Powers and Roots

The symbol x^3 means $x \cdot x \cdot x$. More generally, if n is any strictly positive integer, then x^n means the product $\overbrace{x \cdot x \cdots x}^{n \text{ factors}}$.

If x is a positive number, then $\sqrt{x} = x^{\frac{1}{2}}$ is used to denote the positive number that obeys $\sqrt{x}\sqrt{x} = x$. For example $2 \times 2 = 4$, so $\sqrt{4} = 2$. The equation $x^2 = 137$ has two solutions. The positive one is denoted $\sqrt{137}$ and the negative one $-\sqrt{137}$. So the general solution to $x^2 = 137$ is $x = \pm\sqrt{137}$. It is possible to define the square root of a negative number. But this involves enlarging the real number system to the complex number system and will not be covered in this course.

If x is a positive number and n is a strictly positive integer, then $\sqrt[n]{x} = x^{\frac{1}{n}}$ is used to denote the positive number that obeys $(\sqrt[n]{x})^n = x$. For example $2 \cdot 2 \cdot 2 \cdot 2 = 16$ so $\sqrt[4]{16} = 2$.

Call

$$p(x) = x^n \qquad\qquad r(x) = x^{\frac{1}{n}}$$

So p is the symbol for a machine (let's call it a powifier) that outputs x^n in response to the input x and r is the symbol for a machine (let's call it a rootifier) that outputs $x^{\frac{1}{n}}$ in response to the input x. If you put x into the input hopper of the rootifier you get the output $r(x) = \sqrt[n]{x}$. If you take this output of the rootifier and put it into the input hopper of the powifier, the output of the powifier will be

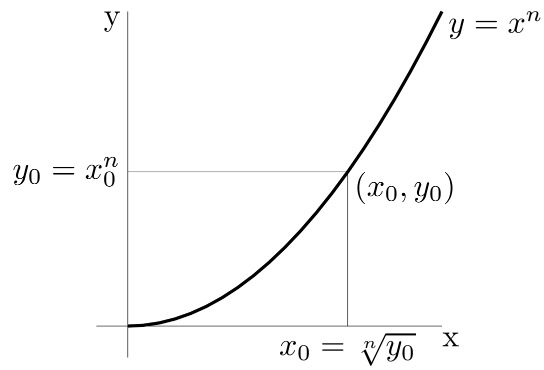
$$p(r(x)) = \big(r(x)\big)^n = \big(\sqrt[n]{x}\big)^n = x$$

The last equality was a consequence of the definition of $\sqrt[n]{x}$. Similarly, if you feed x into the powifer first and then feed the resulting output into the rootifier, the output of the rootifier is

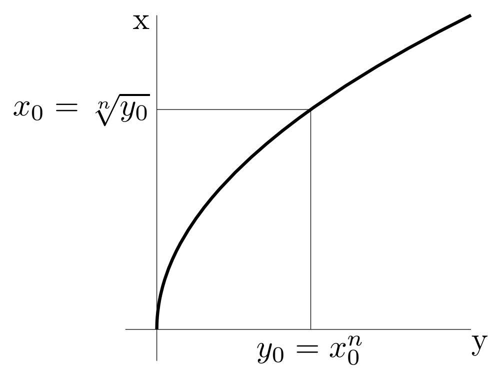
$$r(p(x)) = \sqrt[n]{p(x)} = \sqrt[n]{x^n} = x$$

The equations $r(p(x)) = p(r(x)) = x$ say that the rootifier undoes whatever the powifier does and vice versa. Consequently, p and r are called inverse functions.

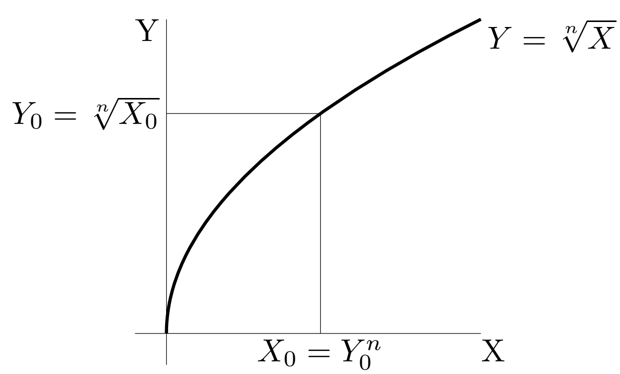
Because p and r are inverse functions, it is very easy to convert the graph of $p(x)$ into the graph of $r(x)$. Here is how. Graph $y = x^n$. Observe that if (x_0, y_0) is any point on the graph, then x_0 and y_0 are related by $y_0 = x_0^n$. Consequently, by the definition of the n^{th} root, $x_0 = \sqrt[n]{y_0}$.



Now redraw the same graph, but this time flip it over so that the y–axis runs horizontally and the x–axis runs vertically. This is a little unorthodox, but perfectly legal.



Then replace every x with a Y and every Y with an X.



Any point on the curve has its X and Y coordinates related by $Y = \sqrt[n]{X}$. So the curve is the graph of $\sqrt[n]{X}$ against X.

