

Warnings

These notes highlight number of common, but serious, first year calculus errors.

Warning 1. The formula

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

is valid only under the hypothesis $\lim_{x \rightarrow a} g(x) \neq 0$.

Discussion. In some cases, when $\lim_{x \rightarrow a} g(x) = 0$, the limit $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ does not exist at all. But in other cases it does exist and when it does, it can take any value at all. Here are two simple examples. In both examples $g(x) = x - a$, which has the property that $\lim_{x \rightarrow a} g(x) = 0$. In the first example, $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ does not exist at all (because $\frac{f(x)}{g(x)}$ is huge and positive for $x - a$ small and positive and huge and negative for $x - a$ small and negative). In the second it does exist and takes the value 123.45.

$$f(x) = 1 \quad g(x) = x - a \quad \Rightarrow \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} \text{ does not exist}$$

$$f(x) = 123.45(x - a) \quad g(x) = x - a \quad \Rightarrow \quad \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{123.45(x - a)}{x - a} = \lim_{x \rightarrow a} 123.45 = 123.45$$

Warning 2. L'Hôpital's rule,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

is not valid if the limit $\lim_{x \rightarrow a} g(x)$ exists and is nonzero.

Discussion. Here is an example with $a = 0$. If $f(x) = 3x$ and $g(x) = 4 + 5x$, then

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 0} \frac{3x}{4+5x} = \frac{3 \times 0}{4+5 \times 0} = 0 \\ \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} &= \lim_{x \rightarrow 0} \frac{3}{5} = \frac{3}{5} \end{aligned}$$

Warning 3. $\frac{d}{dx}[f(a)]$ is **not** the same as $f'(a)$.

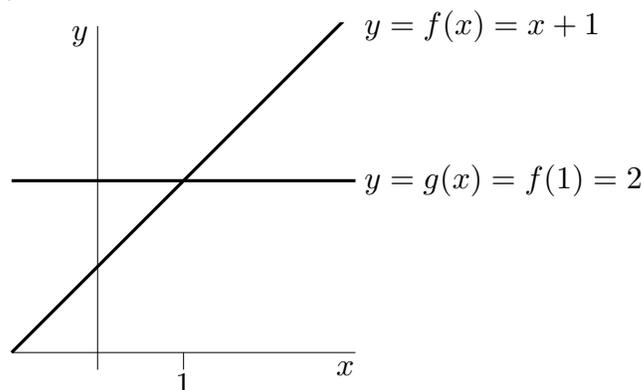
Discussion. The expression $\frac{d}{dx}[f(a)]$ is, by definition, the derivative of the constant function $g(x) = f(a)$. In other words, $\frac{d}{dx}[f(a)]$ means

- first evaluate $f(x)$ at $x = a$, to get a constant, $f(a)$.
- Then differentiate the constant to get zero.

On the other hand, $f'(a)$ means

- first find the derivative $f'(x)$.
- Then evaluate the derivative at $x = a$. Typically, this is not zero.

Here is an example. Let $f(x) = x + 1$ and $a = 1$. Then $f'(x) = 1$ for all x . In particular, $f'(1) = 1$. But if we first evaluate $f(x)$ at $x = 1$, we get $f(1) = 2$. The derivative of the constant function $g(x) = 2$ is $g'(x) = 0$.



The two functions $f(x)$ and $f(1)$ (which I am also calling $g(x)$ here) both take the value 1 when $x = 1$, but they have different slopes and hence have different derivatives: $f'(1) = 1$ but $\frac{d}{dx}[f(1)] = \frac{d}{dx}1 = 0$.

Warning 4. The usual formula for the derivative of $\sin x$, namely $\frac{d}{dx} \sin x = \cos x$, is valid only if x is measured in radians.

Discussion. Any derivative measures the instantaneous rate of change of one quantity per unit change of a second quantity. If you change the units of either quantity, the numerical value of the derivative changes. For example, consider the average rate of change of $\sin x$ as x moves from 0 to a right angle. If x is measured in radians,

$$\text{average rate of change} = \frac{\text{change in } \sin x}{\text{change in } x} = \frac{1 - 0}{\frac{\pi}{2} - 0} = \frac{2}{\pi}$$

because $\sin x$ changes from 0 to 1 as x changes from 0 radians to $\frac{\pi}{2}$ radians. But, if x is measured in degrees,

$$\text{average rate of change} = \frac{\text{change in } \sin x}{\text{change in } x} = \frac{1 - 0}{90 - 0} = \frac{1}{90}$$

The two answers are different even though x moved from 0 to a right angle in both cases.

Here is a sample problem in which one gets the wrong answer by expressing an angle in the wrong units. Suppose that you wish to compute, approximately, the sin of two degrees, but that you can't just use your calculator. (Pretend that you are the person designing the software used by the calculator.) Recall that

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

gives an approximate value for $f(x)$ for all x close to x_0 . Apply this approximation to $f(x) = \sin(x)$ with $x_0 = 0$. Then

$$\begin{aligned} f(x) &= \sin x & f(x_0) &= f(0) = 0 \\ f'(x) &= \cos x & f'(x_0) &= f'(0) = 1 \end{aligned}$$

so that $\sin x = f(x) \approx f(x_0) + f'(x_0)(x - x_0) = 0 + 1(x - 0) = x$. If we apply this with $x = 2$ we get

$$\sin 2 \approx 2$$

which is both wrong and ridiculous, as $\sin x$ is never bigger than one in magnitude. The problem arose because we used $\frac{d}{dx} \sin x = \cos x$ in the computation

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = 0 + 1(x - 0) = x$$

thereby implicitly assuming that x is in radians. We should have first rewritten $2^\circ = 2 \frac{\pi}{180} = 0.0349066$ radians and then applied

$$\sin(2^\circ) = \sin(0.0349066 \text{ rad}) \approx [f(x_0) + f'(x_0)(x - x_0)]_{x=0.0349066} = x \Big|_{x=0.0349066} = 0.0349066$$

For comparison purposes, the exact value of sin of 2 degrees is 0.0348995 to 7 decimal places. Our rather simple minded approximation scheme gave an error of $0.0349066 - 0.0348995 = 0.0000071$, which is a relative error of $100 \frac{0.0000071}{0.0348995} \% = 0.021\%$.

Warning 5. $\frac{d}{dx} a^x$ is not the same as $\frac{d}{dx} x^a$.

Discussion. $\frac{d}{dx} f(x)$ is the rate of change of $f(x)$ per unit change in x . In $\frac{d}{dx} a^x$, the exponent changes as x changes, **with the base, a, remaining fixed**. In $\frac{d}{dx} x^a$, the base changes as x

changes, **with the exponent, a, remaining fixed**. As a concrete example, suppose $a = 4$. As x changes from 4 to 5,

○ $a^x = 4^x$ changes from $4^4 = 256$ to $4^5 = 1024$

⇒ the average rate of change of $a^x = 4^x$ from $x = 4$ to $x = 5$ is $\frac{1024-256}{5-4} = 768$

○ $x^a = x^4$ changes from $4^4 = 256$ to $5^4 = 625$

⇒ the average rate of change of $x^a = x^4$ from $x = 4$ to $x = 5$ is $\frac{625-256}{5-4} = 369$

The correct formulae for the instantaneous rates of change of a^x and x^a are

$$\frac{d}{dx}a^x = \ln a \ a^x \qquad \frac{d}{dx}x^a = ax^{a-1}$$

Warning 6. The slope of the curve $y = f(x)$ at $x = a$ is not $f'(x)$. It is $f'(a)$.

Discussion. Consider the problem “Find the tangent line to the curve $y = x^2$ at the point $(1, 1)$ ”. The tangent line is a straight line. Any straight line has only one slope. In this case the slope of the tangent line is the same as the slope of $y = x^2$ at $x = 1$, which is $\left. \frac{d}{dx}x^2 \right|_{x=1} = 2x \Big|_{x=1} = 2$. So the tangent line is $y - 1 = 2(x - 1)$. If you neglect to evaluate $\left. \frac{d}{dx}x^2 \right|_{x=1} = 2x$ at $x = 1$, you get $y - 1 = (2x)(x - 1)$ instead. This is not a straight line at all, because as x changes the quantity you put in for the slope ($2x$) also changes. As you move along a straight line, the slope is not allowed to change.