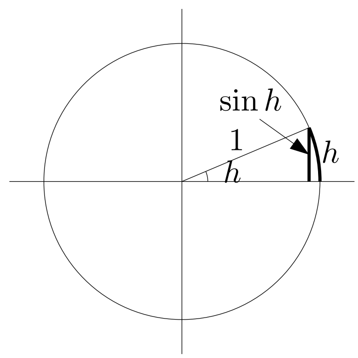
The Derivative of $\sin x$ at $x=0$

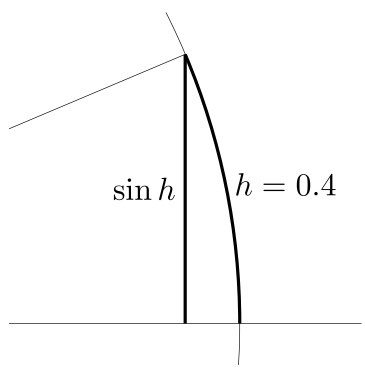
By definition, the derivative of $\sin x$ evaluated at $x = 0$ is

$$\lim_{h\to 0} \frac{\sin h - \sin 0}{h} = \lim_{h\to 0} \frac{\sin h}{h}$$

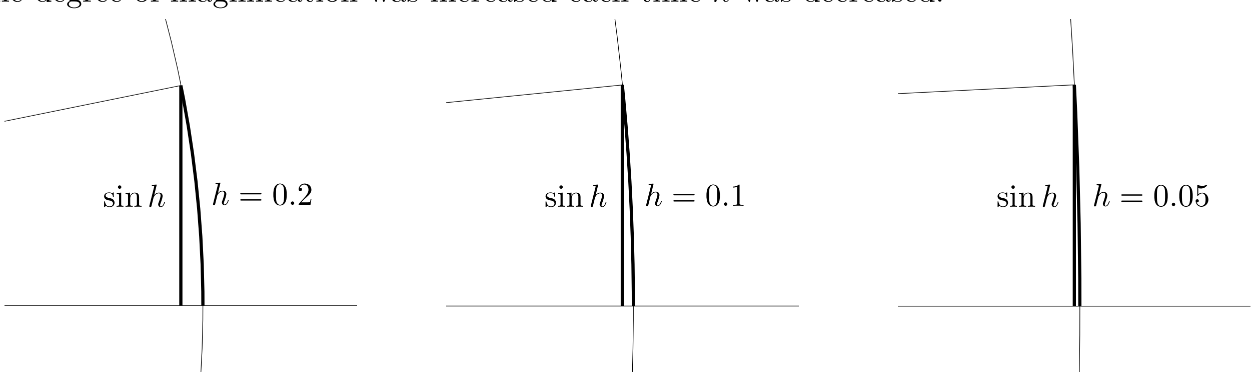
The figure below contains a circle of radius 1. Recall that an arc of length h on such a circle subtends an angle of h **radians** at the center of the circle. So the darkened arc in the figure has length h and the darkened vertical line in the figure has length $\sin h$. We must determine what happens to the ratio of the lengths of the darkened vertical line and darkened arc as h tends to zero.



Here is a magnified version of the part of the above figure that contains the darkened arc and vertical line.



This particular figure has been drawn with $h = .4$ radians. Here are three more such blow ups. In each successive figure, I have used a smaller value of h. To make the figures clearer, the degree of magnification was increased each time h was decreased.



As we make h smaller and smaller and look at the figure with ever increasing magnification, the arc of length h and vertical line of length $\sin h$ look more and more alike. We would guess from this that

$$\lim_{h\to 0}\frac{\sin h}{h}=1$$

The tables of values

h	$\sin h$	$\frac{\sin h}{h}$
0.4	.3894	.9735
0.2	.1987	.9934
0.1	.09983	.9983
0.05	.049979	.99958
0.01	.00999983	.999983
0.001	.0099999983	.9999983

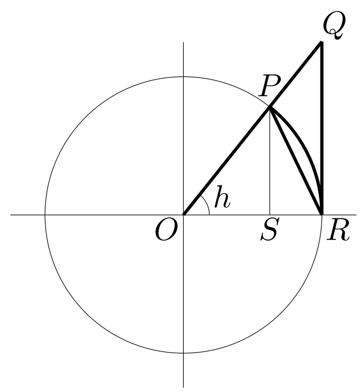
h	$\sin h$	$\frac{\sin h}{h}$
−0.4	−.3894	.9735
−0.2	−.1987	.9934
−0.1	−.09983	.9983
−0.05	−.049979	.99958
−0.01	−.00999983	.999983
−0.001	−.0099999983	.9999983

suggest the same guess.

$$\boxed{\frac{d}{dx}\sin x\Big|_{x=0}=\lim_{h\to 0}\frac{\sin h}{h}=1}$$

Here is an argument that shows that the guess really is correct.

Proof that $\lim_{h\to 0}\frac{\sin h}{h}=1$:



The circle in the figure above has radius 1. Hence

$$|OP|=|OR|=1\quad |PS|=\sin h\quad |QR|=\tan h$$

The triangle OPR had base 1 and height $\sin h$ and hence area $\frac{1}{2}\times 1\times\sin h$. The triangle OQR had base 1 and height $\tan h$ and hence area $\frac{1}{2}\times 1\times\tan h$. The piece of pie OPR is the fraction $\frac{h}{2\pi}$ of the whole circle, which has area $\pi 1^2$. So the piece of pie OPR has area $\frac{h}{2\pi}\times\pi 1^2=\frac{h}{2}$. The triangle OPR is contained in and hence has smaller area than the piece of pie OPR, which in turn is contained in and hence has smaller area than the triangle OQR. The inequalities stating this are

$$\tfrac{1}{2}\sin h\le\tfrac{h}{2}\le\tfrac{1}{2}\tan h\quad\Longrightarrow\quad\sin h\le h\le\tfrac{\sin h}{\cos h}\quad\Longrightarrow\quad\cos h\le\tfrac{\sin h}{h}\le 1$$

As h tends to 0, $\cos h$ approaches one. Because $\frac{\sin h}{h}$ is sandwiched between $\cos h$ and 1, it must also approach 1.