}}_1}\big(\tfrac{a}{L^2}Q^*Q+\mathfrak{Q}_n\big)^{-1}_{\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1),\mathbb{L}^{-1}(\ell'_1))\;B_1(k,\ell'_1) \\
&= \mathfrak{Q}_n\big(\mathbb{L}^{-1}(k+\ell_1)\big)^{-1}\;B_1(k,\ell_1) + \sum_{\ell'_1\in\hat{\mathcal{B}}_1}C(k,\ell_1,\ell'_1)B_1(k,\ell'_1)
\end{aligned}$$

where

$$B_1(k,\ell_1') = u_1(k+\ell_1')^{\mathfrak{q}}\Big\{\tfrac{a}{L^2} + \mathfrak{Q}_n\big(\mathbb{L}^{-1}(k+\ell_1')\big)\sum_{\ell_n\in\hat{\mathcal{B}}_n} B_2(k,\ell_1',\ell_n)\Big\}$$

$$B_2(k,\ell_1',\ell_n) = u_n(\mathbb{L}^{-1}(k+\ell_1')+\ell_n)^{2\mathfrak{q}}\ \mathbf{D}_{n+1}^{-1(*)}\big(k+\ell_1'+\mathbb{L}(\ell_n)\big)\ \Delta^{(n+1)(*)}(k)$$

$$C(k,\ell_1,\ell_1') = \big(\tfrac{a}{L^2}Q^*Q+\mathfrak{Q}_n\big)^{-1}_{\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1),\mathbb{L}^{-1}(\ell_1')) - \mathfrak{Q}_n\big(\mathbb{L}^{-1}(k+\ell_1)\big)^{-1}\delta_{\ell_1,\ell_1'}$$

(a) Choose $m_{10} = \min\{2, m_1, \bar{m}(\pi)\}$. Then

$$\Big|\big(\tfrac{a}{L^2}Q^*Q+\mathfrak{Q}_n\big)^{-1}_{\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1),\mathbb{L}^{-1}(\ell_1'))\Big| \le \tfrac{6}{5}a\,\delta_{\ell_1,\ell_1'} + \tfrac{2}{aL^2}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_{1,\nu}|+\pi}\big)^{\mathfrak{q}}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_{1,\nu}'|+\pi}\big)^{\mathfrak{q}}$$

(by Lemma 6.3.b, Proposition 2.4.a)

$$\big|u_1(k+\ell_1')^{\mathfrak{q}}\big| \le \prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_{1,\nu}'|+\pi}\big)^{\mathfrak{q}} \qquad \text{(by Lemma 2.2.a)}$$

$$\big|\mathfrak{Q}_n(\mathbb{L}^{-1}(k+\ell_1'))\big| \le \tfrac{6}{5}a \qquad \text{(by Proposition 2.4.a)}$$

$$\big|u_n(\mathbb{L}^{-1}(k+\ell_1')+\ell_n)^{2\mathfrak{q}}\big| \le \prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_{n,\nu}|+\pi}\big)^{2\mathfrak{q}} \qquad \text{(by Lemma 2.2.a)}$$

$$\big|\mathbf{D}_{n+1}^{-1(*)}\big(k+\ell_1'+\mathbb{L}(\ell_n)\big)\big| \le \tfrac{1}{\gamma_1\pi} \text{ if } (\ell_1',\ell_n)\ne(0,0) \qquad \text{(by Lemma 3.2.d)}$$

$$\big|\Delta^{(n+1)(*)}(k)\big| \le 2a \qquad \text{(by Lemma 4.2.c)}$$

$$\big|\mathbf{D}_{n+1}^{-1(*)}(k)\ \Delta^{(n+1)(*)}(k)\big| \le \Gamma_6 \qquad \text{(by Lemma 4.2.f)}$$

So

$$\begin{aligned}
&\Big|\big(A_{\psi_{(*)}\theta_{(*)}}\big)_{\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1))\Big| \\
&\quad\le \sum_{\ell_1'\in\hat{\mathcal{B}}_1}\Big\{\tfrac{6}{5}a\,\delta_{\ell_1,\ell_1'} + \tfrac{2}{aL^2}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_{1,\nu}|+\pi}\big)^{\mathfrak{q}}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_{1,\nu}'|+\pi}\big)^{\mathfrak{q}}\Big\} \\
&\qquad\qquad \prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_{1,\nu}'|+\pi}\big)^{\mathfrak{q}}\Big\{\tfrac{a}{L^2} + \tfrac{6}{5}a\sum_{\ell_n\in\hat{\mathcal{B}}_n}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_n|+\pi}\big)^{2\mathfrak{q}}\max\big\{\Gamma_6,\tfrac{2a}{\gamma_1\pi}\big\}\Big\} \\
&\quad\le \text{const}\sum_{\ell_1'\in\hat{\mathcal{B}}_1}\Big\{\tfrac{6}{5}a\,\delta_{\ell_1,\ell_1'} + \tfrac{2}{aL^2}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_{1,\nu}|+\pi}\big)^{\mathfrak{q}}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_{1,\nu}'|+\pi}\big)^{\mathfrak{q}}\Big\}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_{1,\nu}'|+\pi}\big)^{\mathfrak{q}} \\
&\quad\le \Gamma_{19}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_{1,\nu}|+\pi}\big)^{\mathfrak{q}}
\end{aligned}$$

(b) Using the bounds of part (a) together with

$$\left|u_1(k+\ell_1')^{\mathfrak{q}}\right| \le |k| \prod_{\nu=0}^{3} \left(\tfrac{24}{|\ell'_{1,\nu}|+\pi}\right)^{\mathfrak{q}} \quad \text{if } \ell_1' \ne 0 \qquad \text{(by Lemma 2.2.b)}$$

$$\left|u_n(\mathbb{L}^{-1}(k+\ell_1')+\ell_n)^{2\mathfrak{q}}\right| \le |\mathbb{L}^{-1}(k+\ell_1')| \prod_{\nu=0}^{3} \left(\tfrac{24}{|\ell_n|+\pi}\right)^{2\mathfrak{q}} \text{ if } \ell_n \ne 0 \qquad \text{(by Lemma 2.2.b)}$$

we have, if $\ell_1' \ne 0$,

$$\begin{aligned}\left|B_1(k,\ell_1')\right| &\le |k| \prod_{\nu=0}^{3} \left(\tfrac{24}{|\ell'_{1,\nu}|+\pi}\right)^{\mathfrak{q}} \Big\{ \tfrac{a}{L^2} + \tfrac{6}{5}a \sum_{\ell_n \in \hat{\mathcal{B}}_n} \prod_{\nu=0}^{3} \left(\tfrac{24}{|\ell_n|+\pi}\right)^{2\mathfrak{q}} \max\left\{\Gamma_6, \tfrac{2a}{\gamma_1 \pi}\right\}\Big\} \\ &\le \text{const}|k| \prod_{\nu=0}^{3} \left(\tfrac{24}{|\ell'_{1,\nu}|+\pi}\right)^{\mathfrak{q}}\end{aligned}$$

and

$$\left|B_1(k,0)\right| \le \prod_{\nu=0}^{3} \left(\tfrac{24}{\pi}\right)^{\mathfrak{q}} \Big\{ \tfrac{a}{L^2} + \tfrac{6}{5}a \sum_{\ell_n \in \hat{\mathcal{B}}_n} \prod_{\nu=0}^{3} \left(\tfrac{24}{|\ell_n|+\pi}\right)^{2\mathfrak{q}} \max\left\{\Gamma_6, \tfrac{2a}{\gamma_1 \pi}\right\}\Big\} \le \text{const}$$

Using these bounds, the first bound of part (a) and Lemma 6.3.b, and assuming that $\ell_1 \ne 0$,

$$\begin{aligned}\left(A_{\psi_{(*)}\theta_{(*)}}\right)_{\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1)) &= \sum_{\ell_1' \in \hat{\mathcal{B}}_1} \left(\tfrac{a}{L^2}Q^*Q + \mathfrak{Q}_n\right)^{-1}_{\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1), \mathbb{L}^{-1}(\ell_1'))\ B_1(k,\ell_1') \\ &= \left(\tfrac{a}{L^2}Q^*Q + \mathfrak{Q}_n\right)^{-1}_{\mathbb{L}^{-1}(k)}(\mathbb{L}^{-1}(\ell_1),0))\ B_1(k,0) + O\Big(|k| \prod_{\nu=0}^{3} \left(\tfrac{24}{|\ell_{1,\nu}|+\pi}\right)^{\mathfrak{q}}\Big) \\ &= \tfrac{2}{aL^2} \prod_{\nu=0}^{3} \left(\tfrac{24}{|\ell_{1,\nu}|+\pi}\right)^{\mathfrak{q}} \prod_{\nu=0}^{3} \left(\tfrac{24}{\pi}\right)^{\mathfrak{q}} \prod_{\substack{0\le\nu\le 3\\ \ell_{1,\nu}\ne 0}} |k_\nu|\ B_1(k,0) + O\Big(|k| \prod_{\nu=0}^{3} \left(\tfrac{24}{|\ell_{1,\nu}|+\pi}\right)^{\mathfrak{q}}\Big) \\ &= O\Big(|k| \prod_{\nu=0}^{3} \left(\tfrac{24}{|\ell_{1,\nu}|+\pi}\right)^{\mathfrak{q}}\Big)\end{aligned}$$

□

Proof of Proposition 6.1.

Bound on $\left\|\mathbb{L}_*^{-1} A_{\psi,\phi} \mathbb{L}_*\right\|_{m=1}$*:* By [5, Lemma 15.b] and Lemma 6.3.b, if $|\operatorname{Im} k_{\nu'}| \le 2$

for each $0 \le \nu' \le 3$ then

$$\begin{aligned}
&\big|\big[\mathbb{L}_*^{-1}\big\{\big(aL^{-2}Q^*Q+\mathfrak{Q}_n\big)^{-1}-\mathfrak{Q}_n^{-1}\big\}\mathbb{L}_*\big]_k(\ell,\ell')\big| \\
&\qquad = \big|\big[\big(aL^{-2}Q^*Q+\mathfrak{Q}_n\big)^{-1}-\mathfrak{Q}_n^{-1}\big]_{\mathbb{L}^{-1}k}(\mathbb{L}^{-1}\ell,\mathbb{L}^{-1}\ell')\big| \\
&\qquad \le \tfrac{2}{aL^2}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_\nu|+\pi}\big)^{\mathfrak{q}}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell'_\nu|+\pi}\big)^{\mathfrak{q}}
\end{aligned}$$

So, by [5, Lemma 12.b],

$$\big\|\mathbb{L}_*^{-1}\big\{\big(aL^{-2}Q^*Q+\mathfrak{Q}_n\big)^{-1}-\mathfrak{Q}_n^{-1}\big\}\mathbb{L}_*\|_{m=1} \le \mathrm{const}_{\mathfrak{q}}$$

By Proposition 2.4.a, Lemma 2.2.a and [5, Lemmas 12.b,c],

$$\|\mathfrak{Q}_n\|_{m=1},\ \|\mathfrak{Q}_n^{-1}\|_{m=1},\ \|Q_n\|_{m=1} \le \mathrm{const}_{\mathfrak{q}}$$

too. Now just apply [5, Lemmas 15.c and 16.c].

Bound on $\big\|\mathbb{L}_*^{-1}A_{\psi,\phi,\nu}\mathbb{L}_*\big\|_{m=1}$*:* By [5, Lemma 15.b] and Lemma 6.3.c, if $|\mathrm{Im}\,k_{\nu'}| \le 2$ for each $0 \le \nu' \le 3$ then

$$\begin{aligned}
\big|\big[\mathbb{L}_*^{-1}\big(\mathfrak{Q}_n^{-1}Q_{+,\nu}^{(+)}\check{\mathfrak{Q}}_{n+1}Q_{+,\nu}^{(-)}\big)\mathbb{L}_*\big]_k(\ell,\ell')\big| &= \big|\big[\mathfrak{Q}_n^{-1}Q_{+,\nu}^{(+)}\check{\mathfrak{Q}}_{n+1}Q_{+,\nu}^{(-)}\big]_{\mathbb{L}^{-1}k}(\mathbb{L}^{-1}\ell,\mathbb{L}^{-1}\ell')\big| \\
&\le \tfrac{3e^4}{2L^2}\big(\tfrac{24}{\pi}\big)^4\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_\nu|+\pi}\big)^{\mathfrak{q}-1}\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell'_\nu|+\pi}\big)^{\mathfrak{q}}
\end{aligned}$$

As $\mathfrak{q} > 2$, [5, Lemma 12.b] yields

$$\big\|\mathbb{L}_*^{-1}\mathfrak{Q}_n^{-1}Q_{+,\nu}^{(+)}\check{\mathfrak{Q}}_{n+1}Q_{+,\nu}^{(-)}\mathbb{L}_*\|_{m=1} \le \mathrm{const}_{\mathfrak{q}}$$

By Lemma 2.6.b and [5, Lemma 12.c], $\|Q_{n,\nu}^{(-)}\|_{m=1} \le \mathrm{const}_{\mathfrak{q}}$ too, since $\mathfrak{q} > 2$. Now just apply [5, Lemma 16.c].

Bound on $\big\|\mathbb{L}_*^{-1}A_{\psi_{(*)}\theta_{(*)}}\mathbb{L}_*\big\|_{m_8}$*:* This follows from Lemma 6.6.a by [5, Lemma 12.c].

Bound on $\big\|\mathbb{L}_*^{-1}\ \tfrac{a}{L^2}C^{(n)}(\mu)^{(*)}Q^*\ \mathbb{L}_*\big\|_{m_8}$*:* By (6.3)

$$\mathbb{L}_*^{-1}\ \tfrac{a}{L^2}C^{(n)}(\mu)^{(*)}Q^*\ \mathbb{L}_* = \mathbb{L}_*^{-1}A_{\psi_{(*)}\theta_{(*)}}\mathbb{L}_* + L^2\mu\mathbb{L}_*^{-1}A_{\psi,\phi}\mathbb{L}_*S_{n+1}^{(*)}S_{n+1}(L^2\mu)^{(*)}Q_{n+1}^*\mathfrak{Q}_{n+1}$$

Now just apply Proposition 5.1, Lemma 2.2 and Proposition 2.4.c.

Bound on $\big\|\mathbb{L}_*^{-1}A_{\psi_{(*)}\theta_{(*)}\nu}\mathbb{L}_*\big\|_{m_8}$*:* It suffices to bound

- $\mathbb{L}_*^{-1}A_{\psi,\phi,\nu}\mathbb{L}_*$ as above,

- bound $\mathbb{L}_*^{-1}\mathfrak{Q}_n^{-1}Q_{+,\nu}^{(+)}\check{\mathfrak{Q}}_{n+1}\mathbb{L}_*$ using

$$\begin{aligned}\big|\big[\mathbb{L}_*^{-1}\big(\mathfrak{Q}_n^{-1}Q_{+,\nu}^{(+)}\check{\mathfrak{Q}}_{n+1}\big)\mathbb{L}_*\big]_k(\ell)\big| &= \big|\big[\mathfrak{Q}_n^{-1}Q_{+,\nu}^{(+)}\check{\mathfrak{Q}}_{n+1}\big]_{\mathbb{L}^{-1}k}(\mathbb{L}^{-1}\ell)\big| \\ &\le \tfrac{3e^2}{2L^2}\big(\tfrac{24}{\pi}\big)^3\prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_\nu|+\pi}\big)^{\mathfrak{q}-1}\end{aligned}$$

(by [5, Lemma 16.b], (2.12), Proposition 2.4.a and Lemmas 2.6.b, 2.3.a, 6.3.a) and [5, Lemma 12.c], and
- bound $D_{n+1}^{-1(*)}Q_{n+1,\nu}^{(+)}\Delta^{(n+1)(*)}$ using

$$\begin{aligned}&\big|\big(\hat{\mathbf{D}}_{n+1}^{-1(*)}Q_{n+1,\nu}^{(+)}\Delta^{(n+1)(*)}\big)_k(\ell_{n+1})\big| \\ &\quad\le \big|\hat{\mathbf{D}}_{n+1}^{-1(*)}\big(k+\ell_{n+1}\big)\big)\zeta_{n+1,\nu}^{(+)}(k,\ell_{n+1})u_{n+1,\nu}^{(+)}(k+\ell_{n+1})u_{n+1}(k+\ell_{n+1})^{\mathfrak{q}-1} \\ &\qquad\qquad \hat{\Delta}^{(n+1)(*)}(k)\big|\end{aligned}$$

(by (2.11)) and

$$\begin{aligned}\big|\zeta_{n+1,\nu}^{(+)}(k,\ell_{n+1})u_{n+1,\nu}^{(+)}(k+\ell_{n+1})\big| &\le e^2\big(\tfrac{24}{\pi}\big)^3 && \text{(by Lemma 2.6.b)}\\ \big|u_{n+1}(k+\ell_{n+1})^{\mathfrak{q}-1}\big| &\le \prod_{\nu=0}^{3}\big(\tfrac{24}{|\ell_{n+1}|+\pi}\big)^{\mathfrak{q}-1} && \text{(by Lemma 2.2.a)}\\ \big|\hat{\mathbf{D}}_{n+1}^{-1(*)}\big(k+\ell_{n+1}\big)\big| &\le \tfrac{1}{\gamma_1\pi}\ \text{if } \ell_{n+1}\ne 0 && \text{(by Lemma 3.2.d)}\\ \big|\hat{\Delta}^{(n+1)(*)}(k)\big| &\le 2a && \text{(by Lemma 4.2.c)}\\ \big|\hat{\mathbf{D}}_{n+1}^{-1(*)}(k)\ \hat{\Delta}^{(n+1)(*)}(k)\big| &\le \Gamma_6 && \text{(by Lemma 4.2.f)}\end{aligned}$$

and [5, Lemma 12.c].

Bound on $\big\|\mathbb{L}_*^{-1}A_{\psi_{(*)}\theta_{(*)}\nu}(\mu)\mathbb{L}_*\big\|_{m_8}$*:* This follows from the previous bounds of this Proposition, Remark 6.2.b, Proposition 5.1, Lemma 2.6.c and Proposition 2.4.c.

□

A Trigonometric Inequalities

Lemma A.1.

(a) For x, y *real with* $|x| \le \frac{\pi}{2}$,

$$\big|\sin(x+iy)\big| \ge \tfrac{\sqrt{2}}{\pi}|x+iy|$$

(b) For x, y *real with* $|y| \le 1$,

$$\tfrac{|\sin(x+iy)|}{|x+iy|} \le 2\min\left\{1\,,\ \tfrac{1}{|x+iy|}\right\} \qquad \left|\mathrm{Im}\,\tfrac{\sin(x+iy)}{x+iy}\right| \le 2|y|\min\left\{|x|\,,\ \tfrac{2}{|x+iy|}\right\}$$

(c) For $0 < \varepsilon \le 1$ *and* x, y *real with* $|\varepsilon x| \le \pi$, $|y| \le 2$

$$\left|\frac{\sin\frac{1}{2}(x+iy)}{\frac{1}{\varepsilon}\sin\frac{1}{2}\varepsilon(x+iy)}\right| \le 4\min\left\{1, \frac{2}{|x|}\right\} \qquad \left|\mathrm{Im}\,\frac{\sin\frac{1}{2}(x+iy)}{\frac{1}{\varepsilon}\sin\frac{1}{2}\varepsilon(x+iy)}\right| \le 6|y|\min\left\{|x|\,,\ \frac{8}{|x|}\right\}$$

(d) For x *real with* $|x| \le \frac{\pi}{2}$,

$$\tfrac{2}{\pi} \le \tfrac{\sin x}{x} \le 1$$

(e) For any complex number z *obeying* $|z| \le 2$,

$$\left|\tfrac{\sin z}{z} - 1\right| \le \tfrac{1}{2}|z|^2$$

Proof. By the standard trig identity

$$\sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

(a) For x, y real with $|x| \le \frac{\pi}{2}$

$$\begin{aligned}
\big|\mathrm{Re}\,\sin(x+iy)\big| &= \big|\sin(x)\cosh(y)\big| \ge |\sin(x)| \ge \tfrac{2}{\pi}|x| \\
\big|\mathrm{Re}\,\sin(x+iy)\big| &= \big|\sin(x)\cosh(y)\big| \ge \big|\sin(x)\sinh(y)\big| \ge |\sin(x)|\,|y| \\
\big|\mathrm{Im}\,\sin(x+iy)\big| &= \big|\cos(x)\sinh(y)\big| \ge |\cos(x)|\,|y|
\end{aligned}$$

so that

$$\big|\sin(x+iy)\big| \ge \max\left\{\tfrac{2}{\pi}|x|\,,\ |y|\right\} \ge \tfrac{\sqrt{2}}{\pi}|x+iy|$$

(b) For x, y real with $|y| \le 1$,

$$\begin{aligned}
|\sin(x+iy)| &= \big|\sin(x)\cosh(y) + i\cos(x)\sinh(y)\big| \le \cosh(1)\big|\sin(x) + i\cos(x)\big| \\
&= \cosh(1)
\end{aligned}$$

and, since $\big|\sinh(y)\big| = \Big|\sum_{n=0}^\infty \frac{y^{2n+1}}{(2n+1)!}\Big| \le |y|\sum_{n=0}^\infty \frac{|y|^{2n}}{(2n)!} = |y|\cosh(y) \le \cosh(1)|y|$,

$$\begin{aligned}|\sin(x+iy)| &= \big|\sin(x)\cosh(y) + i\cos(x)\sinh(y)\big| \le \cosh(1)\big||x| + i|y|\big| \\ &= \cosh(1)|x+iy|\end{aligned}$$

Thus

$$\tfrac{|\sin(x+iy)|}{|x+iy|} \le \cosh(1)\min\big\{1\,,\ \tfrac{1}{|x+iy|}\big\} \le 2\min\big\{1\,,\ \tfrac{1}{|x+iy|}\big\}$$

giving the first bound.

For the second bound

$$\Big|\mathrm{Im}\,\tfrac{\sin(x+iy)}{x+iy}\Big| = \Big|\tfrac{x\,\mathrm{Im}\,\sin(x+iy) - y\,\mathrm{Re}\,\sin(x+iy)}{x^2+y^2}\Big| = \Big|\tfrac{x\,\cos(x)\sinh(y) - y\,\sin(x)\cosh(y)}{x^2+y^2}\Big|$$

Using

$$\begin{aligned}
\cos x - 1 &= -\int_0^x dt\ \sin t &&= -\int_0^x dt\int_0^t ds\ \cos s \\
\sin x - x &= \int_0^x dt\ [\cos t - 1] &&= -\int_0^x dt\int_0^t ds\int_0^s du\ \cos u \\
\cosh y - 1 &= \int_0^y dt\ \sinh t &&= \int_0^y dt\int_0^t ds\ \cosh s \\
\sinh y - y &= \int_0^y dt\ [\cosh t - 1] &&= \int_0^y dt\int_0^t ds\int_0^s du\ \cosh u
\end{aligned}$$

and $\cosh(1) < 2$, we have

$$\begin{aligned}\cos x &= 1 + \alpha(x)\tfrac{x^2}{2} &\qquad \sin x &= x + \beta(x)\tfrac{x^3}{6} \\ \cosh y &= 1 + \gamma(y)y^2 & \sinh y &= y + \delta(y)\tfrac{y^3}{3}\end{aligned}$$

with, for $|y| \le 1$, $|\alpha(x)|\,,\ |\beta(x)|\,,\ |\gamma(y)|\,,\ |\delta(y)|\ \le 1$. Consequently,

$$\Big|\mathrm{Im}\,\tfrac{\sin(x+iy)}{x+iy}\Big| \le 2|xy|$$

Alternatively, using $|\sin(x)| \le |x|$, $|\sinh(y)| \le 2|y|$ and $|\cosh(y)| \le 2$,

$$\Big|\mathrm{Im}\,\tfrac{\sin(x+iy)}{x+iy}\Big| \le \tfrac{4|xy|}{x^2+y^2} \le \tfrac{4|y|}{\sqrt{x^2+y^2}}$$

(c) For $0 < \varepsilon \le 1$, x, y real and $|\varepsilon x| \le \pi$, $|y| \le 2$,

$$\left|\frac{\sin\frac12(x+iy)}{\frac1\varepsilon\sin\frac12\varepsilon(x+iy)}\right| = \left|\frac{\frac{\sin\frac12(x+iy)}{\frac12(x+iy)}}{\frac{\sin\frac12\varepsilon(x+iy)}{\frac12\varepsilon(x+iy)}}\right| \le \tfrac{\pi}{\sqrt2}\cosh(1)\min\left\{1, \frac{2}{|x+iy|}\right\} \le 4\min\left\{1, \frac{2}{|x|}\right\}$$

and

$$
\begin{aligned}
\left|\mathrm{Im}\,\frac{\sin\frac{1}{2}(x+iy)}{\frac{1}{\varepsilon}\sin\frac{1}{2}\varepsilon(x+iy)}\right| &= \frac{\left|\mathrm{Im}\,\frac{\sin\frac{1}{2}(x+iy)}{\frac{1}{2}(x+iy)}\mathrm{Re}\,\frac{\sin\frac{1}{2}\varepsilon(x+iy)}{\frac{1}{2}\varepsilon(x+iy)} - \mathrm{Re}\,\frac{\sin\frac{1}{2}(x+iy)}{\frac{1}{2}(x+iy)}\mathrm{Im}\,\frac{\sin\frac{1}{2}\varepsilon(x+iy)}{\frac{1}{2}\varepsilon(x+iy)}\right|}{\left|\frac{\sin\frac{1}{2}\varepsilon(x+iy)}{\frac{1}{2}\varepsilon(x+iy)}\right|^2} \\
&\le \frac{|y|\min\left\{\frac{1}{2}|x|\,,\ \frac{4}{|x+iy|}\right\} + 4\min\left\{1, \frac{2}{|x|}\right\}\left|\mathrm{Im}\,\frac{\sin\frac{1}{2}\varepsilon(x+iy)}{\frac{1}{2}\varepsilon(x+iy)}\right|}{\left|\frac{\sin\frac{1}{2}\varepsilon(x+iy)}{\frac{1}{2}\varepsilon(x+iy)}\right|} \\
&\le \frac{|y|\min\left\{\frac{1}{2}|x|\,,\ \frac{4}{|x+iy|}\right\} + 4\min\left\{1, \frac{2}{|x|}\right\}\varepsilon|y|\min\left\{\frac{1}{2}\varepsilon|x|\,,\ \frac{4}{\varepsilon|x+iy|}\right\}}{\frac{\sqrt{2}}{\pi}} \\
&\le \tfrac{\pi}{\sqrt{2}}|y|\min\left\{\left(\tfrac{1}{2}+2\varepsilon^2\right)|x|\,,\ \tfrac{4+16}{|x+iy|}\right\} \\
&\le 6|y|\min\left\{|x|\,,\ \tfrac{8}{|x+iy|}\right\}
\end{aligned}
$$

(d) and (e) are standard.

$\square$

B Lattice and Operator Summary

The following table gives, for most of the operators considered in this paper,

- the definition of the operator
- a reference to where in [7, 8], the operator is introduced and
- the translation invariance properties of the operator.

A later table will specify where, in this paper, bounds on the operators are proven.

Operator		Definition	Tiwrt
$D_0 = -e^{-h_0}\partial_0 + \big[\mathbb{1} - e^{-h_0}\big]$	$: \mathcal{H}_0 \to \mathcal{H}_0$	§1.5	$\mathcal{X}_0$
$D_n = L^{2n}\, \mathbb{L}_*^{-n}\, D_0\, \mathbb{L}_*^n$	$: \mathcal{H}_n \to \mathcal{H}_n$	Def 1.5.a	$\mathcal{X}_n$
$\mathfrak{Q}_n = a\big(\mathbb{1} + \sum_{j=1}^{n-1} \frac{1}{L^{2j}} Q_j Q_j^*\big)^{-1}$	$: \mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$	Def 1.5.b	$\mathcal{X}_0^{(n)}$
$\Delta^{(0)} = D_0$	$: \mathcal{H}_0 \to \mathcal{H}_0$	(1.14)	$\mathcal{X}_0$
$\Delta^{(n)} = \big(\mathbb{1} + \mathfrak{Q}_n Q_n D_n^{-1} Q_n^*\big)^{-1} \mathfrak{Q}_n,\ n \ge 1$	$: \mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$	(1.14)	$\mathcal{X}_0^{(n)}$
$C^{(n)} = \big(\frac{a}{L^2} Q^* Q + \Delta^{(n)}\big)^{-1}$	$: \mathcal{H}_0^{(n)} \to \mathcal{H}_0^{(n)}$	(1.15)	$\mathcal{X}_{-1}^{(n+1)}$
$S_n^{-1} = D_n + Q_n^* \mathfrak{Q}_n Q_n$	$: \mathcal{H}_n \to \mathcal{H}_n$	Thm 1.13	$\mathcal{X}_0^{(n)}$
$S_n(\mu)^{-1} = D_n + Q_n^* \mathfrak{Q}_n Q_n - \mu$	$: \mathcal{H}_n \to \mathcal{H}_n$	Thm 1.13	$\mathcal{X}_0^{(n)}$
$\check{\mathfrak{Q}}_{n+1} = \big(\frac{L^2}{a}\mathbb{1} + Q\mathfrak{Q}_n^{-1} Q^*\big)^{-1}$	$: \mathcal{H}_{-1}^{(n+1)} \to \mathcal{H}_{-1}^{(n+1)}$	Lem 2.4.b	$\mathcal{X}_{-1}^{(n+1)}$
$\check{S}_{n+1}(\mu) = \big\{D_n + \check{Q}_{n+1}^* \check{\mathfrak{Q}}_{n+1} \check{Q}_{n+1} - \mu\big\}^{-1}$	$: \mathcal{H}_n \to \mathcal{H}_n$	(5.1)	$\mathcal{X}_{-1}^{(n+1)}$
$A_{\psi,\phi}$	$: \mathcal{H}_n \to \mathcal{H}_0^{(n)}$	Prop 5.1	$\mathcal{X}_{-1}^{(n+1)}$

The references in the above table are to [7, 8] and "Tiwrt" stands for "translation invariant with respect to".

Operator		Definition	Tiwrt
$S_n Q_n^* = D_n^{-1} Q_n^* \big(\mathbb{1} + \mathfrak{Q}_n Q_n D_n^{-1} Q_n^*\big)^{-1}$	$: \mathcal{H}_0^{(n)} \to \mathcal{H}_n$	Lemma 5.6	$\mathcal{X}_0^{(n)}$
$\check{S}_{n+1} = \check{S}_{n+1}(0)$	$: \mathcal{H}_n \to \mathcal{H}_n$	after (6.2)	$\mathcal{X}_{-1}^{(n+1)}$
$A_{\psi,\phi}$	$: \mathcal{H}_n \to \mathcal{H}_0^{(n)}$	(6.4)	$\mathcal{X}_{-1}^{(n+1)}$
$A_{\psi,\phi,\nu}$	$: \mathcal{H}_n \to \mathcal{H}_0^{(n)}$	Remark 6.2	$\mathcal{X}_{-1}^{(n+1)}$
$A_{\psi_{(*)},\theta_{(*)}} = \frac{a}{L^2} C^{(n)^{(*)}} Q^*$	$: \mathcal{H}_{-1}^{(n+1)} \to \mathcal{H}_0^{(n)}$	before Rmk 6.2	$\mathcal{X}_{-1}^{(n+1)}$
$A_{\psi_*,\theta_*,\nu},\ A_{\psi,\theta,\nu},\ A_{\psi_*,\theta_*,\nu}(\mu),\ A_{\psi,\theta,\nu}(\mu)$	$: \mathcal{H}_{-1}^{(n+1)} \to \mathcal{H}_0^{(n)}$	Remark 6.2	$\mathcal{X}_{-1}^{(n+1)}$

The lattices involved are

$$\begin{aligned}
\mathcal{X}_n &= \big(\varepsilon_n^2\mathbb{Z}/\varepsilon_n^2 L_{\rm tp}\mathbb{Z}\big) \times \big(\varepsilon_n\mathbb{Z}^3/\varepsilon_n L_{\rm sp}\mathbb{Z}^3\big) & \hat{\mathcal{X}}_n &= \big(\tfrac{2\pi}{\varepsilon_n^2 L_{\rm tp}}\mathbb{Z}/\tfrac{2\pi}{\varepsilon_n^2}\mathbb{Z}\big)\times\big(\tfrac{2\pi}{\varepsilon_n L_{\rm sp}}\mathbb{Z}^3/\tfrac{2\pi}{\varepsilon_n}\mathbb{Z}^3\big)\\
\mathcal{X}_0^{(n)} &= \big(\mathbb{Z}/\varepsilon_n^2 L_{\rm tp}\mathbb{Z}\big) \times \big(\mathbb{Z}^3/\varepsilon_n L_{\rm sp}\mathbb{Z}^3\big) & \hat{\mathcal{X}}_0^{(n)} &= \big(\tfrac{2\pi}{\varepsilon_n^2 L_{\rm tp}}\mathbb{Z}/2\pi\mathbb{Z}\big)\times\big(\tfrac{2\pi}{\varepsilon_n L_{\rm sp}}\mathbb{Z}^3/2\pi\mathbb{Z}^3\big)\\
\mathcal{X}_{-1}^{(n+1)} &= \big(L^2\mathbb{Z}/\varepsilon_n^2 L_{\rm tp}\mathbb{Z}\big)\times\big(L\mathbb{Z}^3/\varepsilon_n L_{\rm sp}\mathbb{Z}^3\big) & \hat{\mathcal{X}}_{-1}^{(n+1)} &= \big(\tfrac{2\pi}{\varepsilon_n^2 L_{\rm tp}}\mathbb{Z}/\tfrac{2\pi}{L^2}\mathbb{Z}\big)\times\big(\tfrac{2\pi}{\varepsilon_n L_{\rm sp}}\mathbb{Z}^3/\tfrac{2\pi}{L}\mathbb{Z}^3\big)
\end{aligned}$$

where $\varepsilon_n = \frac{1}{L^n}$. The "single period" lattices are

$$\begin{aligned}
\mathcal{B}_n &= \big(\varepsilon_n^2\mathbb{Z}/\mathbb{Z}\big) \times \big(\varepsilon_n\mathbb{Z}^3/\mathbb{Z}^3\big) & \hat{\mathcal{B}}_n &= \big(2\pi\mathbb{Z}/\tfrac{2\pi}{\varepsilon_n^2}\mathbb{Z}\big)\times\big(2\pi\mathbb{Z}^3/\tfrac{2\pi}{\varepsilon_n}\mathbb{Z}^3\big)\\
\mathcal{B}^+ &= \big(\mathbb{Z}/L^2\mathbb{Z}\big) \times \big(\mathbb{Z}^3/L\mathbb{Z}^3\big) & \hat{\mathcal{B}}^+ &= \big(\tfrac{2\pi}{L^2}\mathbb{Z}/2\pi\mathbb{Z}\big)\times\big(\tfrac{2\pi}{L}\mathbb{Z}^3/2\pi\mathbb{Z}^3\big)
\end{aligned}$$

The following table specifies where, in this paper, bounds on the various operators are proven.

Operator	Bound
Q_n	Lemma 2.2.a
$\mathfrak{Q}_n$	Proposition 2.4
$Q_{n,\nu}^{(\pm)}$	Lemma 2.6
D_n	Lemma 3.2
$\Delta^{(n)}$	Lemma 4.2
$C^{(n)}$	Corollary 4.5
$D^{(n)}$	Corollary 4.5
$S_n(\mu),\ S_n$	Proposition 5.1
$S_{n,\nu}^{(\pm)}(\mu),\ S_{n,\nu}^{(\pm)}$	Proposition 5.1
$A_{\psi,\phi}$	Proposition 6.1
$A_{\psi,\phi,\nu}$	Proposition 6.1
$A_{\psi_{(*)}\theta_{(*)}}$	Proposition 6.1
$A_{\psi_{(*)}\theta_{(*)}\nu}(\mu)$	Proposition 6.1
$\check{S}_{n+1}$	Corollary 6.5

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