More Examples Using Approximations

We have derived three different formulae that are used to approximate a given function $f(x)$ for x near a given point x_0:

$$f(x) \approx f(x_0) \tag{1}$$
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{2}$$
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \tfrac{1}{2} f''(x_0)(x - x_0)^2 \tag{3}$$

Another Notation

Suppose that we have two variables x and y that are related by $y = f(x)$, for some function x. For example, x might be the number of cars manufactured per week in some factory and y the cost of manufacturing those x cars. Let x_0 be some fixed value of x and let $y_0 = f(x_0)$ be the corresponding value of y. Now suppose that x changes by an amount Δx, from x_0 to $x_0 + \Delta x$. As x undergoes this change, y changes from $y_0 = f(x_0)$ to $f(x_0 + \Delta x)$. The change in y that results from the change Δx in x is

$$\Delta y = f(x_0 + \Delta x) - f(x_0)$$

Substituting $x = x_0 + \Delta x$ into the linear approximation (2) yields the approximation

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)(x_0 + \Delta x - x_0) = f(x_0) + f'(x_0)\Delta x$$

for $f(x_0 + \Delta x)$ and consequently the approximation

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \approx f(x_0) + f'(x_0)\Delta x - f(x_0) \ \Rightarrow\ \boxed{\Delta y \approx f'(x_0)\Delta x} \tag{4}$$

for Δy. In the automobile manufacturing example, when the production level is x_0 cars per week, increasing the production level by Δx will cost approximately $f'(x_0)\Delta x$. The additional cost per additional car, $f'(x_0)$, is called the "marginal cost" of a car.

If we use the quadratic approximation (3) in place of the linear approximation (2)

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x + \tfrac{1}{2} f''(x_0)\Delta x^2$$

we arrive at the quadratic approximation

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \approx f(x_0) + f'(x_0)\Delta x + \tfrac{1}{2} f''(x_0)\Delta x^2 - f(x_0) \ \Rightarrow\ \boxed{\Delta y \approx f'(x_0)\Delta x + \tfrac{1}{2} f''(x_0)\Delta x^2} \tag{5}$$

for Δy.

Example 1 (Stewart p181 #30)

Estimate the amount of point needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

Solution. The volume of a hemisphere of radius r is $V(r) = \frac{1}{2}\frac{4}{3}\pi r^3$. The radius of the hemisphere before the paint is applied was $r_0 = 25$m. The corresponding volume was $V_0 = V(25)$. When the paint was applied, the radius increased by $\Delta r = .0005$m to $25 + .0005$m. The volume of paint used was the change, $\Delta V = V(25 + .0005) - V(25)$, in the volume of the hemisphere. By (4)

$$\Delta V \approx V'(r_0)\Delta r = 2\pi r_0^2 \Delta r = 2\pi(25)^2 \times 0.0005 = 1.9634954 \approx \boxed{1.96}$$

We have just computed an approximation to the volume of paint used. In this problem, we can compute the exact volume

$$V(25 + .0005) - V(25) = \tfrac{2}{3}\pi(25 + .0005)^3 - \tfrac{2}{3}\pi(25)^3$$

Applying $(a + b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$ with $a = 25$ and $b = 0.0005$, gives

$$\begin{aligned} V(25 + .0005) - V(25) &= \tfrac{2}{3}\pi[(25)^3 + 3(25)^2(0.0005) + 3(25)(0.0005)^2 + (0.0005)^3 - (25)^3] \\ &= \tfrac{2}{3}\pi[3(25)^2(0.0005) + 3(25)(0.0005)^2 + (0.0005)^3] \end{aligned}$$

The linear approximation, $\Delta V \approx 2\pi(25)^2 \times 0.0005$, is recovered by retaining only the first of the three terms in the square brackets. Thus the error introduced by the linear approximation is obtained by retaining only the last two terms in the square brackets. This error is

$$\tfrac{2}{3}\pi[3(25)(0.0005)^2 + (0.0005)^3] = 0.0000393$$

Example 2

If an aircraft crosses the Atlantic ocean at a speed of u mph, the flight costs the company

$$C(u) = 100 + \tfrac{u}{3} + \tfrac{240{,}000}{u}$$

dollars per passenger. When there is no wind, the aircraft flies at an airspeed of 550mph. Find the approximate savings, per passenger, when there is a 35 mph tail wind. Estimate the cost when there is a 50 mph head wind.

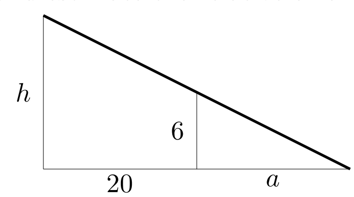
Solution. Let $u_0 = 550$. When the aircraft flies at speed u_0, the cost per passenger is $C(u_0)$. By (4), a change of Δu in the airspeed results in an change of

$$\Delta C \approx C'(u_0)\Delta u = \left[\tfrac{1}{3} - \tfrac{240{,}000}{u_0^2}\right]\Delta u = \left[\tfrac{1}{3} - \tfrac{240{,}000}{550^2}\right]\Delta u \approx -.460\Delta u$$

in the cost per passenger. With the tail wind $\Delta u = 35$ and the resulting $\Delta C \approx -.460 \times 35 = -16.10$, so there is a savings of $\boxed{\$16.10}$. With the head wind $\Delta u = -50$ and the resulting $\Delta C \approx -.4601 \times (-50) = 23.01$, so there is an additional cost of $\boxed{\$23.00}$.

Example 3

To compute the height h of a lamp post, the length a of the shadow of a six–foot pole is measured. The pole is 20 ft from the lamp post. If the length of the shadow was measured to be 15 ft, with an error of at most one inch, find the height of the lamp post and estimate the relative error in the height.



Solution. By similar triangles,

$$\frac{a}{6} = \frac{20+a}{h} \quad\Rightarrow\quad h = (20+a)\frac{6}{a} = \frac{120}{a} + 6$$

The length of the shadow was measured to be $a_0 = 15$ ft. The corresponding height of the lamp post is $h_0 = \frac{120}{a_0} + 6 = \frac{120}{15} + 6 = \boxed{14 \text{ ft}}$. If the error in the measurement of the length of the shadow was Δa, then the exact shadow length was $a = a_0 + \Delta a$ and the exact lamp post height is $h = f(a_0 + \Delta a)$, where $f(a) = \frac{120}{a} + 6$. The error in the computed lamp post height is $\Delta h = h - h_0 = f(a_0 + \Delta a) - f(a_0)$. By (4)

$$\Delta h \approx f'(a_0)\Delta a = -\tfrac{120}{a_0^2}\Delta a = -\tfrac{120}{15^2}\Delta a$$

We are told that $|\Delta a| \le \frac{1}{12}$. Consequently $|\Delta h| \le \frac{120}{15^2}\frac{1}{12} = \frac{10}{225}$ (approximately). The relative error is then

$$\tfrac{|\Delta h|}{h_0} \le \tfrac{10}{225 \times 14} \approx \boxed{0.003} \text{ or } \boxed{0.3\%}$$