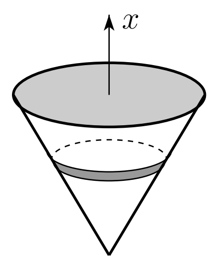
Volumes

One can express volumes of regions in three dimensions as integrals using the same strategy as we used to express areas of regions in two dimensions as integrals — approximate the region by a union of small, simple pieces whose volume we can compute and then then take the limit as the "piece size" tends to zero. Often this results in "multivariable integrals" that are beyond our present scope. But there are some special cases in which this leads to integrals that we can handle. Here are some examples.

Example 1 (Cone)

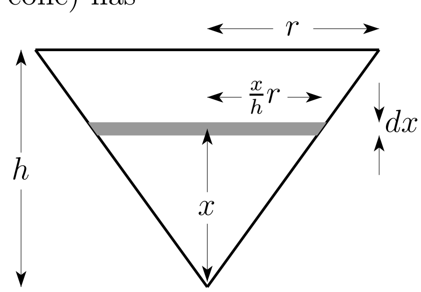
Find the volume of the circular cone of height h and radius r.

Solution. Here is a sketch of the cone. We have called the vertical axis x, just so that we end up with a "dx" integral. To compute its volume, we slice it up into thin horizontal



"pancakes". A typical pancake also appears in the sketch above. The pancake at height x (which is the fraction $\frac{x}{h}$ of the total height of the cone) has

- thickness[1] dx and
- radius $\frac{x}{h}r$ and hence
- cross–sectional area $\pi\left(\frac{x}{h}r\right)^2$ and hence
- volume $\pi\left(\frac{x}{h}r\right)^2\,dx$



As x runs from 0 to h, the total volume is

$$\int_0^h \pi\left(\frac{x}{h}r\right)^2\,dx = \frac{\pi r^2}{h^2}\int_0^h x^2\,dx = \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{1}{3}\pi r^2 h$$

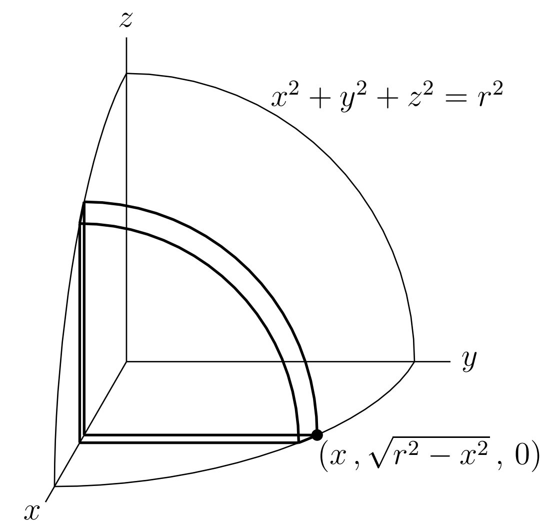
Example 1

[1] Of course what we really do is pick a natural number n, slice the cone into n pancakes each of thickness $\Delta x = \frac{h}{n}$ and then take the limit $n \to \infty$. In the integral Δx is replaced by dx. Knowing that this is what is going to happen, we are just skipping a few steps.

Example 2 (Sphere)

Find the volume of the sphere of radius r.

Solution. We'll find the volume of the part of the sphere in the first octant[2], sketched below. Then we'll multiply by 8. To compute the volume, we slice it up into thin vertical "pancakes".



Each pancake is one quarter of a circular disk. The pancake a distance x from the yz–plane is shown in the sketch above. The radius of that pancake is the distance from the dot shown in the figure to the x–axis, i.e. the y–coordinate of the dot. To get the coordinates of the dot, observe that it lies the xy–plane and so has z–coordinate zero and that it also lies on the sphere, so that its coordinates obey $x^2 + y^2 + z^2 = r^2$. So the pancake at distance x from the yz–plane has

- thickness[3] dx and
- radius $\sqrt{r^2 - x^2}$
- cross–sectional area $\frac{1}{4}\pi\big(\sqrt{r^2 - x^2}\,\big)^2$ and hence
- volume $\frac{\pi}{4}\big(r^2 - x^2\big)\ dx$

As x runs from 0 to r, the total volume of the part of the sphere in the first octant is

$$\int_0^r \frac{\pi}{4}\big(r^2 - x^2\big)\ dx = \frac{\pi}{4}\left[r^2 x - \frac{x^3}{3}\right]_0^r = \frac{1}{6}\pi r^3$$

and the total volume of the whole sphere is eight times that, which is $\frac{4}{3}\pi r^3$, as expected.

Example 2

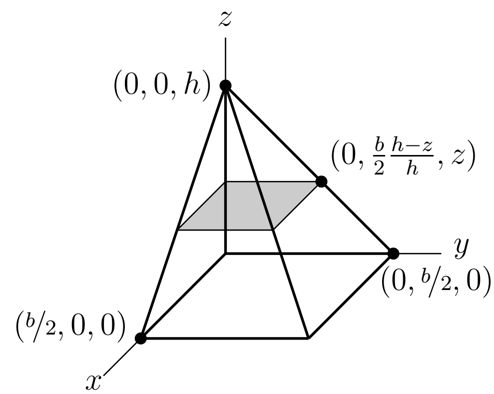
[2] The first octant is the set of all points (x, y, z) with $x \geq 0$, $y \geq 0$ and $z \geq 0$.

[3] Yet again what we really do is pick a natural number n, slice the octant of the sphere into n pancakes each of thickness $\Delta x = \frac{r}{n}$ and then take the limit $n \to \infty$. In the integral Δx is replaced by dx. Knowing that this is what is going to happen, we are again just skipping a few steps.

Example 3 (Pyramid)

Find the volume of the pyramid which has height h and whose base is a square of side b.

Solution. Here is a sketch of the part of the pyramid that is in the first octant. To compute its volume, we slice it up into thin horizontal "square pancakes". A typical pancake also



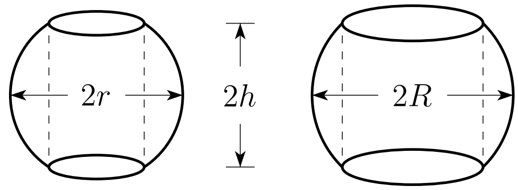
appears in the sketch above. The pancake at height z is the fraction $\frac{h-z}{h}$ of the distance from the peak of the pyramid to its base. So it is a square of side $\frac{h-z}{h}b$. As a check, note that when $z = h$ the pancake has side $\frac{h-h}{h}b = 0$, and when $z = 0$ the pancake has side $\frac{h-0}{h}b = b$. So the pancake has cross-sectional area $\big(\frac{h-z}{h}b\big)^2$ and thickness dz and hence volume $\big(\frac{h-z}{h}b\big)^2\,dz$. The volume of the whole pyramid is

$$\begin{aligned}
\int_0^h \Big(\frac{h-z}{h}b\Big)^2\,dz &= \frac{b^2}{h^2}\int_0^h (h-z)^2\,dz \\
&= \frac{b^2}{h^2}\int_h^0 t^2\,(-dt) \qquad \text{with } t = h-z,\ dt = -dz \\
&= -\frac{b^2}{h^2}\Big[\frac{t^3}{3}\Big]_h^0 = -\frac{b^2}{h^2}\Big[-\frac{h^3}{3}\Big] \\
&= \frac{1}{3}b^2h
\end{aligned}$$

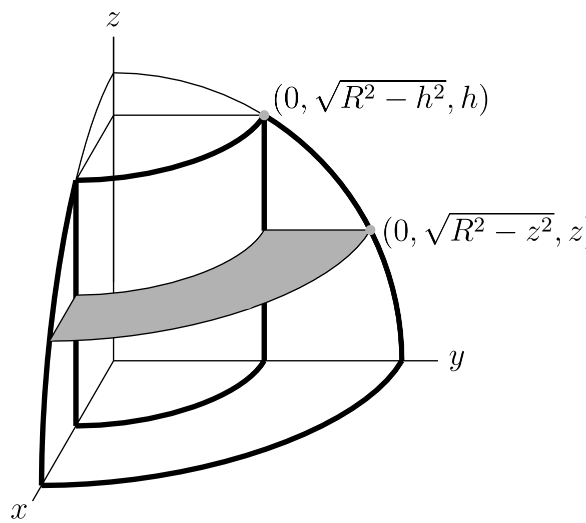
Example 3

Example 4 (Napkin Ring)

Suppose you make two napkin rings by drilling holes with different diameters through two wooden balls — one of radius r and the other of radius R, with $R > r$. The hole diameters are chosen so that both napkin rings have height $2h$. Which ring has more wood in it?



Solution. We'll compute the volume of the napkin ring with radius R. Then, to get the volume of the napkin ring of radius r, we just need to replace R by r. To compute the volume of the napkin ring of radius R, we slice it up into thin horizontal "pancakes". Here is a sketch of the part of the napkin ring in the first octant showing a typical pancake. The coordinates



of the two points marked in the yz–plane of that figure are found by remembering that the equation of the sphere is $x^2+y^2+z^2=R^2$. As the two points are in the yz–plane, $x=0$ for them so that $y=\sqrt{R^2-z^2}$. In particular, at the top of the napkin ring $z=h$ so that $y=\sqrt{R^2-h^2}$. The pancake at height z, shown in the sketch, is a "washer" — a circular disk with a circular hole cut in its center. The outer radius of the washer is $\sqrt{R^2-z^2}$ and the inner radius of the washer is $\sqrt{R^2-h^2}$. So the cross–sectional area of the washer is

$$\pi\big(\sqrt{R^2-z^2}\,\big)^2-\pi\big(\sqrt{R^2-h^2}\,\big)^2=\pi(h^2-z^2)$$

Recall that we pick sliced the napkin ring into thin horizontal pancakes. The pancake at height z has thickness dz and cross–sectional area $\pi(h^2-z^2)$ and hence volume $\pi(h^2-z^2)\,dz$. So the total volume of wood in the napkin ring of radisu R is

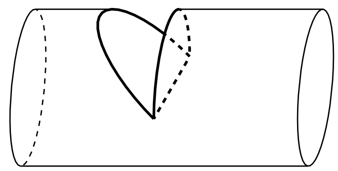
$$\int_{-h}^{h}\pi(h^2-z^2)\,dz=\pi\Big[h^2z-\frac{z^3}{3}\Big]_{-h}^{h}=\pi\Big[\frac{2}{3}h^3-\frac{2}{3}\big(-h\big)^3\Big]=\frac{4\pi}{3}h^3$$

This is independent of R. The napkin ring of radius r contains precisely the same volume of wood as the napkin ring of radius R!

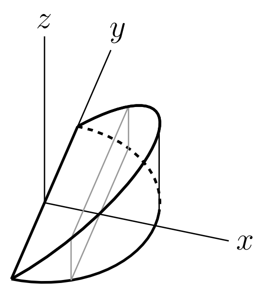
Example 4

Example 5 (Notch)

A 45° notch is cut to the centre of a cylindrical log having radius 20 cm. One plane face of the notch is perpendicular to the axis of the log. See the sketch below. What volume of wood was removed?



Solution 1. Slice the notch into rectangles as in the figure below.



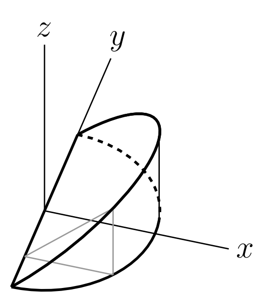
Suppose that the base of the notch is in the xy–plane. Then the circular part of the boundary of the base of the notch has equation $x^2+y^2 = 20^2$. (We're putting the origin of the xy–plane at the centre of the circle.) If our coordinate system is such that x is constant on each slice, then the slice has width $2y = 2\sqrt{20^2 - x^2}$ and height x (since the upper face of the notch is at 45° to the base). So the slice has cross–sectional area $2x\sqrt{20^2 - x^2}$ and the volume is

$$V = \int_0^{20} 2x\sqrt{20^2 - x^2}\,dx$$

Make the change of variables $u = 20^2 - x^2$, $du = -2x\,dx$.

$$V = \int_{20^2}^{0} \sqrt{u}\,(-du) = -\frac{u^{3/2}}{3/2}\bigg|_{20^2}^{0} = \frac{2}{3}20^3 = \frac{16,000}{3}$$

Solution 2. Suppose that the base of the notch is in the xy–plane with the skinny edge along the y–axis. Slice the notch into triangles parallel to the x–axis as in the figure below.



Then the circular part of the boundary of the base of the notch has equation $x^2 + y^2 = 20^2$. Our coordinate system is such that y is constant on each slice, so that the slice has both base and height $x = \sqrt{20^2 - y^2}$ (since the upper face of the notch is at 45° to the base). So the slice has cross–sectional area $\frac{1}{2}\big(\sqrt{20^2 - y^2}\big)^2$ and the volume is

$$V = \tfrac{1}{2}\int_{-20}^{20} (20^2 - y^2)\,dy = \int_0^{20} (20^2 - y^2)\,dy = \Big[20^2 y - \frac{y^3}{3}\Big]_0^{20} = \frac{2}{3}20^3 = \frac{16,000}{3}$$

Example 5

