The Chain Rule

Question

You are walking. Your position at time x is $g(x)$. Your are walking in an environment in which the air temperature depends on position. The temperature at position y is $f(y)$. What instantaneous rate of change of temperature do you feel at time x?

Because your position at time x is $y = g(x)$, the temperature you feel at time x is $F(x) = f\big(g(x)\big)$. The instantaneous rate of change of temperature that you feel is $F'(x)$. We have a complicated function $F(x)$, constructed from two simple functions, $g(x)$ and $f(y)$. We wish to compute the derivative, $F'(x)$, of the complicated function in terms of the derivatives, $g'(x)$ and $f'(y)$, of the two simple functions. This is exactly what the chain rule does.

The Chain Rule

$$\boxed{\text{If } F(x) = f\big(g(x)\big), \text{ then } F'(x) = f'\big(g(x)\big)g'(x).}$$

Special Cases

a) If $f(y) = y^n$, then $f'(y) = ny^{n-1}$, $f\big(g(x)\big) = g(x)^n$ and $f'\big(g(x)\big)g'(x) = ng(x)^{n-1}g'(x)$. So

$$\boxed{\tfrac{d}{dx} g(x)^n = n\, g(x)^{n-1}\, g'(x)}$$

b) If $f(y) = \sin y$, then $f'(y) = \cos y$, $f\big(g(x)\big) = \sin\big(g(x)\big)$ and $f'\big(g(x)\big)g'(x) = \cos\big(g(x)\big)g'(x)$. So

$$\boxed{\tfrac{d}{dx} \sin\big(g(x)\big) = \cos\big(g(x)\big)\, g'(x)}$$

Similarly

$$\boxed{\tfrac{d}{dx} \cos\big(g(x)\big) = -\sin\big(g(x)\big)\, g'(x)}$$

Units

In the question posed above, x has units of seconds, $g(x)$ has units of meters, y has units of meters and $f(y)$ has units of degrees. Consequently, $F(x) = f\big(g(x)\big)$ has units of degrees, $F'(x)$ has units $\frac{\text{degrees}}{\text{second}}$, $f'(y)$ has units $\frac{\text{degrees}}{\text{meter}}$ and $g'(x)$ has units $\frac{\text{meters}}{\text{second}}$. Thus

$f'\big(g(x)\big)g'(x)$ has units $\frac{\text{degrees}}{\text{meter}} \times \frac{\text{meters}}{\text{second}} = \frac{\text{degrees}}{\text{second}}$ which is the same as the units of $F'(x)$. This of course does not prove that $F'(x)$ and $f'\big(g(x)\big)g'(x)$ are the same. But it does provide a consistency check.

Derivation of the Chain Rule

Our goal is to evaluate

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{h}$$

in terms of

$$f'(y) = \lim_{H\to 0} \frac{f(y+H) - f(y)}{H} \qquad \text{and} \qquad g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

The limit we wish to evaluate looks almost like the limit defining $f'(y)$ if we choose $y = g(x)$:

$$f'\big(g(x)\big) = \lim_{H\to 0} \frac{f\big(g(x)+H\big) - f\big(g(x)\big)}{H}$$

We know the answer to the limit $\lim\limits_{H\to 0} \frac{f(g(x)+H)-f(g(x))}{H}$ and we wish to compute the limit $\lim\limits_{h\to 0} \frac{f(g(x+h))-f(g(x))}{h}$. We can make the numerators of the two limits identical just by defining

$$H = g(x+h) - g(x)$$

Substituting this in, and observing that H tends to zero as h tends to zero,

$$\begin{aligned} f'\big(g(x)\big) &= \lim_{H\to 0} \frac{f\big(g(x)+H\big) - f\big(g(x)\big)}{H} = \lim_{h\to 0} \frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{g(x+h) - g(x)} \\ &= \lim_{h\to 0} \frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{h} \frac{1}{\frac{g(x+h)-g(x)}{h}} \\ &= \lim_{h\to 0} \frac{1}{\frac{g(x+h)-g(x)}{h}} \lim_{h\to 0} \frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{h} \\ &= \frac{1}{g'(x)} \lim_{h\to 0} \frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{h} \end{aligned}$$

Cross multiplying by $g'(x)$ gives

$$\lim_{h\to 0} \frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{h} = f'\big(g(x)\big)\, g'(x)$$

which is the chain rule.