Approximating Functions Near a Specified Point

Suppose that you are interested in the values of some function $f(x)$ for x near some fixed point x_0. The function is too complicated to work with directly. So you wish to work instead with some other function $F(x)$ that is both simple and a good approximation to $f(x)$ for x near x_0. We'll consider a couple of examples of this scenario later. First, we develop three different approximations.

First approximation

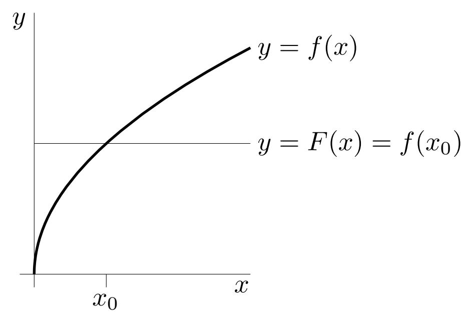
The simplest functions are those that are constants. The first approximation will be by a constant function. That is, the approximating function will have the form $F(x) = A$. To ensure that $F(x)$ is a good approximation for x close to x_0, we chose the constant A so that $f(x)$ and $F(x)$ take exactly the same value when $x = x_0$.

$$F(x) = A \ \Rightarrow\ F(x_0) = A \qquad \text{so} \qquad f(x_0) = F(x_0)\ \Rightarrow\ A = f(x_0)$$

Our first, and crudest, approximation rule is

$$\boxed{f(x) \approx f(x_0)} \tag{1}$$

Here is a figure showing the graphs of a typical $f(x)$ and approximating function $F(x)$. At $x = x_0$, $f(x)$ and



$F(x)$ take the same value. For x very near x_0, the values of $f(x)$ and $F(x)$ remain close together. But the quality of the approximation deteriorates fairly quickly as x moves away from x_0.

Second Approximation – the tangent line, or linear, approximation

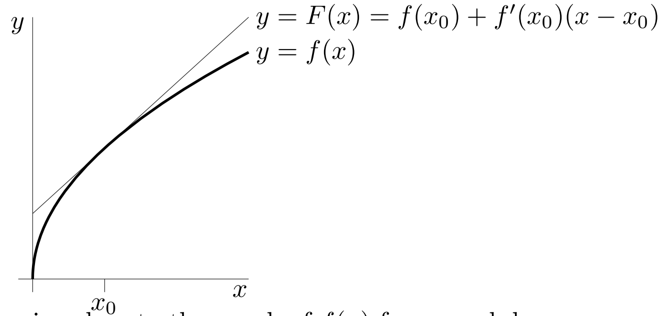
We now develop a better approximation by allowing the approximating function to be a linear function of x and not just a constant function. That is, we allow $F(x)$ to be of the form $A + Bx$. To ensure that $F(x)$ is a good approximation for x close to x_0, we chose the constants A and B so that $f(x_0) = F(x_0)$ and $f'(x_0) = F'(x_0)$. Then $f(x)$ and $F(x)$ will have both the same value and the same slope at $x = x_0$.

$$\begin{array}{llll} F(x) = A + Bx & \Rightarrow F(x_0) = A + Bx_0 & \text{so} & f(x_0) = F(x_0) \ \Rightarrow\ A + Bx_0 = f(x_0) \\ F'(x) = B & \Rightarrow F'(x_0) = B & \text{so} & f'(x_0) = F'(x_0) \ \Rightarrow\ \quad B \quad = f'(x_0) \end{array}$$

Subbing $B = f'(x_0)$ into $A + Bx_0 = f(x_0)$ gives $A = f(x_0) - x_0 f'(x_0)$ and consequently $F(x) = A + Bx = f(x_0) - x_0 f'(x_0) + x f'(x_0) = f(x_0) + f'(x_0)(x - x_0)$. So, our second approximation is

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0)} \tag{2}$$

Here is a figure showing the graphs of a typical $f(x)$ and approximating function $F(x)$. Observe that the graph



of $f(x_0) + f'(x_0)(x - x_0)$ remains close to the graph of $f(x)$ for a much larger range of x than did the graph of $f(x_0)$.

Third approximation – the quadratic approximation

We finally develop a still better approximation by allowing the approximating function be to a quadratic function of x. That is, we allow $F(x)$ to be of the form $A + Bx + Cx^2$. To ensure that $F(x)$ is a good approximation for x close to x_0, we chose the constants A, B and C so that $f(x_0) = F(x_0)$ and $f'(x_0) = F'(x_0)$ and $f''(x_0) = F''(x_0)$.

$$\begin{aligned}
F(x) &= A + Bx + Cx^2 &&\Rightarrow\quad F(x_0) = A + Bx_0 + Cx_0^2 = f(x_0)\\
F'(x) &= B + 2Cx &&\Rightarrow\quad F'(x_0) = B + 2Cx_0 = f'(x_0)\\
F''(x) &= 2C &&\Rightarrow\quad F''(x_0) = 2C = f''(x_0)
\end{aligned}$$

Solve for C first, then B and finally A.

$$\begin{aligned}
C = \tfrac{1}{2} f''(x_0) \ &\Rightarrow\ B = f'(x_0) - 2Cx_0 = f'(x_0) - x_0 f''(x_0)\\
&\Rightarrow\ A = f(x_0) - x_0 B - Cx_0^2 = f(x_0) - x_0[f'(x_0) - x_0 f''(x_0)] - \tfrac{1}{2} f''(x_0) x_0^2
\end{aligned}$$

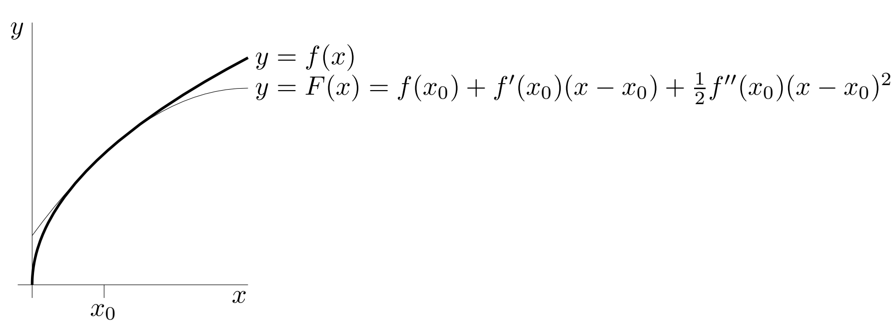
Then build up $F(x)$.

$$\begin{aligned}
F(x) &= f(x_0) - f'(x_0)x_0 + \tfrac{1}{2} f''(x_0)x_0^2 &&\text{(this line is } A)\\
&\quad + f'(x_0)\,x - f''(x_0)x_0 x &&\text{(this line is } Bx)\\
&\quad + \tfrac{1}{2} f''(x_0)x^2 &&\text{(this line is } Cx^2)\\
&= f(x_0) + f'(x_0)(x - x_0) + \tfrac{1}{2} f''(x_0)(x - x_0)^2
\end{aligned}$$

Our third approximation is

$$\boxed{f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \tfrac{1}{2} f''(x_0)(x - x_0)^2} \tag{3}$$

It is called the quadratic approximation. Here is a figure showing the graphs of a typical $f(x)$ and approximating function $F(x)$. The third approximation looks better than both the first and second.



By now you should be able to generate still better approximations on your own. When you do, the algebra will be simpler if you make $F(x)$ of the form

$$a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + \cdots$$

Because x_0 is itself a constant, this is really just a rewriting of $A + Bx + Cx^2 + Dx^3 + \cdots$. For example,

$$\begin{aligned} a + b(x - x_0) + c(x - x_0)^2 &= a + bx - bx_0 + cx^2 - 2cxx_0 + cx_0^2 = (a - bx_0 + cx_0^2) + (b - 2cx_0)x + cx^2 \\ &= A + Bx + Cx^2 \end{aligned}$$

with $A = a - bx_0 + cx_0^2$, $B = b - 2cx_0$ and $C = c$. The advantage of the form $a + b(x - x_0) + c(x - x_0)^2 + \cdots$ is that $x - x_0$ is zero when $x = x_0$, so lots of terms in the computation drop out. Try it!

Another Notation

Suppose that we have two variables x and y that are related by $y = f(x)$, for some function x. For example, x might be the number of cars manufactured per week in some factory and y the cost of manufacturing those x cars. Let x_0 be some fixed value of x and let $y_0 = f(x_0)$ be the corresponding value of y. Now suppose that x changes by an amount Δx, from x_0 to $x_0 + \Delta x$. As x undergoes this change, y changes from $y_0 = f(x_0)$ to $f(x_0 + \Delta x)$. The change in y that results from the change Δx in x is

$$\Delta y = f(x_0 + \Delta x) - f(x_0)$$

Substituting $x = x_0 + \Delta x$ into the linear approximation (2) yields the approximation

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)(x_0 + \Delta x - x_0) = f(x_0) + f'(x_0)\Delta x$$

for $f(x_0 + \Delta x)$ and consequently the approximation

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \approx f(x_0) + f'(x_0)\Delta x - f(x_0) \;\Rightarrow\; \boxed{\Delta y \approx f'(x_0)\Delta x} \tag{4}$$

for Δy. In the automobile manufacturing example, when the production level is x_0 cars per week, increasing the production level by Δx will cost approximately $f'(x_0)\Delta x$. The additional cost per additional car, $f'(x_0)$, is called the "marginal cost" of a car.

If we use the quadratic approximation (3) in place of the linear approximation (2)

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x + \tfrac{1}{2} f''(x_0)\Delta x^2$$

we arrive at the quadratic approximation

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \approx f(x_0) + f'(x_0)\Delta x + \tfrac{1}{2} f''(x_0)\Delta x^2 - f(x_0) \;\Rightarrow\; \boxed{\Delta y \approx f'(x_0)\Delta x + \tfrac{1}{2} f''(x_0)\Delta x^2} \tag{5}$$

for Δy.

Example 1

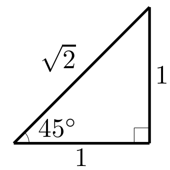
Suppose that you wish to compute, approximately, $\tan 46°$, but that you can't just use your calculator. This will be the case, for example, if the computation is an exercise to help prepare you for designing the software to be used by the calculator.

In this example, we choose $f(x) = \tan x$, $x = 46\frac{\pi}{180}$ radians and $x_0 = 45\frac{\pi}{180} = \frac{\pi}{4}$ radians. This is a good choice for x_0 because

- $x_0 = 45°$ is close to $x = 46°$. Generally, the closer x is to x_0, the better the quality of our three approximations
- We know the values of all trig functions at $45°$.

The first step in applying our approximations is to compute f and its first two derivatives at $x = x_0$.

$$f(x) = \tan x \qquad \Rightarrow \qquad f(x_0) = \tan\tfrac{\pi}{4} = 1$$
$$f'(x) = (\cos x)^{-2} \qquad \Rightarrow \qquad f'(x_0) = \tfrac{1}{\cos^2(\pi/4)} = 2$$
$$f''(x) = -2\tfrac{-\sin x}{\cos^3 x} \qquad \Rightarrow \qquad f''(x_0) = 2\tfrac{\sin(\pi/4)}{\cos^3(\pi/4)} = 2\tfrac{1/\sqrt{2}}{(1/\sqrt{2})^3} = 2\tfrac{1}{1/2} = 4$$



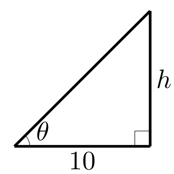
As $x - x_0 = 46\frac{\pi}{180} - 45\frac{\pi}{180} = \frac{\pi}{180}$ radians, the three approximations are

$$\begin{aligned} f(x) &\approx f(x_0) && = 1 \\ f(x) &\approx f(x_0) + f'(x_0)(x - x_0) && = 1 + 2\tfrac{\pi}{180} = 1.034907 \\ f(x) &\approx f(x_0) + f'(x_0)(x - x_0) + \tfrac{1}{2}f''(x_0)(x - x_0)^2 && = 1 + 2\tfrac{\pi}{180} + \tfrac{1}{2}4\big(\tfrac{\pi}{180}\big)^2 = 1.035516 \end{aligned}$$

For comparison purposes, $\tan 46°$ really is 1.035530 to 6 decimal places.

Recall that all of our derivative formulae for trig functions, were developed under the assumption that angles were measured in radians. As our approximation formulae used those derivatives, we were obliged to express $x - x_0$ in radians.

Example 2



Suppose that you are ten meters from a vertical pole. You were contracted to measure the height of the pole. You can't take it down or climb it. So you measure the angle subtended by the top of the pole. You measure $\theta = 30°$, which gives $h = 10\tan 30° = \frac{10}{\sqrt{3}} \approx 5.77$m. But there's a catch. Angles are hard to measure accurately. Your contract specifies that the height must be measured to within an accuracy of 10 cm. How accurate did your measurement of θ have to be?

For simplicity, we are going to assume that the pole is perfectly straight and perfectly vertical and that your distance from the pole was exactly 10 m. Write $h = h_0 + \Delta h$, where h is the exact height and $h_0 = \frac{10}{\sqrt{3}}$ is the computed height. Their difference, Δh, is the error. Similarly, write $\theta = \theta_0 + \Delta\theta$ where θ is the exact angle, θ_0 is the measured angle and $\Delta\theta$ is the error. Then

$$h_0 = 10\tan\theta_0 \qquad h_0 + \Delta h = 10\tan(\theta_0 + \Delta\theta)$$

We apply $\Delta y \approx f'(x_0)\Delta x$, with y replaced by h and x replaced by θ. That is, we apply $\Delta h \approx f'(\theta_0)\Delta\theta$. Choosing $f(\theta) = 10\tan\theta$ and $\theta_0 = 30°$ and subbing in $f'(\theta_0) = 10\sec^2\theta_0 = 10\sec^2 30° = 10\big(\frac{2}{\sqrt{3}}\big)^2 = \frac{40}{3}$, we see that the error in the computed value of h and the error in the measured value of θ are related by

$$\Delta h \approx \tfrac{40}{3}\Delta\theta$$

To achieve $|\Delta h| \le .1$, we better have $|\Delta\theta|$ smaller than $.1\frac{3}{40}$ radians or $.1\frac{3}{40}\frac{180}{\pi} = .43°$.

Example 3

Estimate the amount of point needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

Solution. The volume of a hemisphere of radius r is $V(r) = \frac{1}{2}\frac{4}{3}\pi r^3$. The radius of the hemisphere before the paint is applied was $r_0 = 25$m. The corresponding volume was $V_0 = V(25)$. When the paint was applied, the radius increased by $\Delta r = .0005$m to $25 + .0005$m. The volume of paint used was the change, $\Delta V = V(25 + .0005) - V(25)$, in the volume of the hemisphere. By (4)

$$\Delta V \approx V'(r_0)\Delta r = 2\pi r_0^2\Delta r = 2\pi(25)^2 \times 0.0005 = 1.9634954 \approx \boxed{1.96}$$

We have just computed an approximation to the volume of paint used. In this problem, we can compute the exact volume

$$V(25+.0005) - V(25) = \tfrac{2}{3}\pi(25+.0005)^3 - \tfrac{2}{3}\pi(25)^3$$

Applying $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ with $a = 25$ and $b = 0.0005$, gives

$$\begin{aligned} V(25+.0005) - V(25) &= \tfrac{2}{3}\pi[(25)^3 + 3(25)^2(0.0005) + 3(25)(0.0005)^2 + (0.0005)^3 - (25)^3] \\ &= \tfrac{2}{3}\pi[3(25)^2(0.0005) + 3(25)(0.0005)^2 + (0.0005)^3] \end{aligned}$$

The linear approximation, $\Delta V \approx 2\pi(25)^2 \times 0.0005$, is recovered by retaining only the first of the three terms in the square brackets. Thus the error introduced by the linear approximation is obtained by retaining only the last two terms in the square brackets. This error is

$$\tfrac{2}{3}\pi[3(25)(0.0005)^2 + (0.0005)^3] = 0.0000393$$

Example 4

If an aircraft crosses the Atlantic ocean at a speed of u mph, the flight costs the company

$$C(u) = 100 + \tfrac{u}{3} + \tfrac{240{,}000}{u}$$

dollars per passenger. When there is no wind, the aircraft flies at an airspeed of 550mph. Find the approximate savings, per passenger, when there is a 35 mph tail wind. Estimate the cost when there is a 50 mph head wind.

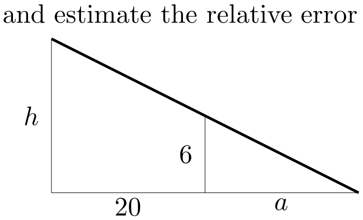
Solution. Let $u_0 = 550$. When the aircraft flies at speed u_0, the cost per passenger is $C(u_0)$. By (4), a change of Δu in the airspeed results in an change of

$$\Delta C \approx C'(u_0)\Delta u = \left[\tfrac{1}{3} - \tfrac{240{,}000}{u_0^2}\right]\Delta u = \left[\tfrac{1}{3} - \tfrac{240{,}000}{550^2}\right]\Delta u \approx -.460\Delta u$$

in the cost per passenger. With the tail wind $\Delta u = 35$ and the resulting $\Delta C \approx -.460 \times 35 = -16.10$, so there is a savings of $\boxed{\$16.10}$. With the head wind $\Delta u = -50$ and the resulting $\Delta C \approx -.4601 \times (-50) = 23.01$, so there is an additional cost of $\boxed{\$23.00}$.

Example 5

To compute the height h of a lamp post, the length a of the shadow of a six–foot pole is measured. The pole is 20 ft from the lamp post. If the length of the shadow was measured to be 15 ft, with an error of at most one inch, find the height of the lamp post and estimate the relative error in the height.



Solution. By similar triangles,

$$\frac{a}{6} = \frac{20+a}{h} \quad \Rightarrow \quad h = (20+a)\frac{6}{a} = \frac{120}{a} + 6$$

The length of the shadow was measured to be $a_0 = 15$ ft. The corresponding height of the lamp post is $h_0 = \frac{120}{a_0} + 6 = \frac{120}{15} + 6 = \boxed{14 \text{ ft}}$. If the error in the measurement of the length of the shadow was Δa, then the exact shadow length was $a = a_0 + \Delta a$ and the exact lamp post height is $h = f(a_0 + \Delta a)$, where $f(a) = \frac{120}{a} + 6$. The error in the computed lamp post height is $\Delta h = h - h_0 = f(a_0 + \Delta a) - f(a_0)$. By (4)

$$\Delta h \approx f'(a_0)\Delta a = -\tfrac{120}{a_0^2}\Delta a = -\tfrac{120}{15^2}\Delta a$$

We are told that $|\Delta a| \le \frac{1}{12}$. Consequently $|\Delta h| \le \frac{120}{15^2}\frac{1}{12} = \frac{10}{225}$ (approximately). The relative error is then

$$\frac{|\Delta h|}{h_0} \le \frac{10}{225\times 14} \approx \boxed{0.003} \text{ or } \boxed{0.3\%}$$

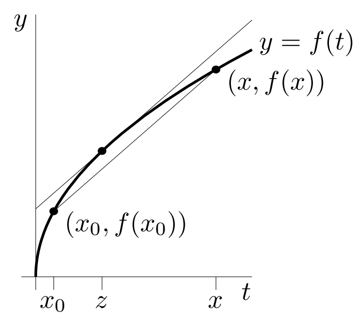
The Error in the Approximations

Any time you make an approximation, it is desirable to have some idea of the size of the error you introduced. We will now develop a formula for the error introduced by the approximation $f(x) \approx f(x_0)$. This formula can be used to get an upper bound on the size of the error, even when you cannot determine $f(x)$ exactly.

By simple algebra

$$f(x) = f(x_0) + \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \tag{6}$$

The coefficient $\frac{f(x)-f(x_0)}{x-x_0}$ of $(x - x_0)$ is the average slope of $f(t)$ as t moves from $t = x_0$ to $t = x$. In the figure below, it is the slope of the secant joining the points $(x_0, f(x_0))$ and $(x, f(x))$. As t moves x_0 to x,



the instantaneous slope $f'(t)$ keeps changing. Sometimes it is larger than the average slope $\frac{f(x)-f(x_0)}{x-x_0}$ and sometimes it is smaller than the average slope. But because $\frac{f(x)-f(x_0)}{x-x_0}$ is the average value of $f'(t)$ as t runs from x_0 to x, there must be some number z between x_0 and x for which $f'(z) = \frac{f(x)-f(x_0)}{x-x_0}$. Subbing this into formula (6)

$$\boxed{f(x) = f(x_0) + f'(z)(x - x_0) \qquad \text{for some } z \text{ between } x_0 \text{ and } x} \tag{7}$$

Thus the error in the approximation $f(x) \approx f(x_0)$ is exactly $f'(z)(x - x_0)$ for some z between x_0 and x.

For example, suppose we approximate $\sin 46^\circ$ by $\sin 45^\circ$. As the derivative of $\sin x$ is $\cos x$ (when x is in radians) and $x - x_0 = 1 \times \frac{\pi}{180}$, the error must be $\frac{\pi}{180}\cos z$ for some z between 45° and 46°. Even though we don't know z exactly, we do know that $|\cos z|$ cannot be larger than 1, so the error in approximating $\sin 46^\circ$ by $\sin 45^\circ$ cannot be larger than $\frac{\pi}{180}$.

There are formulae similar to (7), that can be used to bound the error in our other approximations. One is

$$\boxed{f(x) = f(x_0) + f'(x_0)(x - x_0) + \tfrac{1}{2}f''(z)(x - x_0)^2 \qquad \text{for some } z \text{ between } x_0 \text{ and } x}$$

It implies that the error in the approximation $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$ is exactly $\frac{1}{2}f''(z)(x - x_0)^2$ for some z between x_0 and x. Applying this with $f(x) = \sin x$, $x = 46\frac{\pi}{180}$ and $x_0 = 45\frac{\pi}{180}$ shows that the error in approximating $\sin 46^\circ$ by $\sin 45^\circ + \frac{\pi}{180}\cos 45^\circ$ is $\frac{1}{2}\big(\frac{\pi}{180}\big)^2(-\sin z)$ for some z between 45° and 46°. As $|\sin z| \le 1$, this error cannot be larger than $\frac{1}{2}\big(\frac{\pi}{180}\big)^2$.

Example 2 Revisited

In the second example (measuring the height of the pole), we used the linear approximation

$$f(\theta_0 + \Delta\theta) \approx f(\theta_0) + f'(\theta_0)\Delta\theta \tag{8}$$

with $f(\theta) = 10\tan\theta$ and $\theta_0 = 30\frac{\pi}{180}$ to get

$$\Delta h = f(\theta_0 + \Delta\theta) - f(\theta_0) \approx f'(\theta_0)\Delta\theta \;\Rightarrow\; \Delta\theta \approx \frac{\Delta h}{f'(\theta_0)}$$

While this procedure is fairly reliable, it did involve an approximation. So that you could not 100% guarantee to your client's lawyer that an accuracy of 10 cm was achieved. If we use the exact formula (7), with the replacements $x \to \theta_0 + \Delta\theta$, $x_0 \to \theta_0$, $z \to \phi$,

$$f(\theta_0 + \Delta\theta) = f(\theta_0) + f'(\phi)\Delta\theta \text{ for some } \phi \text{ between } \theta_0 \text{ and } \theta_0 + \Delta\theta$$

in place of the approximate formula (2), this legality is taken care of.

$$\Delta h = f(\theta_0 + \Delta\theta) - f(\theta_0) = f'(\phi)\Delta\theta \;\Rightarrow\; \Delta\theta = \frac{\Delta h}{f'(\phi)} \text{ for some } \phi \text{ between } \theta_0 \text{ and } \theta_0 + \Delta\theta$$

Of course we do not know exactly what ϕ is. But suppose that we know that the angle was somewhere between 25° and 35°. In other words suppose that, even though we don't know precisely what our measurement error was, it was certainly no more than 5°. Then $f'(\phi) = 10\sec^2(\phi)$ must be smaller than $10\sec^2 35^\circ < 14.91$, which means that $\frac{\Delta h}{f'(\phi)}$ must be at least $\frac{.1}{14.91}$ radians or $\frac{.1}{14.91}\frac{180}{\pi} = .38^\circ$. A measurement error of 0.38° is certainly acceptable.