# Differentiation Rules

## Statement

Suppose that $f'(x)$ and $g'(x)$ both exist.

a) If $S(x) = af(x) + bg(x)$, with $a$ and $b$ constants, then $S'(x) = af'(x) + bg'(x)$. That is,

$$\tfrac{d}{dx}[af(x) + bg(x)] = af'(x) + bg'(x)$$

Setting $a = b = 1$ and $b = 0$ gives the two special cases

$$\tfrac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \qquad \tfrac{d}{dx}[af(x)] = af'(x)$$

b) (Product Rule) If $P(x) = f(x)g(x)$, then $P'(x) = f'(x)g(x) + f(x)g'(x)$. That is,

$$\tfrac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

c) (Quotient Rule) If $Q(x) = \frac{f(x)}{g(x)}$, then $Q'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$, provided $g(x) \neq 0$. That is,

$$\frac{d}{dx}\Big[\frac{f(x)}{g(x)}\Big] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Setting $f(x)$ to the constant function $f(x) = 1$ gives the special case

$$\frac{d}{dx}\Big[\frac{1}{g(x)}\Big] = -\frac{g'(x)}{g(x)^2}$$

## Derivation

Define

$$F(x,h) = \frac{f(x+h) - f(x)}{h} \qquad G(x,h) = \frac{g(x+h) - g(x)}{h}$$

Observe that, cross multiplying by $h$,

$$f(x+h) = f(x) + hF(x,h) \qquad g(x+h) = g(x) + hG(x,h)$$

By the definition of the derivative

$$\lim_{h\to 0} F(x,h) = f'(x) \qquad \lim_{h\to 0} G(x,h) = g'(x)$$

a)

$$\begin{aligned} S(x+h)-S(x) &= af(x+h)+bg(x+h)-af(x)-bg(x) \\ &= a[f(x)+hF(x,h)]+b[g(x)+hG(x,h)]-af(x)-bg(x) \\ &= ahF(x,h)+bhG(x,h) \end{aligned}$$

By definition

$$S'(x)=\lim_{h\to 0}\tfrac{S(x+h)-S(x)}{h}=\lim_{h\to 0}\tfrac{ahF(x,h)+bhG(x,h)}{h}=\lim_{h\to 0}[aF(x,h)+bG(x,h)]=af'(x)+bg'(x)$$

b)

$$\begin{aligned} P(x+h)-P(x) &= f(x+h)g(x+h)-f(x)g(x) \\ &= [f(x)+hF(x,h)][g(x)+hG(x,h)]-f(x)g(x) \\ &= hF(x,h)g(x)+f(x)hG(x,h)+h^2F(x,h)G(x,h) \end{aligned}$$

By definition

$$\begin{aligned} P'(x) &= \lim_{h\to 0}\tfrac{P(x+h)-P(x)}{h}=\lim_{h\to 0}\tfrac{hF(x,h)g(x)+hf(x)G(x,h)+h^2F(x,h)G(x,h)}{h} \\ &= \lim_{h\to 0}[F(x,h)g(x)+f(x)G(x,h)+hF(x,h)G(x,h)] \\ &= f'(x)g(x)+f(x)g'(x)+0f'(x)g'(x) \\ &= f'(x)g(x)+f(x)g'(x) \end{aligned}$$

c) By definition,

$$Q'(x)=\lim_{h\to 0}\tfrac{Q(x+h)-Q(x)}{h}=\lim_{h\to 0}\tfrac{1}{h}\big[\tfrac{f(x+h)}{g(x+h)}-\tfrac{f(x)}{g(x)}\big]=\lim_{h\to 0}\tfrac{1}{h}\tfrac{f(x+h)g(x)-f(x)g(x+h)}{g(x)g(x+h)}$$

The numerator

$$\begin{aligned} f(x+h)g(x)-f(x)g(x+h) &= \big[f(x)+hF(x,h)\big]g(x)-f(x)\big[g(x)+hG(x,h)\big] \\ &= hF(x,h)g(x)-f(x)hG(x,h) \end{aligned}$$

Dividing by $hg(x)g(x+h)$

$$\tfrac{f(x+h)g(x)-f(x)g(x+h)}{hg(x)g(x+h)}=\tfrac{hF(x,h)g(x)-hf(x)G(x,h)}{hg(x)g(x+h)}=\tfrac{F(x,h)}{g(x+h)}-\tfrac{f(x)G(x,h)}{g(x)g(x+h)}$$

Hence

$$\begin{aligned} Q'(x) &= \lim_{h\to 0}\tfrac{1}{h}\tfrac{f(x+h)g(x)-f(x)g(x+h)}{g(x)g(x+h)}=\lim_{h\to 0}\big[\tfrac{F(x,h)}{g(x+h)}-\tfrac{f(x)G(x,h)}{g(x)g(x+h)}\big]=\tfrac{f'(x)}{g(x)}-\tfrac{f(x)g'(x)}{g(x)^2} \\ &= \tfrac{f'(x)g(x)-f(x)g'(x)}{g(x)^2} \end{aligned}$$

as desired.

 