Derivatives of Exponentials

Fix any $a > 0$. The definition of the derivative of a^x is

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h} = \lim_{h\to 0} a^x \frac{a^h-1}{h} = a^x \lim_{h\to 0}\frac{a^h-1}{h} = C(a)\ a^x$$

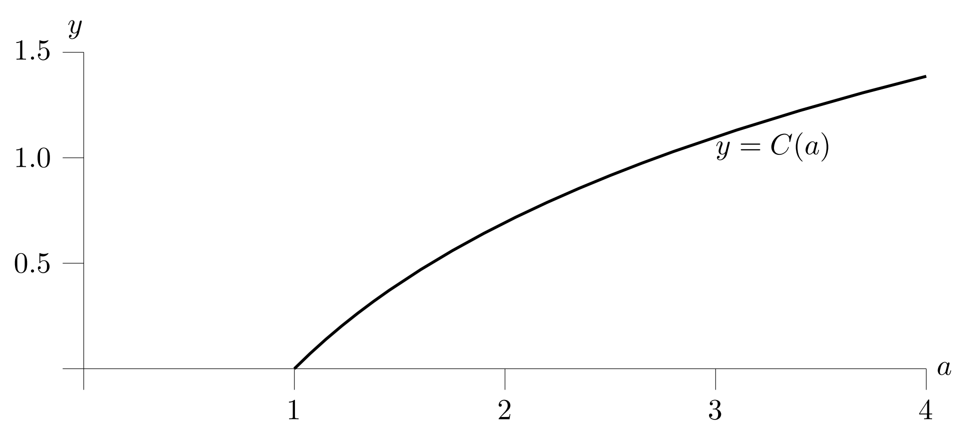
where we are using $C(a)$ to denote the coefficient $\lim\limits_{h\to 0}\frac{a^h-1}{h}$ that appears in the derivative. This coefficient does not depend on x. So, at this stage, we know, for example, that $\frac{d}{dx}2^x$ is 2^x times some fixed number $C(2)$. We will eventually get a formula for $C(a)$. For now, we just try to get an idea of what $C(a)$ looks like by computing $\frac{a^h-1}{h}$ for various values of a and various small values of h. Here is a table of such values. The second row has $a = 2$. So it contains a number of values of $\frac{2^h-1}{h}$ for various values of h. For example, the table entry in the row labeled 2 and column labeled 0.001 is $\frac{2^{0.001}-1}{0.001} = 0.6933874$.

$\frac{a^h-1}{h}$

a \ h	0.1	0.01	0.001	0.0001	0.00001
1	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
2	0.7177346	0.6955550	0.6933874	0.6931712	0.6931494
3	1.1612317	1.1046692	1.0992160	1.0986726	1.0986181
4	1.4869836	1.3959480	1.3872557	1.3863905	1.3863038
10	2.5892541	2.3292992	2.3052381	2.3028502	2.3026115

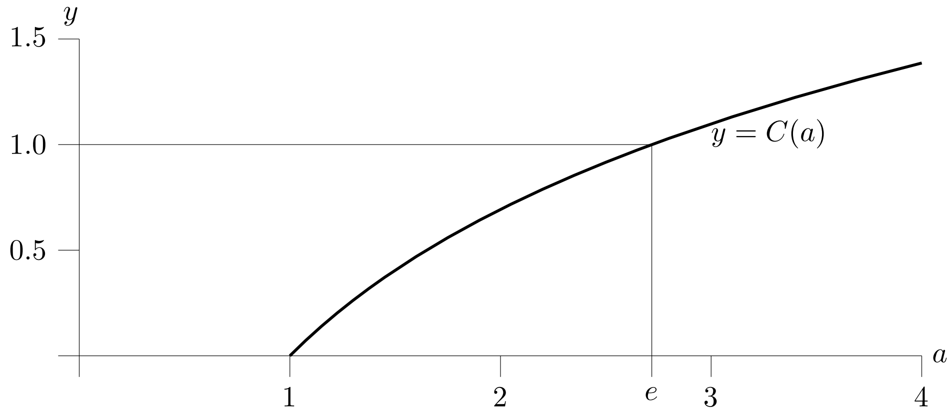
Observe that, if you fix $a = 2$ (i.e. look at the second row) and make h smaller and smaller (i.e. move to the right along row 2), the first four decimal places of the table entries appear to stabilize at 0.6931. So it looks like $C(2) = 0.6931$, to four decimal places, and consequently $\frac{d}{dx}2^x = 0.6931 \times 2^x$.

Similarly, it looks like $C(1) = 0$, $C(3) = 1.0986$, $C(4) = 1.3863$, $C(10) = 2.3026$. One can use a computer to estimate $C(a)$, like this, for many different values of a and thereby plot the graph of $C(a)$ against a. Here it is



Observe that

- $C(1) = 0$. We did not need the graph to see this: $1^h = 1$ for all h. Consequently, $C(1) = \lim\limits_{h\to 0}\frac{1^h-1}{h} = \lim\limits_{h\to 0}\frac{1-1}{h} = \lim\limits_{h\to 0}\frac{0}{h} = 0$.
- $C(a)$ increases as a increases.
- There is exactly one value of a for which $C(a) = 1$. See the following figure.



The value of a for which $C(a) = 1$ is given the name e. That is, e is defined by the condition $C(e) = 1$, or equivalently, by the condition that $\boxed{\frac{d}{dx}e^x = e^x}$.

From the graph, it looks like e is roughly $2\frac{3}{4}$. We can determine e to a much greater degree of accuracy using Newton's method. Recall that Newton's method is an algorithm for finding approximate solutions to equations of the form $f(a) = 0$. The algorithm is

step 1: Make a first guess a_1.

step 2: Construct a second guess by applying the formula $a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$.

step 3: Construct a third guess by applying the formula $a_3 = a_2 - \frac{f(a_2)}{f'(a_2)}$.

and so on. In general, the $n+1^{\text{st}}$ guess is constructed from the n^{th} guess by applying the formula $a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$. Usually, as n increases, a_n very quickly approaches a solution of $f(a) = 0$.

In the present case, $f(x) = C(x) - 1$ and

$$f'(x) = C'(x) = \lim_{h\to 0} \frac{d}{dx}\frac{x^h - 1}{h} = \lim_{h\to 0} \frac{hx^{h-1}}{h} = \lim_{h\to 0} x^{h-1} = \frac{1}{x}$$

so

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)} = a_n - \frac{[C(a_n)-1]}{1/a_n} = a_n - a_n[C(a_n) - 1] = a_n[2 - C(a_n)]$$

Of course, because we cannot compute $C(a)$ exactly, we cannot apply $a_{n+1} = a_n[2 - C(a_n)]$ exactly as it stands. But we can approximate $C(a_n) = \lim_{h\to 0} \frac{a^h - 1}{h}$ by taking a very small value of h, like $h = 0.000001$. Starting with $a_1 = 3$,

$$\begin{aligned}
a_2 &= a_1[2 - C(a_1)] \approx 3\big[2 - \tfrac{3^{0.000001}-1}{0.000001}\big] &= 2.70416 \\
a_3 &= a_2[2 - C(a_2)] \approx a_2\big[2 - \tfrac{a_2^{0.000001}-1}{0.000001}\big] &= 2.71824 \\
a_4 &= a_3[2 - C(a_3)] \approx a_3\big[2 - \tfrac{a_3^{0.000001}-1}{0.000001}\big] &= 2.71828 \\
a_5 &= a_4[2 - C(a_4)] \approx a_4\big[2 - \tfrac{a_4^{0.000001}-1}{0.000001}\big] &= 2.71828
\end{aligned}$$

It looks like the solution of $C(a) = 1$, which we have named e, is about 2.71828. To check this, here is another table of values of $\frac{a^h - 1}{h}$, this time with $a = 2.718275$ and $a = 2.718285$.

$\frac{a^h-1}{h}$

a \ h	0.000001	0.0000001	0.00000001	0.000000001
2.718275	0.9999980	0.9999975	0.9999975	0.9999974
2.718285	1.0000017	1.0000012	1.0000012	1.0000012

The table suggests that $C(2.718275)$ is a little smaller than 1 and $C(2.718285)$ is a little larger than 1, so that e is between 2.718275 and 2.718285.

