Improper Integrals

To this point we have only considered integrals $\int_a^b f(x)\,dx$ with

- the limits of integration a and b finite and
- the integrand $f(x)$ bounded (and in fact continuous except possibly for finitely many jump discontinuities)

An integral having either an infinite limit of integration or an unbounded integrand is called improper. Here are two examples

$$\int_0^\infty \frac{dx}{1+x^2} \qquad \int_0^1 \frac{dx}{x}$$

The first has an infinite domain of integration and the integrand of the second tends to ∞ as x approaches the left end of the domain of integration. We'll start with an example that illustrates the traps that you can fall into if you treat such integrals sloppily. Then we'll see how to treat them carefully.

Example 1

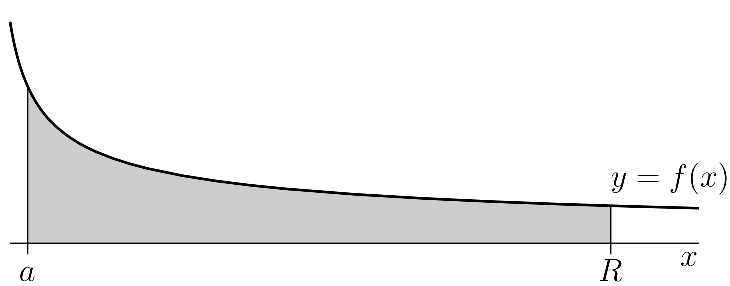
The computation

$$\int_{-1}^1 \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\bigg|_{-1}^1 = \frac{1}{-1} - \frac{-1}{-1} = -2$$

is *wrong*. In fact, the answer is ridiculous. The integrand $\frac{1}{x^2} > 0$, so the integral has to be positive. The flaw in the argument is that the fundamental theorem of calculus, which says that if $F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$, is applicable only when $F'(x)$ exists and equals $f(x)$ for *all* $a \le x \le b$. In this case $F'(x) = \frac{1}{x^2}$ does not exist for $x = 0$. The given integral is improper. We'll see later that the correct answer is $+\infty$.

Example 1

The careful way to treat an integral like $\int_a^\infty \frac{dx}{1+x^2}$ that has a bounded integrand but has a domain of integration that extends to $+\infty$ is to approximate the integral by one with a bounded domain of integration, like $\int_a^R \frac{dx}{1+x^2}$, and then take the limit $R \to \infty$. Here is the



formal definition and some variations. We'll work through some examples in detail shortly.

Definition 2.

(a) If the integral $\int_a^R f(x)\ dx$ exists for all $R > a$, then

$$\int_a^\infty f(x)\ dx = \lim_{R\to\infty} \int_a^R f(x)\ dx$$

when the limit exists (and is finite).

(b) If the integral $\int_r^b f(x)\ dx$ exists for all $r < b$, then

$$\int_{-\infty}^b f(x)\ dx = \lim_{r\to-\infty} \int_r^b f(x)\ dx$$

when the limit exists (and is finite).

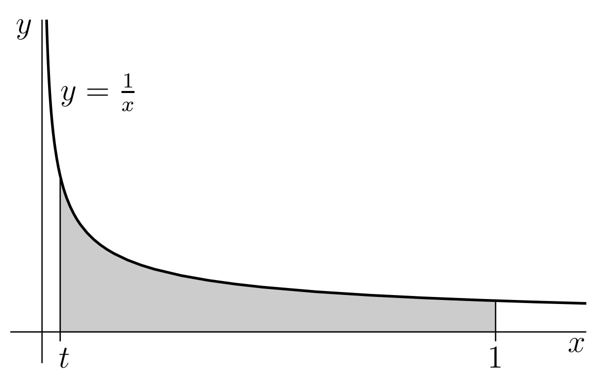
(c) If the integral $\int_r^R f(x)\ dx$ exists for all $r < R$, then

$$\int_{-\infty}^\infty f(x)\ dx = \lim_{r\to-\infty} \int_r^c f(x)\ dx + \lim_{R\to\infty} \int_c^R f(x)\ dx$$

when both limits exist (and are finite). Any c can be used.

When the limit(s) exist, the integral is said to be convergent. Otherwise it is said to be divergent.

The careful way to treat an integral like $\int_0^1 \frac{dx}{x}$ that has a finite domain of integration but whose integrand is unbounded near one limit of integration is to approximate the integral by one with a domain of integration on which the integrand is bounded, like $\int_t^1 \frac{dx}{x}$, with $t > 0$, and then take the limit $t \to 0+$. Here is the formal definition and some variations.



Definition 3.

(a) If the integral $\int_t^b f(x)\,dx$ exists for all $a < t < b$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a+} \int_t^b f(x)\,dx$$

when the limit exists (and is finite).

(b) If the integral $\int_a^T f(x)\,dx$ exists for all $a < T < b$, then

$$\int_a^b f(x)\,dx = \lim_{T\to b-} \int_a^T f(x)\,dx$$

when the limit exists (and is finite).

(c) Let $a < c < b$. If the integrals $\int_a^T f(x)\,dx$ and $\int_t^b f(x)\,dx$ exist for all $a < T < c$ and $c < t < b$, then

$$\int_a^b f(x)\,dx = \lim_{T\to c-} \int_a^T f(x)\,dx + \lim_{t\to c+} \int_t^b f(x)\,dx$$

when both limit exist (and are finite).

When the limit(s) exist, the integral is said to be convergent. Otherwise it is said to be divergent.

If an integral has more than one "source of impropriety", for example an infinite domain of integration and an integrand with an unbounded integrand or multiple infinite discontinuities, then you split it up into a sum of integrals with a single "source of impropriety" in each. For the integral, as a whole, to converge every term in that sum has to converge. For example, the integral $\int_{-\infty}^{\infty} \frac{dx}{(x-2)x^2}$ has a domain of integration that extends to both $+\infty$ and $-\infty$ and, in addition, has an integrand which is singular (i.e. becomes infinite) at $x = 2$ and at $x = 0$. So we would write

$$\int_{-\infty}^{\infty} \frac{dx}{(x-2)x^2} = \int_{-\infty}^{-1} \frac{dx}{(x-2)x^2} + \int_{-1}^{0} \frac{dx}{(x-2)x^2} + \int_{0}^{1} \frac{dx}{(x-2)x^2} + \int_{1}^{2} \frac{dx}{(x-2)x^2} + \int_{2}^{3} \frac{dx}{(x-2)x^2} + \int_{3}^{\infty} \frac{dx}{(x-2)x^2}$$

(Here the "break point" -1 could be replaced by any number strictly less than 0; 1 could be replaced by any number strictly between 0 and 2; and 3 could be replaced by any number strictly bigger than 2.) On the right hand side

- the first integral has domain of integration extending to $-\infty$

- the second integral has an integrand that becomes unbounded as $x \to 0-$,
- the third integral has an integrand that becomes unbounded as $x \to 0+$,
- the fourth integral has an integrand that becomes unbounded as $x \to 2-$,
- the fifth integral has an integrand that becomes unbounded as $x \to 2+$, and
- the last integral has domain of integration extending to $+\infty$.

We are now ready for some (important) examples.

Example 4 ($\int_1^\infty \frac{dx}{x^p}$)

Fix any $p > 0$. The domain of the integral $\int_1^\infty \frac{dx}{x^p}$ extends to $+\infty$ and the integrand $\frac{1}{x^p}$ is continuous and bounded on the whole domain. So the definition of this integral is

$$\int_1^\infty \frac{dx}{x^p} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^p}$$

When $p \neq 1$, one antiderivative of x^{-p} is $\frac{x^{-p+1}}{-p+1}$, and when $p = 1$ and $x > 0$, one antiderivative of x^{-p} is $\ln x$. So

$$\int_1^\infty \frac{dx}{x^p} = \lim_{R\to\infty} \begin{cases} \frac{1}{1-p}\left[R^{1-p} - 1\right] & \text{if } p \neq 1 \\ \ln R & \text{if } p = 1 \end{cases}$$

As $R \to \infty$, $\ln R$ tends to ∞ and R^{1-p} tends to ∞ when $1 - p > 0$ (i.e. the exponent is positive) and tends to 0 when $1 - p < 0$ (i.e. the exponent is negative). So

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \text{divergent} & \text{if } p \leq 1 \\ \frac{1}{p-1} & \text{if } p > 1 \end{cases}$$

Example 4

Example 5 ($\int_0^1 \frac{dx}{x^p}$)

Again fix any $p > 0$. The domain of integration of the integral $\int_0^1 \frac{dx}{x^p}$ is finite, but the integrand $\frac{1}{x^p}$ becomes unbounded as x approaches the left end, 0, of the domain of integration. So the definition of this integral is

$$\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0+} \int_t^1 \frac{dx}{x^p}$$

When $p \neq 1$, one antiderivative of x^{-p} is $\frac{x^{-p+1}}{-p+1}$, and when $p = 1$ and $x > 0$, one antiderivative of x^{-p} is $\ln x$. So

$$\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0+} \begin{cases} \frac{1}{1-p}\left[1 - t^{1-p}\right] & \text{if } p \neq 1 \\ -\ln t & \text{if } p = 1 \end{cases}$$

As $t \to 0+$, $\ln t$ tends to $-\infty$ and t^{1-p} tends to 0 when $1-p > 0$ (i.e. the exponent is positive) and tends to ∞ when $1-p < 0$ (i.e. the exponent is negative). So

$$\int_0^1 \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \text{divergent} & \text{if } p \ge 1 \end{cases}$$

Example 5

Example 6 ($\int_0^\infty \frac{dx}{x^p}$)

Yet again fix $p > 0$. This time the domain of integration of the integral $\int_0^\infty \frac{dx}{x^p}$ extends to $+\infty$, and in addition the integrand $\frac{1}{x^p}$ becomes unbounded as x approaches the left end, 0, of the domain of integration. So we have to split it up.

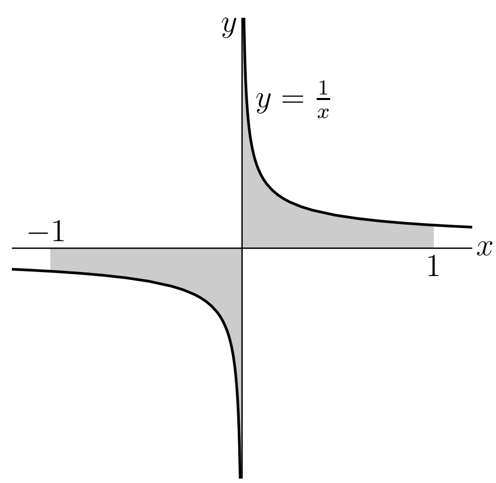
$$\int_0^\infty \frac{dx}{x^p} = \int_0^1 \frac{dx}{x^p} + \int_1^\infty \frac{dx}{x^p}$$

We saw, in Example 5, that the first integral diverged whenever $p \ge 1$, and we also saw, in Example 4, that the second integral diverged whenever $p \le 1$. So the integral $\int_0^\infty \frac{dx}{x^p}$ diverges for all values of p.

Example 6

Example 7 ($\int_{-1}^1 \frac{dx}{x}$)

This is a pretty subtle example. It looks from the sketch below that the signed area to the



left of the y–axis should exactly cancel the area to the right of the y–axis making the value of the integral $\int_{-1}^1 \frac{dx}{x}$ exactly zero. But both of the integrals

$$\int_0^1 \frac{dx}{x} = \lim_{t\to 0+} \int_t^1 \frac{dx}{x} = \lim_{t\to 0+} \Big[\ln x\Big]_t^1 = \lim_{t\to 0+} \ln \frac{1}{t} = +\infty$$

$$\int_{-1}^0 \frac{dx}{x} = \lim_{T\to 0-} \int_{-1}^T \frac{dx}{x} = \lim_{T\to 0-} \Big[\ln |x|\Big]_{-1}^T = \lim_{T\to 0-} \ln |T| = -\infty$$

diverge so $\int_{-1}^{1}\frac{dx}{x}$ diverges.

Don't make the mistake of thinking that $\infty-\infty=0$. It is undefined — and for good reason. For example, we have just seen that the area to the right of the y–axis is

$$\lim_{t\to 0+}\int_t^1\frac{dx}{x}=+\infty$$

and that the area to the left of the y–axis is (substitute $-7t$ for T above)

$$\lim_{t\to 0+}\int_{-1}^{-7t}\frac{dx}{x}=-\infty$$

If $\infty-\infty=0$, the following limit should be 0.

$$\begin{aligned}\lim_{t\to 0+}\left[\int_t^1\frac{dx}{x}+\int_{-1}^{-7t}\frac{dx}{x}\right]&=\lim_{t\to 0+}\Big[\ln\frac{1}{t}+\ln|-7t|\Big]=\lim_{t\to 0+}\Big[\ln\frac{1}{t}+\ln(7t)\Big]=\lim_{t\to 0+}\ln 7\\&=\ln 7\end{aligned}$$

This appears to give $\infty-\infty=\ln 7$. Of course the number 7 was picked at random. You can make $\infty-\infty$ be any number at all, by making a suitable replacement for 7.

Example 7

Example 8 (Example 1 revisited)

The careful computation of the integral of Example 1 is

$$\begin{aligned}\int_{-1}^{1}\frac{1}{x^2}\,dx&=\lim_{T\to 0-}\int_{-1}^{T}\frac{1}{x^2}\,dx+\lim_{t\to 0+}\int_t^1\frac{1}{x^2}\,dx\\&=\lim_{T\to 0-}\Big[-\frac{1}{x}\Big]_{-1}^{T}+\lim_{t\to 0+}\Big[-\frac{1}{x}\Big]_t^1\\&=\infty+\infty\end{aligned}$$

Example 8

Example 9 ($\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$)

Since

$$\begin{aligned}\lim_{R\to\infty}\int_0^R\frac{dx}{1+x^2}&=\lim_{R\to\infty}\Big[\arctan x\Big]_0^R=\lim_{R\to\infty}\arctan R=\frac{\pi}{2}\\\lim_{r\to-\infty}\int_r^0\frac{dx}{1+x^2}&=\lim_{r\to-\infty}\Big[\arctan x\Big]_r^0=\lim_{r\to-\infty}-\arctan r=\frac{\pi}{2}\end{aligned}$$

The integral $\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$ converges and takes the value π.

Example 9

Example 10

For what values of p does $\int_e^\infty \frac{dx}{x(\ln x)^p}$ converge?

Solution. For $x \ge e$, the denominator $x(\ln x)^p$ is never zero. So the integrand is bounded on the entire domain of integration and this integral is improper only because the domain of integration extends to $+\infty$ and we proceed as usual. We have

$$\begin{aligned}
\int_e^\infty \frac{dx}{x(\ln x)^p} &= \lim_{R\to\infty}\int_e^R \frac{dx}{x(\ln x)^p} \\
&= \lim_{R\to\infty}\int_1^{\ln R} \frac{du}{u^p} \qquad \text{with } u = \ln x,\ du = \frac{dx}{x} \\
&= \lim_{R\to\infty} \begin{cases} \frac{1}{1-p}\Big[(\ln R)^{1-p} - 1\Big] & \text{if } p \ne 1 \\ \ln(\ln R) & \text{if } p = 1 \end{cases} \\
&= \begin{cases} \text{divergent} & \text{if } p \le 1 \\ \frac{1}{p-1} & \text{if } p > 1 \end{cases}
\end{aligned}$$

just as in Example 4, but with R replaced by $\ln R$.

Example 10

Example 11 (the gamma function)

The gamma function $\Gamma(x)$ is defined by the improper integral

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\ dx$$

We shall now compute $\Gamma(n)$ for all natural numbers n. To get started, we'll compute

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty e^{-x}\ dx = \lim_{R\to\infty}\int_0^R e^{-x}\ dx = \lim_{R\to\infty}\Big[-e^{-x}\Big]_0^R \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
\Gamma(2) &= \int_0^\infty xe^{-x}\ dx = \lim_{R\to\infty}\int_0^R xe^{-x}\ dx \\
&= \lim_{R\to\infty}\Big[-xe^{-x}\Big|_0^R + \int_0^R e^{-x}\ dx\Big] \\
&\qquad\qquad \text{(by integration by parts with } u = x,\ dv = e^{-x}dx,\ v = -e^{-x},\ du = dx) \\
&= \lim_{R\to\infty}\Big[-xe^{-x} - e^{-x}\Big]_0^R \\
&= 1
\end{aligned}$$

For the last equality, we used that $\lim_{x\to\infty} xe^{-x} = 0$. Now we move on to general n, using the same type of computation as we just used to evaluate $\Gamma(2)$. For any natural number n,

$$\begin{aligned}
\Gamma(n+1) &= \int_0^\infty x^n e^{-x}\ dx = \lim_{R\to\infty} \int_0^R x^n e^{-x}\ dx \\
&= \lim_{R\to\infty} \Big[-x^n e^{-x}\Big|_0^R + \int_0^R n x^{n-1} e^{-x}\ dx \Big] \\
&\qquad\qquad \text{(by integration by parts with } u = x^n,\ dv = e^{-x}dx,\ v = -e^{-x},\ du = nx^{n-1}dx\text{)} \\
&= \lim_{R\to\infty} n \int_0^R x^{n-1} e^{-x}\ dx \\
&= n\Gamma(n)
\end{aligned}$$

To get to the third row, we used that $\lim_{x\to\infty} x^n e^{-x} = 0$.

Now that we know $\Gamma(2) = 1$ and $\Gamma(n+1) = n\Gamma(n)$, for all $n \in \mathbb{N}$, we can compute all of the $\Gamma(n)$'s.

$$\begin{aligned}
\Gamma(2) &= 1 \\
\Gamma(3) &= \Gamma(2+1) = 2\Gamma(2) = 2\cdot 1 \\
\Gamma(4) &= \Gamma(3+1) = 3\Gamma(3) = 3\cdot 2\cdot 1 \\
\Gamma(5) &= \Gamma(4+1) = 4\Gamma(4) = 4\cdot 3\cdot 2\cdot 1 \\
&\ \ \vdots \\
\Gamma(n) &= (n-1)\cdot(n-2)\cdots 4\cdot 3\cdot 2\cdot 1 = (n-1)!
\end{aligned}$$

Example 11

Convergence Tests for Improper Integrals

It is very common to encounter integrals that are too complicated to evaluate explicitly. Numerical approximation schemes, evaluated by computer, are often used instead. You want to be sure that at least the integral converges before feeding it into a computer. Fortunately it is usually possible to determine whether or not an improper integral converges even when you cannot evaluate it explicitly. For pedagogical purposes, we are going to concentrate on the problem of determining whether or not an integral $\int_a^\infty f(x)\ dx$ converges, when $f(x)$ has no singularities for $x \ge a$. Recall that the first step in analyzing any improper integral is to write it as a sum of integrals each of has only a single "source of impropriety" — either a domain of integration that extends to $+\infty$, or a domain of integration that extends to $-\infty$, or an integrand which is singular at one end of the domain of integration. So we are now going to consider only the first of these three possibilities. But the techniques that we are about to see have obvious analogues for the other two possibilities.

Now let's start. Imagine that we have an improper integral $\int_a^\infty f(x)\ dx$, that $f(x)$ has no singularities for $x \ge a$ and that $f(x)$ is complicated enough that we cannot evaluate the integral explicitly. The idea is find another improper integral $\int_a^\infty g(x)\ dx$

- with $g(x)$ simple enough that we can evaluate the integral $\int_a^\infty g(x)\ dx$ explicitly, or at least determine easily whether or not $\int_a^\infty g(x)\ dx$ converges, and
- with $g(x)$ behaving enough like $f(x)$ for large x that the integral $\int_a^\infty f(x)\ dx$ converges if and only if $\int_a^\infty g(x)\ dx$ converges.

So far, this is a pretty vague strategy. Here is a theorem which starts to make it more precise.

Theorem 12 (Comparison)**.**

Let a be a real number. Let f and g be functions that are defined and continuous for all $x \ge a$ and assume that $g(x) \ge 0$ for all $x \ge 0$.

(a) If $|f(x)| \le g(x)$ for all $x \ge a$ and if $\int_a^\infty g(x)\ dx$ converges then $\int_a^\infty f(x)\ dx$ converges.

(b) If $f(x) \ge g(x)$ for all $x \ge a$ and if $\int_a^\infty g(x)\ dx$ diverges then $\int_a^\infty f(x)$ diverges.

We will not prove this theorem, but, hopefully, the following supporting arguments should at least appear reasonable to you.

- If $\int_a^\infty g(x)\ dx$ converges, then the area of $\big\{\ (x,y)\ \big|\ x \ge a,\ 0 \le y \le g(x)\ \big\}$ is finite. When $|f(x)| \le g(x)$, the region $\big\{\ (x,y)\ \big|\ x \ge a,\ 0 \le y \le |f(x)|\ \big\}$ is contained inside $\big\{\ (x,y)\ \big|\ x \ge a,\ 0 \le y \le g(x)\ \big\}$ and so also has finite area. Consequently the areas of both the regions $\big\{\ (x,y)\ \big|\ x \ge a,\ 0 \le y \le f(x)\ \big\}$ and $\big\{\ (x,y)\ \big|\ x \ge a,\ f(x) \le y \le 0\ \big\}$ are finite too. (Look at a graph of $f(x)$ and a graph of $|f(x)|$.) The integral $\int_a^\infty f(x)\ dx$ represents the signed area of the union of $\big\{\ (x,y)\ \big|\ x \ge a,\ 0 \le y \le f(x)\ \big\}$ and $\big\{\ (x,y)\ \big|\ x \ge a,\ f(x) \le y \le 0\ \big\}$.
- If $\int_a^\infty g(x)\ dx$ diverges, then the area of $\big\{\ (x,y)\ \big|\ x \ge a,\ 0 \le y \le g(x)\ \big\}$ is infinite. When $f(x) \ge g(x)$, the region $\big\{\ (x,y)\ \big|\ x \ge a,\ 0 \le y \le f(x)\ \big\}$ contains the region $\big\{\ (x,y)\ \big|\ x \ge a,\ 0 \le y \le g(x)\ \big\}$ and so also has infinite area.

Example 13 ($\int_1^\infty e^{-x^2}\ dx$)

We cannot evaluate the integral $\int_1^\infty e^{-x^2}\ dx$ explicitly. But we would still like to tell if it is finite or not. So we want to find another integral that we can compute and that we can compare to $\int_1^\infty e^{-x^2}\ dx$. To do so we pick an integrand that looks like e^{-x^2}, but whose indefinite integral we know — like e^{-x}. When $x \ge 1$, we have $x^2 \ge x$ and hence $e^{-x^2} \le e^{-x}$. The integral

$$\int_1^\infty e^{-x}\ dx = \lim_{R\to\infty}\int_1^R e^{-x}\ dx = \lim_{R\to\infty}\Big[-e^{-x}\Big]_1^R = \lim_{R\to\infty}\Big[e^{-1} - e^{-R}\Big] = e^{-1}$$

converges. So, by Theorem 12, with $a = 1$, $f(x) = e^{-x^2}$ and $g(x) = e^{-x}$, the integral $\int_1^\infty e^{-x^2}\,dx$ converges too.

Example 13

Example 14 ($\int_{1/2}^\infty e^{-x^2}\,dx$)

Of course the integral $\int_{1/2}^\infty e^{-x^2}\,dx$ is quite similar to the integral $\int_1^\infty e^{-x^2}\,dx$ of Example 13. But we cannot just repeat the argument of Example 13 because it is not true that $e^{-x^2} \le e^{-x}$ when $0 < x < 1$. In fact, for $0 < x < 1$, $x^2 < x$ so that $e^{-x^2} > e^{-x}$. However the difference between $\int_{1/2}^\infty e^{-x^2}\,dx$ and $\int_1^\infty e^{-x^2}\,dx$ is exactly $\int_{1/2}^1 e^{-x^2}\,dx$ which is clearly a well defined finite number. So we would expect that $\int_{1/2}^\infty e^{-x^2}\,dx$ should be the sum of the proper integral integral $\int_{1/2}^1 e^{-x^2}\,dx$ and the convergent integral $\int_1^\infty e^{-x^2}\,dx$ and so should be a convergent integral. This is indeed the case. The Theorem below provides the justification.

Example 14

Theorem 15.

Let $a < c$ and let the function $f(x)$ be continuous for all $x \ge a$. Then the improper integral $\int_a^\infty f(x)\,dx$ converges if and only if the improper integral $\int_c^\infty f(x)\,dx$ converges.

Proof. By definition the improper integral $\int_a^\infty f(x)\,dx$ converges if and only if the limit

$$\begin{aligned}\lim_{R\to\infty}\int_a^R f(x)\,dx &= \lim_{R\to\infty}\left[\int_a^c f(x)\,dx + \int_c^R f(x)\,dx\right]\\ &= \int_a^c f(x)\,dx + \lim_{R\to\infty}\int_c^R f(x)\,dx\end{aligned}$$

exists and is finite. (Remember that, in computing the limit, $\int_a^c f(x)\,dx$ is a finite constant independent of R and so can be pulled out of the limit.) But that is the case if and only if the limit $\lim_{R\to\infty}\int_c^R f(x)\,dx$ exists and is finite, which in turn is the case if and only if the integral $\int_c^\infty f(x)\,dx$ converges. □

Example 16

Does the integral $\int_1^\infty \frac{\sqrt{x}}{x^2+x}\,dx$ converge or diverge?

Solution. Our first task is to identify the potential sources of impropriety for this integral. Certainly the domain of integration extends to $+\infty$. But we must also check to see if the integrand contains any singularities. On the domain of integration $x \ge 1$ so the denominator is never zero and the integrand is continuous. So the only problem is at $+\infty$.

Our second task is to develop some intuition. As the only problem is that the domain of integration extends to infinity, whether or not the integral converges will be determined by the behavior of the integrand for very large x. When x is very large $x \ll x^2$ so that the denominator $x^2 + x \approx x^2$ and the integrand $\frac{\sqrt{x}}{x^2+x} \approx \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$. By Example 4, with $p = 3/2$, the integral $\int_1^\infty \frac{dx}{x^{3/2}}$ converges. So we would expect that $\int_1^\infty \frac{\sqrt{x}}{x^2+x}\, dx$ converges too.

Our final task is to verify that our intuition is correct. To do so, we want to apply part (a) of Theorem 12 with $f(x) = \frac{\sqrt{x}}{x^2+x}$ and $g(x)$ being $\frac{1}{x^{3/2}}$, or possibly some constant times $\frac{1}{x^{3/2}}$. We will be able to apply Theorem 12.a if we can show that $\frac{\sqrt{x}}{x^2+x}$ is smaller than some constant times $\frac{1}{x^{3/2}}$ on the domain of integration, $x \ge 1$. Here goes.

$$x \ge 1 \implies x^2 + x > x^2 \implies \frac{1}{x^2+x} < \frac{1}{x^2} \implies \frac{\sqrt{x}}{x^2+x} < \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$$

So Theorem 12.a and Example 4, with $p = 3/2$ do indeed show that the integral $\int_1^\infty \frac{\sqrt{x}}{x^2+x}\, dx$ converges.

Example 16

There is a variant of the comparison Theorem 12 that is often easier to apply than Theorem 12 itself and that also fits well with the sort of intuition that we developed in Example 16. Let's briefly review what happened In Example 16. We first identified the "problem x's" as the large x's. (The integral was improper only because the domain of integration extended to $+\infty$.) Our integrand, $f(x) = \frac{\sqrt{x}}{x^2+x}$, was relatively complicated but we found that, for large x, $f(x)$ "behaved like" the simple function $g(x) = \frac{1}{x^{3/2}}$. We knew that the integral of $g(x)$, over the domain of integration of interest, converged. We then used Theorem 12 to show that the integral of $f(x)$ converged too.

A good way to formalize "$f(x)$ behaves like $g(x)$ for large x" is to require that the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

exists and is a nonzero number. Suppose that this is the case and call the limit $L \ne 0$. Then $\frac{f(x)}{g(x)}$ must approach L as x tends to $+\infty$ and, in particular, there must be a number c such that $\frac{f(x)}{g(x)}$ lies between $\frac{L}{2}$ and $2L$, that is $f(x)$ lies between $\frac{L}{2}g(x)$ and $2Lg(x)$, for all $x \ge c$. Consequently, the integral of $f(x)$ converges if and only if the integral of $g(x)$ converges, by Theorems 12 and 15. These considerations lead to the following variant of Theorem 12.

Theorem 17 (Limiting comparison).

Let $-\infty < a < \infty$. Let f and g be functions that are defined and continuous for all $x \geq a$ and assume that $g(x) \geq 0$ for all $x \geq a$.

(a) If $\int_a^\infty g(x)\ dx$ converges and the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

exists (in this case it is allowed to be any real number, including 0), then $\int_a^\infty f(x)\ dx$ converges.

(b) If $\int_a^\infty g(x)\ dx$ diverges and the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

exists and is nonzero, then $\int_a^\infty f(x)$ diverges.

There are obvious analogs of this theorem for the other types of improper integrals too. Here is an example of how Theorem 17 is used.

Example 18

Does the integral $\int_1^\infty \frac{x+\sin x}{e^{-x}+x^2}\ dx$ converge or diverge?

Solution. Our first task is to identify the potential sources of impropriety for this integral. The domain of integration extends to $+\infty$. On the domain of integration the denominator is never zero so the integrand is continuous. Thus the only problem is at $+\infty$.

Our second task is to develop some intuition about the behavior of the integrand for very large x. When x is very large

- $e^{-x} \ll x^2$, so that the denominator $e^{-x} + x^2 \approx x^2$ and
- $|\sin x| \leq 1 \ll x$, so that the numerator $x + \sin x \approx x$ and
- the integrand $\frac{x+\sin x}{e^{-x}+x^2} \approx \frac{x}{x^2} = \frac{1}{x}$.

Since $\int_1^\infty \frac{dx}{x}$ diverges, we would expect $\int_1^\infty \frac{x+\sin x}{e^{-x}+x^2}\ dx$ to diverge too.

Our final task is to verify that our intuition is correct. To do so, we set

$$f(x) = \frac{x+\sin x}{e^{-x}+x^2} \qquad g(x) = \frac{1}{x}$$

and compute

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x+\sin x}{e^{-x}+x^2} \div \frac{1}{x} \\
&= \lim_{x\to\infty} \frac{(1+\sin x/x)x}{(e^{-x}/x^2+1)x^2} \times x \\
&= \lim_{x\to\infty} \frac{1+\sin x/x}{e^{-x}/x^2+1} \\
&= 1
\end{aligned}$$

Since $\int_1^\infty g(x)\ dx = \int_1^\infty \frac{dx}{x}$ diverges, by Example 4 with $p=1$, Theorem 17.b now tells us that $\int_1^\infty f(x)\ dx = \int_1^\infty \frac{x+\sin x}{e^{-x}+x^2}\ dx$ diverges too.

Example 18