Power Series Representations for Complex Bosonic Effective Actions.
III. Substitution and Fixed Point Equations

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Abstract

In [3, 4, 5] we developed a polymer–like expansion that applies when the (effective) action in a functional integral is an analytic function of the fields being integrated. Here, we develop methods to aid the application of this technique when the method of steepest descent is used to analyze the functional integral. We develop a version of the Banach fixed point theorem that can be used to

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construct and control the critical fields, as analytic functions of external fields, and substitution formulae to control the change in norms that occurs when one replaces the integration fields by the sum of the critical fields and the fluctuation fields.

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1 Introduction

In [3, 4, 5], we developed a power series representation, norms and estimates for an effective action of the form

$$\ln \frac{\int e^{f(\alpha_1,\cdots,\alpha_s;z^*,z)}\, d\mu(z^*,z)}{\int e^{f(0,\cdots,0;z^*,z)}\, d\mu(z^*,z)}$$

Here, $f(\alpha_1,\cdots,\alpha_s;z_*,z)$ is an analytic function of the complex fields $\alpha_1(\mathbf{x})$, $\cdots$, $\alpha_s(\mathbf{x})$, $z_*(\mathbf{x})$, $z(\mathbf{x})$ indexed by $\mathbf{x}$ in a finite set X, and $d\mu(z^*,z)$ is a compactly supported product measure. This framework has been used in [6].

In [8, 9] we combine these power series methods with the technique of the block spin renormalization group for functional integrals [12, 1, 11, 2, 10] to see, for a many particle system of weakly interacting Bosons in three space dimensions, the formation of a potential well of the type that typically leads to symmetry breaking in the thermodynamic limit. (For an overview, see [7].) A basic ingredient of this block spin/functional integral approach is a stationary phase argument for the effective actions. For this, it is necessary to construct and analyze "critical fields" at each step. These critical fields are themselves functions of some external fields. The "background fields" of the block spin approach arise as compositions of critical fields at several renormalization group steps and are also functions of some external fields.

In our construction [8, 9], the "background fields" and "critical fields" are analytic maps that are defined on a neighbourhood of the origin in an appropriate Hilbert space of fields and that take values in another Hilbert space of fields. We call such objects "field maps". See Definition 2.3, where we also generalize the definition of the norm of a (complex valued) analytic function of fields [4, Definition 2.6] to field maps.

In § 3 we prove bounds on compositions like

$$\tilde{h}(\alpha_1,\cdots,\alpha_s) = h\Big(A_1(\alpha_1,\cdots,\alpha_s),\cdots,A_r(\alpha_1,\cdots,\alpha_s)\Big)$$

in terms of bounds on h and the A_j's. Here, h is a function of r fields and $A_1,\cdots,A_r$ are field maps. See Proposition 3.2 and Corollary 3.3.

The critical fields for each block spin renormalization group transformation are critical configurations for some action. The equations that determine these critical configurations can be expressed as (systems of) implicit equations of the type

$$\gamma = F(\alpha_1,\cdots,\alpha_s;\gamma)$$

which have to be solved for γ as a function $\alpha_1,\cdots,\alpha_s$. In §4, we prove the existence and uniqueness of, and bounds on, solutions to systems of equations of that type. See Proposition 4.1.

2 Field Maps

For an abstract framework, we consider analytic functions $f(\alpha_1, \cdots, \alpha_s)$ of the complex fields $\alpha_1, \cdots, \alpha_s$ (none of which are "history" or source fields, in the terminology of [4]) on a finite set X. Here are some associated definitions and notation from [4].

Definition 2.1 (n–tuples)**.**

(a) Let $n \in \mathbb{Z}$ with $n \geq 0$ and $\vec{\mathbf{x}} = (\mathbf{x}_1, \cdots, \mathbf{x}_n) \in X^n$ be an ordered n–tuple of points of X. We denote by $n(\vec{\mathbf{x}}) = n$ the number of components of $\vec{\mathbf{x}}$. Set $\alpha(\vec{\mathbf{x}}) = \alpha(\mathbf{x}_1) \cdots \alpha(\mathbf{x}_n)$. If $n(\vec{\mathbf{x}}) = 0$, then $\alpha(\vec{\mathbf{x}}) = 1$.

(b) For each $s \in \mathbb{N}$, we denote[1]

$$\mathbf{X}^{(s)} = \bigcup_{n_1, \cdots, n_s \geq 0} X^{n_1} \times \cdots \times X^{n_s}$$

If $(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_{s-1}) \in \mathbf{X}^{(s-1)}$ then $(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_{s-1}, -)$ denotes the element of $\mathbf{X}^{(s)}$ having $n(\vec{\mathbf{x}}_s) = 0$. In particular, $X^0 = \{-\}$ and $\alpha(-) = 1$.

(c) We define the concatenation of $\vec{\mathbf{x}} = (\mathbf{x}_1, \cdots, \mathbf{x}_n) \in X^n$ and $\vec{\mathbf{y}} = (\mathbf{y}_1, \cdots, \mathbf{y}_m) \in X^m$ to be

$$\vec{\mathbf{x}} \circ \vec{\mathbf{y}} = \big(\mathbf{x}_1, \cdots, \mathbf{x}_n, \mathbf{y}_1, \cdots, \mathbf{y}_m) \in X^{n+m}$$

For $(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s),\ (\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_s) \in \mathbf{X}^{(s)}$

$$(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s) \circ (\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_s) = (\vec{\mathbf{x}}_1 \circ \vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{x}}_s \circ \vec{\mathbf{y}}_s)$$

Definition 2.2 (Coefficient Systems)**.**

(a) A coefficient system of length s is a function $a(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)$ which assigns a complex number to each $(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s) \in \mathbf{X}^{(s)}$. It is called symmetric if, for each $1 \leq j \leq s$, $a(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)$ is invariant under permutations of the components of $\vec{\mathbf{x}}_j$.

(b) Let $f(\alpha_1, \cdots, \alpha_s)$ be a function which is defined and analytic on a neighbourhood of the origin in $\mathbb{C}^{s|X|}$. Then f has a unique expansion of the form

$$f(\alpha_1, \cdots, \alpha_s) = \sum_{(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s) \in \mathbf{X}^{(s)}} a(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\ \alpha_1(\vec{\mathbf{x}}_1) \cdots \alpha_s(\vec{\mathbf{x}}_s)$$

[1] We distinguish between $X^{n_1} \times \cdots \times X^{n_s}$ and $X^{n_1 + \cdots + n_s}$. We use $X^{n_1} \times \cdots \times X^{n_s}$ as the set of possible arguments for $\alpha_1(\vec{\mathbf{x}}_1) \cdots \alpha_s(\vec{\mathbf{x}}_s)$, while $X^{n_1 + \cdots + n_s}$ is the set of possible arguments for $\alpha_1(\vec{\mathbf{x}}_1 \circ \cdots \circ \vec{\mathbf{x}}_s)$, where $\circ$ is the concatenation operator of part (c).

with $a(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)$ a symmetric coefficient system. This coefficient system is called the symmetric coefficient system of f.

We assume that we are given a metric d on a finite set X and *constant* weight factors $\kappa_1, \cdots, \kappa_s$. In this environment [4, Definition 2.6], for the norm of the function

$$f(\alpha_1, \cdots, \alpha_s) = \sum_{(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s) \in \mathbf{X}^{(s)}} a(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\ \alpha_1(\vec{\mathbf{x}}_1) \cdots \alpha_s(\vec{\mathbf{x}}_s)$$

with $a(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)$ a symmetric coefficient system, simplifies to

$$\|f\|_w = \big|a(-)\big| + \sum_{\substack{n_1, \cdots, n_s \geq 0 \\ n_1 + \cdots + n_s \geq 1}} \max_{\mathbf{x} \in X} \max_{\substack{1 \leq j \leq s \\ n_j \neq 0}} \max_{1 \leq i \leq n_j} \sum_{\substack{\vec{\mathbf{x}}_\ell \in X^{n_\ell} \\ 1 \leq \ell \leq s \\ (\vec{\mathbf{x}}_j)_i = \mathbf{x}}} \big|a(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\big| \kappa_1^{n_1} \cdots \kappa_s^{n_s} e^{\tau_d(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)} \tag{2.1}$$

where $\tau_d(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)$ denotes the length of the shortest tree in X whose set of vertices contains all of the points in the $\vec{x}_j$'s. The family of functions

$$w(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s) = \kappa_1^{n(\vec{\mathbf{x}}_1)} \cdots \kappa_s^{(\vec{\mathbf{x}}_s)} e^{\tau_d(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)}$$

is called the weight system with metric d that associates the weight factor κ_j to the field α_j.

We need to extend these definitions to functions $A(\alpha_1, \cdots, \alpha_s)$ that take values in $\mathbb{C}^X$, rather than $\mathbb{C}$. That is, which map fields $\alpha_1, \cdots, \alpha_s$ to another field $A(\alpha_1, \cdots, \alpha_s)$. A trivial example would be $A(\alpha)(\mathbf{x}) = \alpha(\mathbf{x})$.

Definition 2.3.

(a) An s–field map kernel is a function

$$A : (\mathbf{x}; \vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s) \in X \times \mathbf{X}^{(s)} \mapsto A(\mathbf{x}; \vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s) \in \mathbb{C}$$

which obeys $A(\mathbf{x}; -, \cdots, -) = 0$ for all $\mathbf{x} \in X$.

(b) If A is an s–field map kernel, we define the "s–field map" $(\alpha_1, \cdots, \alpha_s) \mapsto A(\alpha_1, \cdots, \alpha_s)$ by

$$A(\alpha_1, \cdots, \alpha_s)(\mathbf{x}) = \sum_{(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s) \in \mathbf{X}^{(s)}} A(\mathbf{x}; \vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\ \alpha_1(\vec{\mathbf{x}}_1) \cdots \alpha_s(\vec{\mathbf{x}}_s)$$

(c) We define the norm $|||A|||_w$ of the s–field map kernel A by

$$|||A|||_w = \sum_{\substack{n_1,\cdots,n_s\ge 0\\ n_1+\cdots+n_s\ge 1}} \big\|A\big\|_{w;n_1,\cdots,n_s}$$

where

$$\big\|A\big\|_{w;n_1,\cdots,n_s} = \max\big\{L(A;w;n_1,\cdots,n_s)\,,\ R(A;w;n_1,\cdots,n_s)\big\}$$

and

$$L(A;w;n_1,\cdots,n_s) = \max_{\mathbf{x}\in X} \sum_{\substack{\vec{\mathbf{x}}_\ell\in X^{n_\ell}\\ 1\le\ell\le s}} \big|A(\mathbf{x};\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_s)\big|\kappa_1^{n_1}\cdots\kappa_s^{n_s} e^{\tau_d(\mathbf{x},\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_s)}$$

$$R(A;w;n_1,\cdots,n_s) = \max_{\mathbf{x}'\in X}\max_{\substack{1\le j\le s\\ n_j\ne 0}}\max_{1\le i\le n_j}\sum_{\mathbf{x}\in X}\sum_{\substack{\vec{\mathbf{x}}_\ell\in X^{n_\ell}\\ 1\le\ell\le s\\ (\vec{\mathbf{x}}_j)_i=\mathbf{x}'}} \big|A(\mathbf{x};\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_s)\big|\kappa_1^{n_1}\cdots\kappa_s^{n_s} e^{\tau_d(\mathbf{x},\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_s)}$$

We also denote the norm of the corresponding s–field map $A(\alpha_1,\cdots,\alpha_s)$ by $|||A|||_w$.

Remark 2.4. We associate to each s–field map kernel A the analytic function

$$\begin{aligned} f_A(\beta;\alpha_1,\cdots,\alpha_s) &= \sum_{\mathbf{x}\in X}\beta(\mathbf{x})A(\alpha_1,\cdots,\alpha_s)(\mathbf{x}) \\ &= \sum_{\substack{(\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_s)\in\mathbf{X}^{(s)}\\ \mathbf{x}\in X}} A(\mathbf{x};\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_s)\ \beta(\mathbf{x})\alpha_1(\vec{\mathbf{x}}_1)\cdots\alpha_s(\vec{\mathbf{x}}_s)\end{aligned}$$

Denote by $\hat{w}$ the weight system with metric d that associates the weight factor κ_j to α_j, for each $1\le j\le s$, and the weight factor 1 to β. Then

$$\|f_A\|_{\hat{w}} = |||A|||_w$$

Lemma 2.5 (Young's Inequality)**.** *Let $d_1,\cdots,d_s\ge 0$ be integers.*

(a) Let $f(\alpha_1,\cdots,\alpha_s)$ be a function which is defined and analytic on a neighbourhood of the origin in $\mathbb{C}^{s|X|}$ and is of degree at least d_i in the field α_i. Furthermore let $p_1,\cdots,p_s\in(0,\infty]$ be such that $\sum\limits_{j=1}^{s}\frac{d_j}{p_j}=1$. Then, for all fields $\alpha_1,\cdots,\alpha_s$ such that $|\alpha_j(\mathbf{x})|\le\kappa_j$ for all $\mathbf{x}\in X$ and $1\le j\le s$,

$$\big|f(\alpha_1,\cdots,\alpha_s)\big| \le \|f\|_w \prod_{j=1}^{s}\big(\tfrac{1}{\kappa_j}\|\alpha_j\|_{p_j}\big)^{d_j}$$

where $\|\alpha\|_p = \big(\sum\limits_{x\in X}|\alpha(x)|^p\big)^{1/p}$ denotes the L^p norm of α.

(b) Let $(\alpha_1, \cdots, \alpha_s) \mapsto A(\alpha_1, \cdots, \alpha_s)$ be an s–field map which is of degree at least d_i in the field α_i. Furthermore let $p, p_1, \cdots, p_s \in (0, \infty]$ be such that $\sum_{j=1}^{s} \frac{d_j}{p_j} = \frac{1}{p}$. Then, for fields $\alpha_1, \cdots, \alpha_s$ such that $|\alpha_j(\mathbf{x})| \le \kappa_j$ for all $\mathbf{x} \in X$ and $1 \le j \le s$, the L^p norm of the field $A(\alpha_1, \cdots, \alpha_s)$ is bounded by

$$\big\|A(\alpha_1, \cdots, \alpha_s)\big\|_p \le |||A|||_w \prod_{j=1}^{s} \big(\tfrac{1}{\kappa_j}\|\alpha_j\|_{p_j}\big)^{d_j}$$

In particular

$$\max_{\mathbf{x}\in X} \big|A(\alpha_1, \cdots, \alpha_s)(\mathbf{x})\big| \le |||A|||_w$$

Proof. (a) By the definition (2.1) of $\|f\|_w$, we may assume that f is of the form

$$f(\alpha_1, \cdots, \alpha_s) = \sum_{\substack{\vec{\mathbf{x}}_\ell \in X^{n_\ell} \\ 1 \le \ell \le s}} a(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\ \alpha_1(\vec{\mathbf{x}}_1) \cdots \alpha_s(\vec{\mathbf{x}}_s)$$

with a symmetric coefficient a and $n_\ell \ge d_\ell$. Now apply Lemma A.1 with $K = a \prod_{j=1}^{s} \kappa_j^{d_j}$, where we use the L^{p_j} norm for the first d_j components of the variable $\vec{x}_j$, and the L^∞ norm for the last $n_j - d_j$ components of this variable.

(b) As in Remark 2.4 set

$$f_A(\beta; \alpha_1, \cdots, \alpha_s) = \sum_{\mathbf{x}\in X} \beta(\mathbf{x}) A(\alpha_1, \cdots, \alpha_s)(\mathbf{x})$$

As in [13, Theorem 4.2] choose

$$\beta(\mathbf{x}) = e^{-i\theta(x)} |A(\alpha_1, \cdots, \alpha_s)(\mathbf{x})|^{p/p'}$$

where $\theta(x)$ is defined by $A(\alpha_1, \cdots, \alpha_s)(x) = e^{i\theta(x)} |A(\alpha_1, \cdots, \alpha_s)(x)|$ and $\frac{1}{p} + \frac{1}{p'} = 1$. By part (a) and Remark 2.4

$$\begin{aligned}\big\|A(\alpha_1, \cdots, \alpha_s)\big\|_p^p = \big|f_A(\beta; \alpha_1, \cdots, \alpha_s)\big| &\le |||A|||_w \, \|\beta\|_{p'} \prod_{j=1}^{s} \big(\tfrac{1}{\kappa_j}\|\alpha_j\|_{p_j}\big)^{d_j} \\ &= |||A|||_w \, \big\|A(\alpha_1, \cdots, \alpha_s)\big\|_p^{p/p'} \prod_{j=1}^{s} \big(\tfrac{1}{\kappa_j}\|\alpha_j\|_{p_j}\big)^{d_j}\end{aligned}$$

□

Remark 2.6. A linear map $L : \mathbb{C}^X \to \mathbb{C}^X$ can be thought of as a 1–field map kernel. The relation between the norm $|||L|||_w$ as a field map kernel and the norm $|||L|||$ as in [4, Definition A.1] is

$$|||L|||_w = \kappa_1 |||L|||$$

The field $L(\alpha_1)$ is

$$L(\alpha_1)(\mathbf{x}) = \sum_{\mathbf{y} \in X} L(\mathbf{x}, \mathbf{y}) \alpha_1(\mathbf{y})$$

Remark 2.7. In Definition 2.3, we have assumed, for simplicity, that the field map A maps fields $\alpha_1, \cdots, \alpha_s$ on a set X to a field $A(\alpha_1, \cdots, \alpha_s)$ on the same set X. We will apply this definition and the results later in this paper when the input fields $\alpha_1, \cdots, \alpha_s$ are defined on a subset $X_1 \subset X$ and the output field $A(\alpha_1, \cdots, \alpha_s)$ is defined on a, possibly different, subset $X_2 \subset X$. We extend Definition 2.3 and the results later in this paper to cover this setting by viewing $\alpha_1, \cdots, \alpha_s$ and $A(\alpha_1, \cdots, \alpha_s)$ to be fields on X — set $\alpha_1, \cdots, \alpha_s$ to zero on $X \setminus X_1$ and $A(\alpha_1, \cdots, \alpha_s)$ to zero on $X \setminus X_2$.

3 Substitution

We now proceed to prove bounds on compositions like

$$\tilde{h}(\alpha_1, \cdots, \alpha_s) = h\Big(A_1(\alpha_1, \cdots, \alpha_s), \cdots, A_r(\alpha_1, \cdots, \alpha_s)\Big)$$

in terms of bounds on h and the A_j's.

Lemma 3.1. *Let $\lambda_1, \ \cdots, \ \lambda_s$ be constant weight factors and let w_δ be the weight system with metric d that associates the weight factor κ_j to α_j and λ_j to a field δ_j. Fix any $\sigma \geq 1$ and let w_σ be the weight system with metric d that associates the weight factor $\kappa_j + \sigma\lambda_j$ to α_j.*

(a) Let $f(\alpha_1, \cdots, \alpha_s)$ be an analytic function on a neighbourhood of the origin in $\mathbb{C}^{s|X|}$. Set

$$\delta f\big(\alpha_1, \cdots, \alpha_s, \delta_1, \cdots, \delta_s\big) = f\big(\alpha_1 + \delta_1, \cdots, \alpha_s + \delta_s\big) - f\big(\alpha_1, \cdots, \alpha_s\big)$$

Then

$$\|\delta f\|_{w_\delta} \leq \tfrac{1}{\sigma}\|f\|_{w_\sigma}$$

More generally, if $p \in \mathbb{N}$ and $\delta f^{(\geq p)}\big(\alpha_1, \cdots, \alpha_s, \delta_1, \cdots, \delta_s\big)$ is the part of δf that is of degree at least p in $\big(\delta_1, \cdots, \delta_s\big)$, then

$$\|\delta f^{(\geq p)}\|_{w_\delta} \leq \tfrac{1}{\sigma^p}\|f\|_{w_\sigma}$$

(b) Let A be an s–field map and define the $2s$–field map δA by

$$\delta A\big(\alpha_1, \cdots, \alpha_s, \delta_1, \cdots, \delta_s\big) = A\big(\alpha_1 + \delta_1, \cdots, \alpha_s + \delta_s\big) - A\big(\alpha_1, \cdots, \alpha_s\big)$$

Then

$$|||\delta A|||_{w_\delta} \leq \tfrac{1}{\sigma}|||A|||_{w_\sigma}$$

Proof. Let $a(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)$ be a symmetric coefficient system for f. Since a is invariant under permutation of its $\vec{\mathbf{x}}_j$ components,

$$\begin{aligned} f\big(\alpha_1 + \delta_1, \cdots, \alpha_s + \delta_s\big) &= \sum_{(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s) \in \mathbf{X}^{(s)}} a(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\ (\alpha_1 + \delta_1)(\vec{\mathbf{x}}_1) \cdots (\alpha_s + \delta_s)(\vec{\mathbf{x}}_s) \\ &= \sum_{\substack{(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s) \in \mathbf{X}^{(s)} \\ (\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_s) \in \mathbf{X}^{(s)}}} a(\vec{\mathbf{x}}_1 \circ \vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{x}}_s \circ \vec{\mathbf{y}}_s)\ \prod_{j=1}^{s} \tbinom{n(\vec{\mathbf{x}}_j) + n(\vec{\mathbf{y}}_j)}{n(\vec{\mathbf{y}}_j)} \alpha_j(\vec{\mathbf{x}}_j)\delta_j(\vec{\mathbf{y}}_j) \end{aligned}$$

so that

$$\delta a_p(\vec{\mathbf{x}}_1,\cdots\mathbf{x}_s;\vec{\mathbf{y}}_1,\cdots,\mathbf{y}_s)$$
$$= \chi\big(n(\vec{\mathbf{y}}_1)+\cdots+n(\vec{\mathbf{y}}_s)\ge p\big)a(\vec{\mathbf{x}}_1\circ\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{x}}_s\circ\vec{\mathbf{y}}_s)\ \prod_{j=1}^{s}\tbinom{n(\vec{\mathbf{x}}_j)+n(\vec{\mathbf{y}}_j)}{n(\vec{\mathbf{y}}_j)}$$

is a symmetric coefficient system for $\delta f^{(\ge p)}$. Of course $\delta f=\delta f^{(\ge 1)}$. By definition

$$\begin{aligned}\|\delta f^{(\ge p)}\|_{w_\delta} &= \sum_{\substack{k_1,\cdots,k_s\ge0\\ \ell_1,\cdots,\ell_s\ge0\\ \ell_1+\cdots+\ell_s\ge p}} \max_{\mathbf{x}\in X}\max_{\substack{1\le j\le s\\ k_j+\ell_j\ne0}}\max_{1\le i\le k_j+\ell_j}\sum_{\substack{\vec{\mathbf{x}}_m\in X^{k_m}\\ \vec{\mathbf{y}}_m\in X^{\ell_m}\\ (\vec{\mathbf{x}}_j\circ\vec{\mathbf{y}}_j)_i=\mathbf{x}}}\big|\delta a(\vec{\mathbf{x}}_1,\cdots\mathbf{x}_s;\vec{\mathbf{y}}_1,\cdots,\mathbf{y}_s)\big| \\ &\qquad\qquad e^{\tau_d(\vec{\mathbf{x}}_1\circ\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{x}}_s\circ\vec{\mathbf{y}}_s)}\prod_{j=1}^{s}\kappa_j^{k_j}\lambda_j^{\ell_j} \\ &= \sum_{\substack{k_1,\cdots,k_s\ge0\\ \ell_1,\cdots,\ell_s\ge0\\ \ell_1+\cdots+\ell_s\ge p}} \omega(k_1+\ell_1,\cdots,k_s+\ell_s)\prod_{j=1}^{s}\tbinom{k_j+\ell_j}{\ell_j}\prod_{j=1}^{s}\kappa_j^{k_j}\lambda_j^{\ell_j} \\ &= \sum_{\substack{n_1,\cdots,n_s\\ n_1+\cdots+n_s\ge p}}\omega(n_1,\cdots,n_s)c_p(n_1,\cdots,n_s)\end{aligned}$$

where

$$\omega(n_1,\cdots,n_s)=\max_{\substack{1\le j\le s\\ n_j\ne0}}\max_{1\le i\le n_j}\sum_{\substack{\vec{\mathbf{z}}_p\in X^{n_p}\\ (\vec{\mathbf{z}}_j)_i=\mathbf{x}}}\big|a(\vec{\mathbf{z}}_1,\cdots,\vec{\mathbf{z}}_s)\big|e^{\tau_d(\vec{\mathbf{z}}_1,\cdots,\vec{\mathbf{z}}_s)}$$

and

$$c_p(n_1,\cdots,n_s)=\sum_{\substack{k_j,\ell_j\ge0\\ k_j+\ell_j=n_j\\ \ell_1+\cdots+\ell_s\ge p}}\prod_{j=1}^{s}\tbinom{n_j}{\ell_j}\kappa_j^{k_j}\lambda_j^{\ell_j}\le\tfrac{1}{\sigma^p}\prod_{j=1}^{s}(\kappa_j+\sigma\lambda_j)^{n_j}$$

For the last inequality, apply the binomial expansion to each $(\kappa_j+\sigma\lambda_j)^{n_j}$ and compare the two sides of the inequality term by term. This proves part (a). Part (b) follows by Remark 2.4. $\square$

Proposition 3.2. *Let $h(\gamma_1,\cdots,\gamma_r)$ be an analytic function on a neighbourhood of the origin in $\mathbb{C}^{r|X|}$, and let A_j, δA_j, $1\le j\le r$ be s–field maps. Furthermore let $\lambda_1,\ \cdots,\ \lambda_r$ be constant weight factors and let w_λ be the weight system with metric d that associates the weight factor λ_j to the field γ_j.*

(a) Set

$$\tilde{h}(\alpha_1, \cdots, \alpha_s) = h\Big(A_1(\alpha_1, \cdots, \alpha_s), \cdots, A_r(\alpha_1, \cdots, \alpha_s)\Big)$$

Assume that

$$|||A_j|||_w \le \lambda_j$$

for each $1 \le j \le r$. Then

$$\|\tilde{h}\|_w \le \|h\|_{w_\lambda}$$

(b) Assume that there is a $\sigma \ge 1$ such that

$$|||A_j|||_w + \sigma |||\delta A_j|||_w \le \lambda_j$$

for all $1 \le j \le r$. Set

$$\begin{aligned}
&\widetilde{\delta h}(\alpha_1, \cdots, \alpha_s) \\
&\quad = h\Big(A_1(\alpha_1, \cdots, \alpha_s) + \delta A_1(\alpha_1, \cdots, \alpha_s), \cdots, A_r(\alpha_1, \cdots, \alpha_s) + \delta A_r(\alpha_1, \cdots, \alpha_s)\Big) \\
&\qquad\qquad - h\Big(A_1(\alpha_1, \cdots, \alpha_s), \cdots, A_r(\alpha_1, \cdots, \alpha_s)\Big)
\end{aligned}$$

More generally, if $p \in \mathbb{N}$ and $\delta h^{(\ge p)}$ is the part of

$$\delta h(\gamma_1, \cdots, \gamma_r; \delta_1, \cdots, \delta_r) = h(\gamma_1 + \delta_1, \cdots, \gamma_r + \delta_r) - h(\gamma_1, \cdots, \gamma_r)$$

that is of degree at least p in $(\delta_1, \cdots, \delta_r)$, set

$$\begin{aligned}
&\widetilde{\delta h}^{(\ge p)}(\alpha_1, \cdots, \alpha_s) \\
&\quad = \delta h^{(\ge p)}\Big(A_1(\alpha_1, \cdots, \alpha_s), \cdots, A_r(\alpha_1, \cdots, \alpha_s)\,; \\
&\qquad\qquad \delta A_1(\alpha_1, \cdots, \alpha_s), \cdots, \delta A_r(\alpha_1, \cdots, \alpha_s)\Big)
\end{aligned}$$

Then

$$\|\widetilde{\delta h}\|_w \le \tfrac{1}{\sigma}\|h\|_{w_\lambda} \qquad \big\|\widetilde{\delta h}^{(\ge p)}\big\|_w \le \tfrac{1}{\sigma^p}\|h\|_{w_\lambda}$$

Proof. (a) Let $a(\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_r)$ be a symmetric coefficient system for h. Define, for each $n(\vec{\mathbf{x}}_i) = n_i \ge 0$, $1 \le i \le s$,

$$\begin{aligned}
&\tilde{a}(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s) \\
&= \sum_{m_1, \cdots, m_r \ge 0} \ \sum_{\substack{n_{i,j,k} \ge 0 \text{ for} \\ 1 \le i \le s,\, 1 \le j \le r,\, 1 \le k \le m_j \\ \text{with } \Sigma_{j,k} n_{i,j,k} = n_i}} \ \sum_{\substack{\vec{\mathbf{y}}_1 \in X^{m_1} \\ \vdots \\ \vec{\mathbf{y}}_r \in X^{m_r}}} a(\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_r) \prod_{j=1}^r \Big[\prod_{k=1}^{m_j} A_j\big((\vec{\mathbf{y}}_j)_k; \vec{\mathbf{x}}_{1,j,k}, \cdots, \vec{\mathbf{x}}_{s,j,k}\big)\Big]
\end{aligned}$$

where $(\vec{\mathbf{y}}_j)_k$ is the $k^{\rm th}$ component of $\vec{\mathbf{y}}_j$ and the $\vec{\mathbf{x}}_{ijk}$'s are determined by the conditions that $n(\vec{\mathbf{x}}_{ijk}) = n_{ijk}$ and

$$\vec{\mathbf{x}}_i = \circ_{j,k}\vec{\mathbf{x}}_{ijk} = \vec{\mathbf{x}}_{i11} \circ \vec{\mathbf{x}}_{i12} \circ \cdots \circ \vec{\mathbf{x}}_{i1m_1} \circ \vec{\mathbf{x}}_{i21} \circ \cdots \circ \vec{\mathbf{x}}_{i2m_2} \circ \cdots \circ \vec{\mathbf{x}}_{irm_r} \tag{3.1}$$

Then $\tilde{a}(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)$ is a (not necessarily symmetric) coefficient system for $\tilde{h}$. Since

$$\begin{aligned} &\tau_d\big(\operatorname{supp}(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\big) \\ &\qquad \le \tau_d\big(\operatorname{supp}(\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_s)\big) + \sum_{\substack{1\le j\le r\\ \le k\le m_j}} \tau_d\big(\operatorname{supp}((\vec{\mathbf{y}}_j)_k, \vec{\mathbf{x}}_{1,j,k}, \cdots, \vec{\mathbf{x}}_{s,j,k})\big) \end{aligned}$$

we have

$$\begin{aligned} &w(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\, \big|\tilde{a}(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\big| \\ &\qquad \le \sum_{m_1,\cdots,m_r\ge 0} \ \sum_{\substack{n_{i,j,k}\ge 0 \text{ for}\\ 1\le i\le s,\, 1\le j\le r,\, 1\le k\le m_j\\ \text{with } \Sigma_{j,k} n_{i,j,k}=n_i}} \ \sum_{\substack{\vec{\mathbf{y}}_1\in X^{m_1}\\ \vdots \\ \vec{\mathbf{y}}_r\in X^{m_r}}} w_\lambda(\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_r)\big|a(\vec{\mathbf{y}}_1, \cdots, \vec{\mathbf{y}}_r)\big| \\ &\qquad\qquad \prod_{j=1}^{r} \Big[\prod_{k=1}^{m_j} B_j((\vec{\mathbf{y}}_j)_k; \vec{\mathbf{x}}_{1,j,k}, \cdots, \vec{\mathbf{x}}_{s,j,k})\Big] \end{aligned} \tag{3.2}$$

where

$$B_j(\mathbf{y}; \vec{\mathbf{x}}'_1, \cdots, \vec{\mathbf{x}}'_s) = \tfrac{1}{\lambda_j}|A_j(\mathbf{y}; \vec{\mathbf{x}}'_1, \cdots, \vec{\mathbf{x}}'_s)|\kappa_1^{n(\vec{\mathbf{x}}'_1)} \cdots \kappa_s^{n(\vec{\mathbf{x}}'_s)} e^{\tau_d(\operatorname{supp}(\mathbf{y}, \vec{\mathbf{x}}'_1, \cdots, \vec{\mathbf{x}}'_s))}$$

We first observe that when $\vec{\mathbf{x}}_1 = \cdots = \vec{\mathbf{x}}_s = -$, we have $\tilde{a}(-, \cdots, -) = a(-, \cdots, -)$ so that the corresponding contributions to $\|\tilde{h}\|_w$ and $\|h\|_{w_\lambda}$ are identical. Therefore we may assume, without loss of generality, that $h(0, \cdots, 0) = 0$.

We are to bound

$$\|\tilde{h}\|_w = \sum_{\substack{n_1,\cdots,n_s\ge 0\\ n_1+\cdots+n_s\ge 1}} \max_{\mathbf{x}\in X} \max_{\substack{1\le \bar{\jmath}\le s\\ n_{\bar{\jmath}}\ne 0}} \max_{1\le \bar{\imath}\le n_{\bar{\jmath}}} \sum_{\substack{(\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_s)\in X^{n_1}\times\cdots\times X^{n_s}\\ (\vec{\mathbf{x}}_{\bar{\jmath}})_{\bar{\imath}}=\mathbf{x}}} w(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\, \big|\tilde{a}(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\big|$$

First fix any $n_1,\ \cdots,\ n_s \ge 0$ with $n_1 + \cdots + n_s \ge 1$. We claim that

$$\begin{aligned} &\max_{\mathbf{x}\in X} \max_{\substack{1\le \bar{\jmath}\le s\\ n_{\bar{\jmath}}\ne 0}} \max_{1\le \bar{\imath}\le n_{\bar{\jmath}}} \sum_{\substack{(\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_s)\in X^{n_1}\times\cdots\times X^{n_s}\\ (\vec{\mathbf{x}}_{\bar{\jmath}})_{\bar{\imath}}=\mathbf{x}}} w(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\, \big|\tilde{a}(\vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_s)\big| \\ &\qquad \le \sum_{m_1,\cdots,m_r\ge 0} \big\|w_\lambda a\big\|_{m_1,\cdots,m_r} \sum_{\substack{n_{i,j,k}\ge 0 \text{ for}\\ 1\le i\le s,\, 1\le j\le r,\, 1\le k\le m_j\\ \text{with } \Sigma_{j,k} n_{i,j,k}=n_i}} \ \prod_{\substack{1\le j\le r\\ 1\le k\le m_j}} \Big[\tfrac{1}{\lambda_j}\big\|A_j\big\|_{w;n_{1,j,k},\cdots,n_{s,j,k}}\Big] \end{aligned} \tag{3.3}$$

Here, as in [4, Definition 2.6],

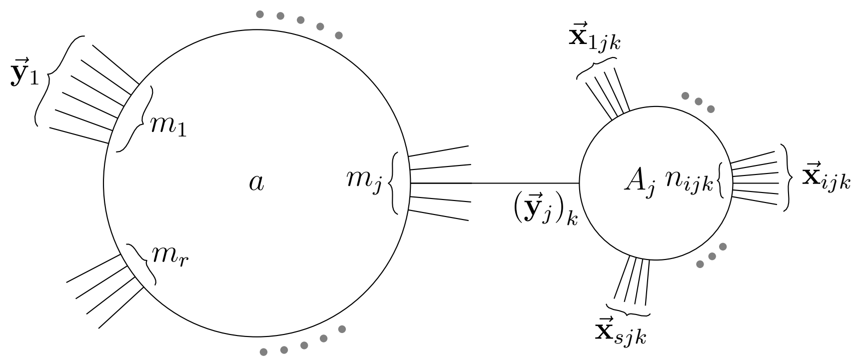
$$\|b\|_{m_1,\cdots,m_r} = \max_{\mathbf{y}\in X} \max_{\substack{1\le j\le r\\ m_j\neq 0}} \max_{1\le i\le m_j} \sum_{\substack{\vec{\mathbf{y}}_\ell\in X^{m_\ell}\\ 1\le\ell\le r\\ (\vec{\mathbf{y}}_j)_i=\mathbf{y}}} \big|b(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_r)\big|$$

To prove (3.3), fix any $\mathbf{x}\in X$ and assume, without loss of generality that $n_1\ge 1$ and $\bar{\jmath}=\bar{\imath}=1$. By (3.2), (the meaning of the $\hat{\jmath},\hat{k}$ introduced after the "=" below is explained immediately following this string of inequalities)

$$\begin{aligned}
&\sum_{\substack{(\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_s)\in X^{n_1}\times\cdots\times X^{n_s}\\ (\vec{\mathbf{x}}_1)_1=\mathbf{x}}} w(\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_s)\ \big|\tilde{a}(\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_s)\big| \\
&\le \sum_{\substack{(\vec{\mathbf{x}}_1,\cdots,\vec{\mathbf{x}}_s)\in X^{n_1}\times\cdots\times X^{n_s}\\ (\vec{\mathbf{x}}_1)_1=\mathbf{x}}} \sum_{m_1,\cdots,m_r\ge 0} \sum_{\substack{n_{i,j,k}\ge 0 \text{ for}\\ 1\le i\le s,\,1\le j\le r,\,1\le k\le m_j\\ \text{with } \Sigma_{j,k} n_{i,j,k}=n_i}} \sum_{\substack{\vec{\mathbf{y}}_1\in X^{m_1}\\ \vdots\\ \vec{\mathbf{y}}_r\in X^{m_r}}} w_\lambda(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_r)\big|a(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_r)\big| \\
&\qquad\qquad \prod_{j=1}^r \Big[\prod_{k=1}^{m_j} B_j\big((\vec{\mathbf{y}}_j)_k;\vec{\mathbf{x}}_{1,j,k},\cdots,\vec{\mathbf{x}}_{s,j,k}\big)\Big] \\
&= \sum_{m_1,\cdots,m_r\ge 0} \sum_{\substack{n_{i,j,k}\ge 0 \text{ for}\\ 1\le i\le s,\,1\le j\le r,\,1\le k\le m_j\\ \text{with } \Sigma_{j,k} n_{i,j,k}=n_i}} \sum_{\substack{\vec{\mathbf{x}}_{i,j,k}\in X^{n_{i,j,k}} \text{ for}\\ 1\le i\le s,\,1\le j\le r,\,1\le k\le m_j\\ \text{with } (\vec{\mathbf{x}}_{1,\hat{\jmath},\hat{k}})_1=\mathbf{x}}} \sum_{\substack{\vec{\mathbf{y}}_1\in X^{m_1}\\ \vdots\\ \vec{\mathbf{y}}_r\in X^{m_r}}} w_\lambda(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_r)\big|a(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_r)\big| \\
&\qquad\qquad \prod_{j=1}^r \Big[\prod_{k=1}^{m_j} B_j\big((\vec{\mathbf{y}}_j)_k;\vec{\mathbf{x}}_{1,j,k},\cdots,\vec{\mathbf{x}}_{s,j,k}\big)\Big] \\
&\le \sum_{m_1,\cdots,m_r\ge 0} \sum_{\substack{n_{i,j,k}\ge 0 \text{ for}\\ 1\le i\le s,\,1\le j\le r,\,1\le k\le m_j\\ \text{with } \Sigma_{j,k} n_{i,j,k}=n_i}} \sum_{\substack{\vec{\mathbf{x}}_{i,\hat{\jmath},\hat{k}}\in X^{n_{i,\hat{\jmath},\hat{k}}} \text{ for}\\ 1\le i\le s\\ \text{with } (\vec{\mathbf{x}}_{1,\hat{\jmath},\hat{k}})_1=\mathbf{x}}} \sum_{\substack{\vec{\mathbf{y}}_1\in X^{m_1}\\ \vdots\\ \vec{\mathbf{y}}_r\in X^{m_r}}} w_\lambda(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_r)\big|a(\vec{\mathbf{y}}_1,\cdots,\vec{\mathbf{y}}_r)\big| \\
&\qquad B_{\hat{\jmath}}\big((\vec{\mathbf{y}}_{\hat{\jmath}})_{\hat{k}};\vec{\mathbf{x}}_{1,\hat{\jmath},\hat{k}},\cdots,\vec{\mathbf{x}}_{s,\hat{\jmath},\hat{k}}\big) \prod_{\substack{1\le j\le r\\ 1\le k\le m_j\\ (j,k)\neq(\hat{\jmath},\hat{k})}} \Big[\tfrac{1}{\lambda_j} L\big(A_j;w;\{n_{i,j,k}\}_{1\le i\le s}\big)\Big] \\
&\le \sum_{m_1,\cdots,m_r\ge 0} \sum_{\substack{n_{i,j,k}\ge 0 \text{ for}\\ 1\le i\le s,\,1\le j\le r,\,1\le k\le m_j\\ \text{with } \Sigma_{j,k} n_{i,j,k}=n_i}} \sum_{\substack{\vec{\mathbf{x}}_{i,\hat{\jmath},\hat{k}}\in X^{n_{i,\hat{\jmath},\hat{k}}} \text{ for}\\ 1\le i\le s\\ \text{with } (\vec{\mathbf{x}}_{1,\hat{\jmath},\hat{k}})_1=\mathbf{x}}} \sum_{\mathbf{y}\in X} \big\|w_\lambda a\big\|_{m_1,\cdots,m_r} \\
&\qquad B_{\hat{\jmath}}\big(\mathbf{y};\vec{\mathbf{x}}_{1,\hat{\jmath},\hat{k}},\cdots,\vec{\mathbf{x}}_{s,\hat{\jmath},\hat{k}}\big) \prod_{\substack{1\le j\le r\\ 1\le k\le m_j\\ (j,k)\neq(\hat{\jmath},\hat{k})}} \Big[\tfrac{1}{\lambda_j} L\big(A_j;w;\{n_{i,j,k}\}_{1\le i\le s}\big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{m_1,\cdots,m_r\ge 0} \|w_\lambda a\|_{m_1,\cdots,m_r} \sum_{\substack{n_{i,j,k}\ge 0 \text{ for} \\ 1\le i\le s,\, 1\le j\le r,\, 1\le k\le m_j \\ \text{with } \Sigma_{j,k} n_{i,j,k}=n_i}} \tfrac{1}{\lambda_{\hat{j}}} R\big(A_{\hat{j}}; w; \{n_{i,\hat{j},\hat{k}}\}_{1\le i\le s}\big) \\
&\qquad\qquad \prod_{\substack{1\le j\le r \\ 1\le k\le m_j \\ (j,k)\ne(\hat{j},\hat{k})}} \Big[\tfrac{1}{\lambda_j} L\big(A_j; w; \{n_{i,j,k}\}_{1\le i\le s}\big)\Big] \\
&\le \sum_{m_1,\cdots,m_r\ge 0} \|w_\lambda a\|_{m_1,\cdots,m_r} \sum_{\substack{n_{i,j,k}\ge 0 \text{ for} \\ 1\le i\le s,\, 1\le j\le r,\, 1\le k\le m_j \\ \text{with } \Sigma_{j,k} n_{i,j,k}=n_i}} \prod_{\substack{1\le j\le r \\ 1\le k\le m_j}} \Big[\tfrac{1}{\lambda_j} \|A_j\|_{w;n_{1,j,k},\cdots,n_{s,j,k}}\Big]
\end{aligned}
$$

Here, for each $\{n_{1,j,k}\}_{\substack{1\le j\le r \\ 1\le k\le m_j}}$, the pair $(\hat{j},\hat{k})$ is the first (j,k), using the lexicographical ordering of (3.1), for which $n_{1,j,k}\ne 0$.



Having completed the proof of (3.3), we now have, recalling the hypothesis that each $|||A_j|||_w \le \lambda_j$,

$$
\begin{aligned}
\|\tilde{h}\|_w &\le \sum_{\substack{n_1,\cdots,n_s\ge 0 \\ n_1+\cdots+n_s\ge 1}} \sum_{m_1,\cdots,m_r\ge 0} \|w_\lambda a\|_{m_1,\cdots,m_r} \sum_{\substack{n_{i,j,k}\ge 0 \text{ for} \\ 1\le i\le s,\, 1\le j\le r,\, 1\le k\le m_j \\ \text{with } \Sigma_{j,k} n_{i,j,k}=n_i}} \prod_{\substack{1\le j\le r \\ 1\le k\le m_j}} \Big[\tfrac{1}{\lambda_j} \|A_j\|_{w;n_{1,j,k},\cdots,n_{s,j,k}}\Big] \\
&\le \sum_{m_1,\cdots,m_r\ge 0} \|w_\lambda a\|_{m_1,\cdots,m_r} \sum_{\substack{n_{i,j,k}\ge 0 \text{ for} \\ 1\le i\le s,\, 1\le j\le r,\, 1\le k\le m_j}} \prod_{1\le j\le r \stackrel{1}{\le} k\le m_j} \Big[\tfrac{1}{\lambda_j} \|A_j\|_{w;n_{1,j,k},\cdots,n_{s,j,k}}\Big] \\
&\le \sum_{m_1,\cdots,m_r\ge 0} \|w_\lambda a\|_{m_1,\cdots,m_r} \\
&= \|h\|_{w_\lambda}
\end{aligned}
$$

(b) Let w_δ be the weight system with metric d that associates the weight factor $|||A_j|||_w$ to γ_j and the weight factor $|||\delta A_j|||_w$ to δ_j. By part (a) of Lemma 3.1, with

$$\begin{aligned}
&f \to h \\
&s \to r \\
&\alpha_j \text{ with weight } \kappa_j \to \gamma_j \text{ with weight } |||A_j|||_w \\
&\delta_j \text{ with weight } \lambda_j \to \delta_j \text{ with weight } |||\delta A_j|||_w
\end{aligned}$$

we have

$$\|\delta h\|_{w_\delta} \le \tfrac{1}{\sigma}\|h\|_{w_\lambda} \qquad \|\delta h^{(\ge p)}\|_{w_\delta} \le \tfrac{1}{\sigma^p}\|h\|_{w_\lambda}$$

Now $\widetilde{\delta h}$ and $\widetilde{\delta h}^{(\ge p)}$ are obtained from δh and $\delta h^{(\ge p)}$, respectively, by the substitutions

$$\gamma_j = A_j(\alpha_1, \cdots, \alpha_s) \qquad \delta_j = \delta A_j(\alpha_1, \cdots, \alpha_s)$$

and the statement follows by part (a). □

Corollary 3.3. *Let B be an r–field map and let A_j, $1 \le j \le r$, be s–field maps. Define the s–field map $\tilde{B}$ by*

$$\tilde{B}(\alpha_1, \cdots, \alpha_s) = B\Big(A_1(\alpha_1, \cdots, \alpha_s), \cdots, A_r(\alpha_1, \cdots, \alpha_s)\Big)$$

Furthermore let $\lambda_1, \cdots, \lambda_r$ be constant weight factors and let w_λ be the weight system with metric d that associates the weight factor λ_j to the $j^{\rm th}$ field of B. Assume that

$$|||A_j|||_w \le \lambda_j$$

for each $1 \le j \le r$. Then

$$|||\tilde{B}|||_w \le |||B|||_{w_\lambda}$$

Proof. This follows from Proposition 3.2 and Remark 2.4. □

Definition 3.4. Denote by $w_{\kappa,\lambda}$ the weight system with metric d that associates the constant weight factor κ_i to the field α_i and the constant weight factor λ_j to the field γ_j. Let $B(\vec{\alpha}, \vec{\gamma})$ be an $(s+r)$–field map with $|||B|||_{w_{\kappa,\lambda}} < \infty$.

(a) Set, for each r–tuple of nonnegative integers $n_{s+1}, \cdots, n_{s+r}$,

$$\begin{aligned}
&B_{n_{s+1},\cdots,n_{s+r}}(\mathbf{x}; \vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_{s+r}) \\
&\qquad = \begin{cases} B(\mathbf{x}; \vec{\mathbf{x}}_1, \cdots, \vec{\mathbf{x}}_{s+r}) & \text{if } n(\vec{\mathbf{x}}_{s+j}) = n_{s+j} \text{ for all } 1 \le j \le r \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Then

$$B = \sum_{n_{s+1},\cdots,n_{s+r}\ge 0} B_{n_{s+1},\cdots,n_{s+r}} \quad \text{and} \quad |||B|||_{w_{\kappa,\lambda}} = \sum_{n_{s+1},\cdots,n_{s+r}\ge 0} |||B_{n_{s+1},\cdots,n_{s+r}}|||_{w_{\kappa,\lambda}}$$

B is said to have minimum degree at least $d_{\min}$ and maximum degree at most $d_{\max} \le \infty$ in its last r arguments if

$$B_{n_{s+1},\cdots,n_{s+r}} = 0 \quad \text{unless} \quad d_{\min} \le n_{s+1} + \cdots + n_{s+r} \le d_{\max}$$

Set

$$|||B|||'_{w_{\kappa,\lambda}} = \sum_{n_{s+1},\cdots,n_{s+r}\ge 0} \big(n_{s+1} + \cdots + n_{s+r}\big) |||B_{n_{s+1},\cdots,n_{s+r}}|||_{w_{\kappa,\lambda}}$$

Think of $|||B|||'_{w_{\kappa,\lambda}}$ as a bound on the derivative of $B(\vec{\alpha},\vec{\gamma})$ with respect to $\vec{\gamma}$. See Lemma 3.7.

(b) Denote by $\mathcal{B}$ the Banach space of all r–tuples $\vec{\Gamma} = (\Gamma_1,\cdots,\Gamma_r)$ of s–field maps with the norm

$$\|\vec{\Gamma}\| = \max_{1\le j\le r} \tfrac{1}{\lambda_j} |||\Gamma_j|||_w$$

Also, for each $\rho > 0$, denote by $\mathcal{B}_\rho$, the closed ball in $\mathcal{B}$ of radius ρ.

(c) For each r–tuple $\vec{\Gamma} \in \mathcal{B}_1$, we define the s–field map $\tilde{B}(\vec{\Gamma})$ by

$$\big(\tilde{B}(\vec{\Gamma})\big)(\vec{\alpha}) = B\big(\vec{\alpha},\vec{\Gamma}(\vec{\alpha})\big)$$

Remark 3.5. Let B be an $(s+r)$–field map with minimum degree at least $d_{\min}$ and maximum degree at most $d_{\max} < \infty$ in its last r arguments.

(a) $d_{\min}|||B|||_{w_{\kappa,\lambda}} \le |||B|||'_{w_{\kappa,\lambda}} \le d_{\max}|||B|||_{w_{\kappa,\lambda}}$

(b) If $d_{\min} = d_{\max} = 1$, B is said to be linear. In this case, for any fixed $\alpha_1,\cdots,\alpha_s$, the map

$$(\gamma_1,\cdots,\gamma_s) \mapsto B(\alpha_1,\cdots,\alpha_s,\gamma_1,\cdots\gamma_r)$$

is linear and $|||B|||'_{w_{\kappa,\lambda}} = |||B|||_{w_{\kappa,\lambda}}$

Example 3.6. A simple example with $s = 0$ and $r = 1$ is the truncated exponential

$$B\big(\gamma\big)(\mathbf{x}) = E_n\big(a\gamma(\mathbf{x})\big) \qquad \text{where} \qquad E_n(z) = \sum_{\ell=n}^{\infty} \tfrac{1}{\ell!} z^\ell$$

and a is a constant. In this example, B is a local function of γ, so that all of the kernels of B are just delta functions. Hence

$$|||B|||_{w_{\kappa,\lambda}} = \sum_{\ell=n}^{\infty} \tfrac{1}{\ell!} a^\ell \lambda^\ell = E_n(a\lambda) \le \tfrac{a^n\lambda^n}{n!} e^{a\lambda}$$
$$|||B|||'_{w_{\kappa,\lambda}} = \sum_{\ell=n}^{\infty} \tfrac{1}{(\ell-1)!} a^\ell \lambda^\ell = a\lambda E_{n-1}(a\lambda) \le \tfrac{a^n\lambda^n}{(n-1)!} e^{a\lambda}$$

Lemma 3.7. *Let B be an $(s+r)$–field map with $|||B|||'_{w_{\kappa,\lambda}} < \infty$. Assume that B has minimum degree at least $d_{\min}$ in its last r arguments. Then, for each $\vec\Gamma, \vec\Gamma' \in \mathcal{B}_1$,*

$$|||\tilde B(\vec\Gamma) - \tilde B(\vec\Gamma')|||_w \le \|\vec\Gamma - \vec\Gamma'\| \, \max\big\{\|\vec\Gamma\|\,,\,\|\vec\Gamma'\|\big\}^{d_{\min}-1} |||B|||'_{w_{\kappa,\lambda}}$$

Proof. Write

$$B = \sum_{\substack{n_{s+1},\cdots,n_{s+r}\ge 0 \\ n_{s+1}+\cdots+n_{s+r}\ge d_{\min}}} B_{n_{s+1},\cdots,n_{s+r}}$$

as in Definition 3.4. Since

$$|||B|||_{w_{\kappa,\lambda}} = \sum_{\substack{n_{s+1},\cdots,n_{s+r}\ge 0 \\ n_{s+1}+\cdots+n_{s+r}\ge d_{\min}}} |||B_{n_{s+1},\cdots,n_{s+r}}|||_{w_{\kappa,\lambda}}$$
$$|||B|||'_{w_{\kappa,\lambda}} = \sum_{\substack{n_{s+1},\cdots,n_{s+r}\ge 0 \\ n_{s+1}+\cdots+n_{s+r}\ge d_{\min}}} |||B_{n_{s+1},\cdots,n_{s+r}}|||'_{w_{\kappa,\lambda}}$$

we may assume, without loss of generality, that at most one $B_{n_{s+1},\cdots,n_{s+r}}$ is nonvanishing. By renaming the γ fields and changing the value of r, we may assume that $n_{s+1} = \cdots = n_{s+r} = 1$. Then $B\big(\vec\alpha, \gamma_1, \cdots \gamma_r\big)$ is multilinear in $\gamma_1, \cdots, \gamma_r$ so that

$$\begin{aligned}\tilde B(\vec\Gamma)(\vec\alpha) - \tilde B(\vec\Gamma')(\vec\alpha) &= B\big(\vec\alpha, \Gamma_1(\vec\alpha), \cdots, \Gamma_r(\vec\alpha)\big) - B\big(\vec\alpha, \Gamma'_1(\vec\alpha), \cdots, \Gamma'_r(\vec\alpha)\big) \\ &= \sum_{j=1}^{r} B\Big(\vec\alpha\,,\, \Gamma_1(\vec\alpha), \cdots, \Gamma_{j-1}(\vec\alpha), \Gamma_j(\vec\alpha) - \Gamma'_j(\vec\alpha), \Gamma'_{j+1}(\vec\alpha), \cdots, \Gamma'_r(\vec\alpha)\Big)\end{aligned}$$

So, by Corollary 3.3,

$$\begin{aligned}|||\tilde B(\vec\Gamma) - \tilde B(\vec\Gamma')|||_w &\le \sum_{j=1}^{r} \Big(\prod_{k=1}^{j-1} \tfrac{|||\Gamma_k|||_w}{\lambda_k}\Big) \tfrac{|||\Gamma_j - \Gamma'_j|||_w}{\lambda_j} \Big(\prod_{k=j+1}^{r} \tfrac{|||\Gamma'_k|||_w}{\lambda_k}\Big) |||B|||_{w_{\kappa,\lambda}} \\ &\le r \;\max\big\{\|\vec\Gamma\|\,,\,\|\vec\Gamma'\|\big\}^{r-1} \;\|\vec\Gamma - \vec\Gamma'\| \;|||B|||_{w_{\kappa,\lambda}} \\ &\le \max\big\{\|\vec\Gamma\|\,,\,\|\vec\Gamma'\|\big\}^{r-1} \;\|\vec\Gamma - \vec\Gamma'\| \;|||B|||'_{w_{\kappa,\lambda}}\end{aligned}$$

The claim follows since $\max\left\{\|\vec{\Gamma}\|\,,\,\|\vec{\Gamma}'\|\right\} \le 1$ and $r \ge d_{\min}$. $\square$

Lemma 3.8 (Product Rule)**.** *Let $A(\vec{\alpha},\vec{\gamma})$ and $B(\vec{\alpha},\vec{\gamma})$ be $(s+r)$–field maps with $|||A|||'_{w_{\kappa,\lambda}}, |||B|||'_{w_{\kappa,\lambda}} < \infty$. Define*

$$C(\vec{\alpha},\vec{\gamma})(\mathbf{x}) = A(\vec{\alpha},\vec{\gamma})(\mathbf{x})\ B(\vec{\alpha},\vec{\gamma})(\mathbf{x})$$

Then

$$|||C|||'_{w_{\kappa,\lambda}} \le |||A|||'_{w_{\kappa,\lambda}}|||B|||_{w_{\kappa,\lambda}} + |||A|||_{w_{\kappa,\lambda}}|||B|||'_{w_{\kappa,\lambda}}$$

Proof. For convenience of notation, write $\vec{n} = (n_{s+1}, \cdots, n_{s+r})$, $|\vec{n}| = n_{s+1}+\cdots+n_{s+r}$ and $\vec{n} \ge 0$ for $n_{s+1}, \cdots, n_{s+r} \ge 0$. Then, in the notation of Definition 3.4.a,

$$C = \sum_{\vec{N}\ge 0} C_{\vec{N}} \qquad \text{with} \qquad C_{\vec{N}} = \sum_{\substack{vecn,\vec{m}\ge 0\\ \vec{n}+\vec{m}=\vec{N}}} A_{\vec{n}} B_{\vec{m}}$$

and

$$\begin{aligned} |||C|||'_{w_{\kappa,\lambda}} &= \sum_{\vec{N}\ge 0} |\vec{N}|\ |||C_{\vec{N}}|||_{w_{\kappa,\lambda}} \\ &\le \sum_{\vec{n},\vec{m}\ge 0} \big(|\vec{n}| + |\vec{m}|\big)\ |||A_{\vec{n}}B_{\vec{m}}|||_{w_{\kappa,\lambda}} \end{aligned}$$

So the claim follows from

$$|||A_{\vec{n}}B_{\vec{m}}|||_{w_{\kappa,\lambda}} \le |||A_{\vec{n}}|||_{w_{\kappa,\lambda}}|||B_{\vec{m}}|||_{w_{\kappa,\lambda}}$$

$\square$

4 Solving Equations

In this section we consider systems of $r \geq 1$ implicit equations of the form

$$\gamma_j = f_j(\vec{\alpha}) + L_j(\vec{\alpha}, \vec{\gamma}) + B_j\big(\vec{\alpha}, \vec{\gamma}\big) \tag{4.1.a}$$

for "unknown" fields $\gamma_1, \cdots, \gamma_r$ as a function of fields $\alpha_1, \cdots, \alpha_s$. In the above equation, $\vec{\alpha} = \big(\alpha_1, \cdots, \alpha_s\big)$, $\vec{\gamma} = \big(\gamma_1, \cdots, \gamma_r\big)$, and for each $1 \leq j \leq r$,

- f_j is an s–field map,
- L_j is an $(s + r)$–field map that is linear in its last r arguments, and
- B_j is an $(s + r)$–field map.

We write the system (4.1.a) in the shorthand notation

$$\vec{\gamma} = \vec{f}(\vec{\alpha}) + \vec{L}(\vec{\alpha}, \vec{\gamma}) + \vec{B}\big(\vec{\alpha}, \vec{\gamma}\big) \tag{4.1.b}$$

Example 4.2, below, is of this form and is a simplified version of the kind of equations that occur as equations for "background fields" and "critical fields" in [8, 9]. The following proposition gives conditions under which this system of equations has a solution $\vec{\gamma} = \vec{\Gamma}(\vec{\alpha})$, estimates on the solution, and a uniqueness statement.

Proposition 4.1. *Let $\kappa_1, \cdots, \kappa_s$ and $\lambda_1, \cdots, \lambda_r$ be constant weight factors for the fields $\alpha_1, \cdots, \alpha_s$ and $\gamma_1, \cdots, \gamma_r$, respectively. As in Definition 3.4 set $\mathcal{B}_1 = \big\{ \vec{\Gamma} \bigm| \|\vec{\Gamma}\| \leq 1 \big\}$ where $\|\vec{\Gamma}\| = \max\limits_{1\leq j\leq r} \frac{1}{\lambda_j} |||\Gamma_j|||_{w_\kappa}$. Let $0 < \mathfrak{c} < 1$ be a contraction factor. Assume that, for each $1 \leq j \leq r$, the $(s+r)$–field map $B_j(\vec{\alpha}; \vec{\gamma})$ has minimum degree at least 2 in its last r arguments (that is, in $\vec{\gamma}$). Also assume that for $1 \leq j \leq r$*

$$\begin{aligned} |||f_j|||_{w_\kappa} + |||L_j|||_{w_{\kappa,\lambda}} + |||B_j|||_{w_{\kappa,\lambda}} &\leq \lambda_j \\ |||L_j|||_{w_{\kappa,\lambda}} + |||B_j|||'_{w_{\kappa,\lambda}} &\leq \mathfrak{c}\lambda_j \end{aligned}$$

(a) Then there is a unique $\vec{\Gamma} \in \mathcal{B}_1$ for which

$$\vec{\Gamma}(\vec{\alpha}) = \vec{f}(\vec{\alpha}) + \vec{L}\big(\vec{\alpha}, \vec{\Gamma}(\vec{\alpha})\big) + \vec{B}\big(\vec{\alpha}, \vec{\Gamma}(\vec{\alpha})\big)$$

That is, which solves (4.1)*. Furthermore*

$$\max_j \tfrac{1}{\lambda_j} |||\Gamma_j|||_w \leq \tfrac{1}{1-\mathfrak{c}} \max_j \tfrac{1}{\lambda_j} |||f_j|||_w \qquad \max_j \tfrac{1}{\lambda_j} |||\Gamma_j - f_j|||_w \leq \tfrac{\mathfrak{c}}{1-\mathfrak{c}} \max_j \tfrac{1}{\lambda_j} |||f_j|||_w$$

(b) Assume, in addition, that

$$|||f_j|||_w \leq (1 - \mathfrak{c})^2 \, \lambda_j \qquad \text{for all } 1 \leq j \leq r$$

Denote by $\vec{\Gamma}$ the solution of part (a) and by $\vec{\Gamma}^{(1)}$ the unique element of $\mathcal{B}_1$ that solves $\gamma_j = f_j(\vec{\alpha}) + L_j(\vec{\alpha}, \vec{\gamma})$ for $1 \le j \le r$. Then

$$\|\vec{\Gamma}^{(1)}\| \le \tfrac{1}{1-\mathfrak{c}}\|\vec{f}\| \qquad \|\vec{\Gamma}^{(1)} - \vec{f}\| \le \tfrac{\mathfrak{c}}{1-\mathfrak{c}}\|\vec{f}\|$$

and

$$\|\vec{\Gamma} - \vec{\Gamma}^{(1)}\| \le \tfrac{\|\vec{f}\|^2}{(1-\mathfrak{c})^3} \max_{1\le j\le r} \tfrac{1}{\lambda_j}|||B_j|||_{w_{\kappa,\lambda}} \le \max_{1\le j\le r} \tfrac{1}{\lambda_j}|||B_j|||_{w_{\kappa,\lambda}}$$

Proof. (a) Define $F(\vec{\Gamma})$ by

$$\vec{F}(\vec{\Gamma}) = \begin{bmatrix} f_1 + \tilde{L}_1(\vec{\Gamma}) + \tilde{B}_1(\vec{\Gamma}) \\ \vdots \\ f_r + \tilde{L}_r(\vec{\Gamma}) + \tilde{B}_r(\vec{\Gamma}) \end{bmatrix}$$

Recall, from Definition 3.4, that

$$\big(\tilde{L}_j(\vec{\Gamma})\big)(\vec{\alpha}) = L_j\big(\vec{\alpha}, \vec{\Gamma}(\vec{\alpha})\big) \quad \text{and} \quad \big(\tilde{B}_j(\vec{\Gamma})\big)(\vec{\alpha}) = B_j\big(\vec{\alpha}, \vec{\Gamma}(\vec{\alpha})\big)$$

By Corollary 3.3 and the hypothesis $|||f_j|||_{w_\kappa} + |||L_j|||_{w_{\kappa,\lambda}} + |||B_j|||_{w_{\kappa,\lambda}} \le \lambda_j$, $\vec{F}$ maps $\mathcal{B}_1$ into $\mathcal{B}_1$. By Lemma 3.7 and Remark 3.5.b, $\|\vec{F}(\vec{\Gamma}) - \vec{F}(\vec{\Gamma}')\| \le \mathfrak{c}\|\vec{\Gamma} - \vec{\Gamma}'\|$ so that $\vec{F}$ is a strict contraction. The claims are now a consequence of the contraction mapping theorem.

(b) The first two bounds are special cases of part (a) with $B_j = 0$. Since L_j is linear in its last r arguments, $\delta\vec{\Gamma} = \vec{\Gamma} - \vec{\Gamma}^{(1)}$ obeys

$$\delta\Gamma_j(\vec{\alpha}) = L_j\big(\vec{\alpha}\,,\, \delta\vec{\Gamma}(\vec{\alpha})\big) + B_j\big(\vec{\alpha}\,,\, \vec{\Gamma}^{(1)}(\vec{\alpha}) + \delta\vec{\Gamma}(\vec{\alpha})\big)$$

for $1 \le j \le r$. View this a fixed point equation determining $\vec{\delta}\Gamma$. The equation is of the form $\vec{\delta} = \vec{G}(\vec{\delta})$ where

$$\vec{G}(\vec{\delta}) = \begin{bmatrix} \tilde{L}_1(\vec{\delta}) + \tilde{B}_1(\vec{\Gamma}^{(1)} + \vec{\delta}) \\ \vdots \\ \tilde{L}_r(\vec{\delta}) + \tilde{B}_r(\vec{\Gamma}^{(1)} + \vec{\delta}) \end{bmatrix}$$

If $\|\vec{\delta}\| \le \mathfrak{c}$ then $\|\vec{\Gamma}^{(1)} + \vec{\delta}\| \le 1$. Therefore, by Corollary 3.3, $\vec{G}$ maps $\mathcal{B}_\mathfrak{c}$ into $\mathcal{B}_\mathfrak{c}$. By Lemma 3.7, $\vec{G}$ is a strict contraction. Apply the contraction mapping theorem. Since $G_j(\vec{0}) = \tilde{B}_j(\vec{\Gamma}^{(1)})$ and

$$\|\vec{\Gamma}^{(1)}\| \le \tfrac{1}{1-\mathfrak{c}}\|\vec{f}\| \implies |||\Gamma_j^{(1)}|||_w \le \tfrac{\|\vec{f}\|}{1-\mathfrak{c}}\lambda_j$$

for each $1 \le j \le r$ and B_j is of degree at least two in its last r arguments we have $|||\tilde{B}_j(\vec{\Gamma}^{(1)})|||_w \le \big(\frac{\|\vec{f}\|}{1-\mathfrak{c}}\big)^2 |||B_j|||_{w_{\kappa,\lambda}}$ so that $\|\vec{G}(\vec{0})\| \le \big(\frac{\|\vec{f}\|}{1-\mathfrak{c}}\big)^2 \max\limits_{1\le j\le r} \frac{1}{\lambda_j} |||B_j|||_{w_{\kappa,\lambda}}$. Therefore the fixed point $\vec{\delta} = \delta\vec{\Gamma}$ obeys

$$\|\delta\vec{\Gamma}\| \le \tfrac{1}{1-\mathfrak{c}} \|\vec{G}(\vec{0})\| \le \tfrac{\|\vec{f}\|^2}{(1-\mathfrak{c})^3} \max_{1\le j\le r} \tfrac{1}{\lambda_j} |||B_j|||_{w_{\kappa,\lambda}} \le (1-\mathfrak{c}) \max_{1\le j\le r} \tfrac{1}{\lambda_j} |||B_j|||_{w_{\kappa,\lambda}}$$

□

Example 4.2. We assume that X is a finite lattice of the form $X = \mathfrak{L}_1/\mathfrak{L}_2$, where $\mathfrak{L}_1$ is a lattice in $\mathbb{R}^d$ and $\mathfrak{L}_2$ is a sublattice of $\mathfrak{L}_1$ of finite index. The Euclidean distance on $\mathbb{R}^d$ induces a distance $|\cdot|$ on X.
Let $W_1, W_2 : X^3 \to \mathbb{C}$ and set, for complex fields ϕ_1, ϕ_2 on X

$$\begin{aligned}
\mathcal{W}_1(\phi_1,\phi_2)(x) &= \sum_{y,z\in X} W_1(x,y,z)\,\phi_1(y)\,\phi_2(z)\\
\mathcal{W}_2(\phi_1,\phi_2)(x) &= \sum_{y,z\in X} W_2(x,y,z)\,\phi_1(y)\,\phi_2(z)
\end{aligned}$$

Aso let S_1 and S_2 be two invertible operators on $L^2(X)$. Pretend that S_1^{-1} and S_2^{-1} are "differential operators". Suppose that we are interested in solving

$$\begin{aligned}
S_1^{-1}\phi_1 + \mathcal{W}_1(\phi_1,\phi_2) &= \alpha_1\\
S_2^{-1}\phi_2 + \mathcal{W}_2(\phi_1,\phi_2) &= \alpha_2
\end{aligned} \tag{4.2}$$

for ϕ_1, ϕ_2 as functions of complex fields α_1, α_2. Suppose further that we are thinking of the $\mathcal{W}_j$'s as small. We would like to write the solution as a perturbation of the $\mathcal{W}_1 = \mathcal{W}_2 = 0$ solution $\phi_1 = S_1\alpha_1$, $\phi_2 = S_2\alpha_2$. So we substitute

$$\phi_1 = S_1\big(\alpha_1 + \gamma_1\big) \qquad \phi_2 = S_2\big(\alpha_2 + \gamma_2\big)$$

into (4.2), giving

$$\begin{aligned}
\gamma_1 + \mathcal{W}_1\big(S_1(\alpha_1+\gamma_1)\,,\, S_2(\alpha_2+\gamma_2)\big) &= 0\\
\gamma_2 + \mathcal{W}_2\big(S_1(\alpha_1+\gamma_1)\,,\, S_2(\alpha_2+\gamma_2)\big) &= 0
\end{aligned}$$

This is of the form (4.1) with

$$\vec{f}(\vec{\alpha}) = \begin{bmatrix} -\mathcal{W}_1\big(S_1\alpha_1\,,\ S_2\alpha_2\big) \\ -\mathcal{W}_2\big(S_1\alpha_1\,,\ S_2\alpha_2\big) \end{bmatrix}$$

$$\vec{L}(\vec{\alpha},\vec{\gamma}) = \begin{bmatrix} -\mathcal{W}_1\big(S_1\gamma_1\,,\ S_2\alpha_2\big) - \mathcal{W}_1\big(S_1\alpha_1\,,\ S_2\gamma_2\big) \\ -\mathcal{W}_2\big(S_1\gamma_1\,,\ S_2\alpha_2\big) - \mathcal{W}_2\big(S_1\alpha_1\,,\ S_2\gamma_2\big) \end{bmatrix}$$

$$\vec{B}(\vec{\alpha},\vec{\gamma})(u) = \begin{bmatrix} -\mathcal{W}_1\big(S_1\gamma_1\,,\ S_2\gamma_2\big) \\ -\mathcal{W}_2\big(S_1\gamma_1\,,\ S_2\gamma_2\big) \end{bmatrix}$$

To apply Proposition 4.1 to Example 4.2, fix any $\mathfrak{m},\mathfrak{k} > 0$ and use the norm $|||\phi_j|||$ with metric $\mathfrak{m}|\cdot|$ and weight factors $\mathfrak{k}$ to measure analytic maps like $\phi_j(\alpha_1,\alpha_2)$. See Definition 2.3.c. The weight factor $\mathfrak{k}$ is used for both α_1 and α_2. Like in [3, §IV] and [4, Definition 4.2] we define, for any linear operator $S : L^2(X) \to L^2(X)$, the "weighted" ℓ^1–ℓ^∞ norm

$$\|S\|_{\mathfrak{m}} = \max\Big\{ \sup_{y\in X}\sum_{x\in X} |S(x,y)|e^{\mathfrak{m}|y-x|}\ ,\ \sup_{x\in X}\sum_{y\in X} |S(x,y)|e^{\mathfrak{m}|y-x|}\Big\}$$

Proposition 4.1 can be applied to this situation:

Corollary 4.3. *Let $K > 0$. Write $\bar{S} = \max_{j=1,2}\|S_j\|_{\mathfrak{m}}$ and $\bar{W} = \max_{j=1,2}\|W_j\|_{\mathfrak{m}}$ and assume that*

$$\bar{S}^2\bar{W}\,\mathfrak{k} < \min\big\{\tfrac{1}{12}\,,\ \tfrac{1}{2K}\big\}$$

Then there are field maps $\phi_1^{(\geq 2)}$, $\phi_2^{(\geq 2)}$ such that

$$\begin{aligned} \phi_1(\alpha_1,\alpha_2) &= S_1\alpha_1 + \phi_1^{(\geq 2)}(\alpha_1,\alpha_2) \\ \phi_2(\alpha_1,\alpha_2) &= S_2\alpha_2 + \phi_2^{(\geq 2)}(\alpha_1,\alpha_2) \end{aligned}$$

solves the equations (4.2) *of Example 4.2 and obeys*

$$|||\phi_j^{(\geq 2)}||| \leq 2\bar{S}^3\,\bar{W}\,\mathfrak{k}^2$$

Furthermore $\phi_j^{(\geq 2)}$ is of degree at least two in (α_1,α_2). The solution is unique in

$$\Big\{\ (\phi_1,\phi_2) \in L^2(X)\times L^2(X)\ \Big|\ |||S_1^{-1}\phi_1|||, |||S_2^{-1}\phi_2||| \leq K\mathfrak{k}\ \Big\}$$

Proof. In Example 4.2 we wrote the equations (4.2) in the form

$$\vec{\gamma} = \vec{f}(\vec{\alpha}) + \vec{L}(\vec{\alpha},\vec{\gamma}) + \vec{B}\big(\vec{\alpha},\vec{\gamma}\big) \tag{4.3}$$

Now apply Proposition 4.1.a and Remark 3.5.a with $r = s = 2$ and

$$d_{\max} = 2 \qquad \mathfrak{c} = \tfrac{1}{2} \qquad \kappa_1 = \kappa_2 = \lambda_1 = \lambda_2 = \mathfrak{k}$$

Since

$$\begin{aligned} |||f_j|||_w &\le \|S_1\|_{\mathfrak{m}}\|S_2\|_{\mathfrak{m}}\|W_j\|_{\mathfrak{m}}\ \kappa_1\kappa_2 \\ |||L_j|||_{w_{\kappa,\lambda}} &\le \|S_1\|_{\mathfrak{m}}\|S_2\|_{\mathfrak{m}}\|W_j\|_{\mathfrak{m}}\ \big(\lambda_1\kappa_2 + \kappa_1\lambda_2\big) \\ |||B_j|||_{w_{\kappa,\lambda}} &\le \|S_1\|_{\mathfrak{m}}\|S_2\|_{\mathfrak{m}}\|W_j\|_{\mathfrak{m}}\lambda_1\lambda_2 \end{aligned}$$

By hypothesis, $|||f_j|||_w$, $|||L_j|||_{w_{\kappa,\lambda}}$, $|||B_j|||_{w_{\kappa,\lambda}} < \frac{1}{6}\lambda_j$ and Proposition 4.1.a gives a solution $\vec{\Gamma}(\vec{\alpha})$ to (4.3) that obeys the bound

$$|||\Gamma_j|||_w \le 2\|S_1\|_{\mathfrak{m}}\|S_2\|_{\mathfrak{m}}\|W_j\|_{\mathfrak{m}}\mathfrak{k}^2$$

Setting

$$\begin{aligned} \phi_1(\alpha_1,\alpha) &= S_1\alpha_1 + S_1\Gamma_1(\alpha_1,\alpha_2) \qquad & \phi_1^{(\ge 2)}(\alpha_1,\alpha_2) &= S_1\Gamma_1(\alpha_1,\alpha_2) \\ \phi_2(\alpha_1,\alpha) &= S_2\alpha_2 + S_2\Gamma_2(\alpha_1,\alpha_2) \qquad & \phi_2^{(\ge 2)}(\alpha_1,\alpha_2) &= S_2\Gamma_2(\alpha_1,\alpha_2) \end{aligned}$$

we have all of the claims, except for uniqueness.

We now prove uniqueness. Assume that $\phi_j = S_j\Phi_j$ and that $\phi_j = S_j(\Phi_j + \delta\Phi_j)$ both solve (4.2), with $|||\Phi_j + \delta\Phi_j||| \le K\mathfrak{k}$ and with $S_j\Phi_j$ being the solution constructed above. Then $\delta\Phi_j$ is a solution of

$$\begin{aligned} \delta\Phi_1 &= -\mathcal{W}_1\big(S_1(\Phi_1+\delta\Phi_1)\,,\ S_2(\Phi_2+\delta\Phi_2)\big) + \mathcal{W}_1\big(S_1\Phi_*\,,\ S_2\Phi\big) \\ \delta\Phi_2 &= -\mathcal{W}_2\big(S_2(\Phi_2+\delta\Phi_2)\,,\ S_1(\Phi_1+\delta\Phi_1)\big) + \mathcal{W}_2\big(S_2\Phi\,,\ S_1\Phi_1\big) \end{aligned}$$

Since

$$\Big|\Big|\Big|\mathcal{W}_j\big(S_1\alpha_1\,,\ S_2\alpha_2\big)\Big|\Big|\Big| \le \|W_j\|_{\mathfrak{m}}\,|||S_1\alpha_1|||\,|||S_2\alpha_2||| \le \|W_j\|_{\mathfrak{m}}\,\|S_1\|_{\mathfrak{m}}\|S_2\|_{\mathfrak{m}}\,|||\alpha_1|||\,|||\alpha_2|||$$

we have

$$\begin{aligned} |||\delta\Phi_1||| &\le \|W_1\|_{\mathfrak{m}}\,\|S_1\|_{\mathfrak{m}}\,\|S_2\|_{\mathfrak{m}}\,\big\{|||\delta\Phi_1|||\,|||\Phi_2+\delta\Phi_2||| + |||\Phi_1|||\,|||\delta\Phi_2|||\big\} \\ |||\delta\Phi_2||| &\le \|W_2\|_{\mathfrak{m}}\,\|S_1\|_{\mathfrak{m}}\,\|S_2\|_{\mathfrak{m}}\,\big\{|||\delta\Phi_1|||\,|||\Phi_2+\delta\Phi_2||| + |||\Phi_1|||\,|||\delta\Phi_2|||\big\} \end{aligned}$$

By hypothesis

$$|||\Phi_1||| \le \mathfrak{k} + 2\|S_1\|_{\mathfrak{m}}\|S_2\|_{\mathfrak{m}}\|W_j\|_{\mathfrak{m}}\mathfrak{k}^2 \le \tfrac{7}{6}\mathfrak{k} \qquad |||\Phi_2 + \delta\Phi_2||| \le K\mathfrak{k}$$

so that

$$\begin{aligned} |||\delta\Phi_1||| + |||\delta\Phi_2||| &\le \big(\|W_1\|_{\mathfrak{m}} + \|W_2\|_{\mathfrak{m}}\big)\, \|S_1\|_{\mathfrak{m}}\, \|S_2\|_{\mathfrak{m}}\, \max\big\{\tfrac{7}{6}, K\big\}\mathfrak{k}\, \big(|||\delta\Phi_1||| + |||\delta\Phi_2|||\big) \\ &\le \bar{S}^2\bar{W}\mathfrak{k}\; 2\max\big\{\tfrac{7}{6}, K\big\}\, \big(|||\delta\Phi_1||| + |||\delta\Phi_2|||\big) \end{aligned}$$

thereby forcing $|||\delta\Phi_*||| = |||\delta\Phi||| = 0$. □

A A Generalisation of Young's Inequality

Lemma A.1. *Let $n \in \mathbb{N}$. For each $1 \le \ell \le n$, let*

- $(X_\ell, d\mu_\ell)$ *be a measure space,*
- $f_\ell : X_\ell \to \mathbb{C}$ *be measureable and*
- $p_\ell \in (0, \infty]$.

Let $K : \mathop{\times}\limits_{\ell=1}^{n} X_\ell \to \mathbb{C}$ have finite L^1–L^∞ norm and assume that $\sum\limits_{\ell=1}^{n} \frac{1}{p_\ell} = 1$. Then

$$\left| \int_{\mathop{\times}\limits_{\ell=1}^{n} X_\ell} K(x_1, \cdots, x_n) \prod_{\ell=1}^{n} f_\ell(x_\ell) \prod_{\ell=1}^{n} d\mu_\ell(x_\ell) \right| \le \|K\|_{L^1-L^\infty} \prod_{\ell=1}^{n} \|f_\ell\|_{L^{p_\ell}(d\mu_\ell)}$$

Proof. We'll use the short hand notations $dm(x_1, \cdots, x_n) = \prod\limits_{\ell=1}^{n} d\mu_\ell(x_\ell)$ and $X = \mathop{\times}\limits_{\ell=1}^{n} X_\ell$. By Hölder (with the usual interpretations when some $p_\ell = \infty$),

$$\begin{aligned}
&\left| \int_X K(x_1, \cdots, x_n) \prod_{\ell=1}^{n} f_\ell(x_\ell) \; dm(x_1, \cdots, x_n) \right| \\
&\qquad \le \int_X \prod_{\ell=1}^{n} \left\{ \big| K(x_1, \cdots, x_n) \big|^{1/p_\ell} \, |f_\ell(x_\ell)| \right\} dm(x_1, \cdots, x_n) \\
&\qquad = \prod_{\ell=1}^{n} \left[\int_X \big| K(x_1, \cdots, x_n) \big| \, |f_\ell(x_\ell)|^{p_\ell} \; \prod_{\ell=1}^{n} d\mu_\ell(x_\ell) \right]^{1/p_\ell} \\
&\qquad \le \prod_{\ell=1}^{n} \left[\|K\|_{L^1-L^\infty} \int_{X_\ell} |f_\ell(x_\ell)|^{p_\ell} \; d\mu_\ell(x_\ell) \right]^{1/p_\ell} \\
&\qquad = \|K\|_{L^1-L^\infty} \prod_{\ell=1}^{n} \|f_\ell\|_{L^{p_\ell}(d\mu_\ell)}
\end{aligned}$$

$\square$

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