Integration of $\sec\theta$, $\csc\theta$, $\sec^3\theta$ and $\csc^3\theta$

These notes show several ways to integrate $\sec\theta$, $\csc\theta$, $\sec^3\theta$ and $\csc^3\theta$.

$\int \sec\theta\ d\theta$ — by trickery

The standard trick used to integrate $\sec\theta$ is to multiply the integrand by $1 = \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}$ and then substitute $y = \sec\theta + \tan\theta$, $dy = (\sec\theta\tan\theta + \sec^2\theta)\ d\theta$.

$$\begin{aligned}\int \sec\theta\ d\theta &= \int \sec\theta\ \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\ d\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\ d\theta = \int \frac{dy}{y} = \ln|y| + C \\ &= \ln|\sec\theta+\tan\theta| + C\end{aligned}$$

$\int \sec\theta\ d\theta$ — by partial fractions

Another method for integrating $\int \sec\theta\ d\theta$, that is more tedious, but less dependent on a memorized trick, is to convert $\int \sec\theta\, d\theta$ into the integral of a rational function using the substitution $y = \sin\theta$, $dy = \cos\theta\ d\theta$ and then use partial fractions.

$$\begin{aligned}\int \sec\theta\ d\theta &= \int \frac{1}{\cos\theta}\ d\theta = \int \frac{\cos\theta}{\cos^2\theta}\ d\theta = \int \frac{\cos\theta}{1-\sin^2\theta}\ d\theta = \int \frac{1}{1-y^2}\ dy \\ &= \int \frac{1}{2}\Big[\frac{1}{1+y} + \frac{1}{1-y}\Big]\ dy = \frac{1}{2}\int \Big[\frac{1}{y+1} - \frac{1}{y-1}\Big]\ dy \\ &= \frac{1}{2}\big[\ln|y+1| - \ln|y-1|\big] + C = \frac{1}{2}\ln\Big|\frac{y+1}{y-1}\Big| + C \\ &= \frac{1}{2}\ln\Big|\frac{\sin\theta+1}{\sin\theta-1}\Big| + C\end{aligned}$$

To see that this answer is really the same as the one above, note that

$$\begin{aligned}&\frac{\sin\theta+1}{\sin\theta-1} = \frac{(\sin\theta+1)^2}{\sin^2\theta-1} = \frac{(\sin\theta+1)^2}{-\cos^2\theta} \\ \implies\ &\frac{1}{2}\ln\Big|\frac{\sin\theta+1}{\sin\theta-1}\Big| = \frac{1}{2}\ln\Big|\frac{(\sin\theta+1)^2}{-\cos^2\theta}\Big| = \frac{1}{2}\ln\Big|\frac{(\sin\theta+1)^2}{\cos^2\theta}\Big| = \ln\Big|\frac{\sin\theta+1}{\cos\theta}\Big| \\ &= \ln|\tan\theta+\sec\theta|\end{aligned}$$

$\int \csc\theta\ d\theta$ — by the $x = \tan\frac{\theta}{2}$ substitution

The integral $\int \csc\theta\ d\theta$ may also be evaluated by both the methods above. The answers are

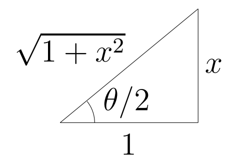
$$\int \csc\theta\ d\theta = \ln|\cot\theta - \csc\theta| + C = -\frac{1}{2}\ln\Big|\frac{\cos\theta+1}{\cos\theta-1}\Big| + C$$

Since $\csc\theta$ is a rational function of $\sin\theta$ and $\cos\theta$, the substitution

$$x = \tan\frac{\theta}{2} \qquad \theta = 2\tan^{-1}x \qquad d\theta = \frac{2}{1+x^2}\,dx$$

$$\sin\theta = 2\,\sin\frac{\theta}{2}\,\cos\frac{\theta}{2} = 2\,\frac{x}{\sqrt{1+x^2}}\,\frac{1}{\sqrt{1+x^2}} = \frac{2x}{1+x^2}$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \frac{1}{1+x^2} - \frac{x^2}{1+x^2} = \frac{1-x^2}{1+x^2}$$



converts $\int \csc\theta\,d\theta$ into the integral of a rational function.

$$\begin{aligned}\int \csc\theta\;d\theta &= \int \frac{1}{\sin\theta}\;d\theta = \int \frac{1+x^2}{2x}\;\frac{2}{1+x^2}dx = \int \frac{1}{x}\;dx = \ln|x| + C \\ &= \ln\left|\tan\frac{\theta}{2}\right| + C\end{aligned}$$

To see that this answer is really the same as the one above, note that

$$\cot\theta - \csc\theta = \frac{\cos\theta - 1}{\sin\theta} = \frac{-2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = -\tan\frac{\theta}{2} \tag{1}$$

$\int \sec^3 x\;dx$ — by trickery

The standard trick used to evaluate $\int \sec^3 x\;dx$ is integration by parts with $u = \sec x$, $dv = \sec^2 x\;dx$, $du = \sec x \tan x\;dx$, $v = \tan x$.

$$\int \sec^3 x\;dx = \int \sec x\;\;\sec^2 x\;dx = \sec x \tan x - \int \tan x\;\;\sec x \tan x\;dx$$

Since $\tan^2 x + 1 = \sec^2 x$, we have $\tan^2 x = \sec^2 x - 1$ and

$$\int \sec^3 x\,dx = \sec x \tan x - \int [\sec^3 x - \sec x]\;dx = \sec x \tan x + \ln|\sec x + \tan x| + C - \int \sec^3 x\,dx$$

where we used $\int \sec x\;dx = \ln|\sec x + \tan x| + C$. Now moving the $\int \sec^3 x\;dx$ from the right hand side to the left hand side

$$\begin{aligned} 2\int \sec^3 x\;dx &= \sec x \tan x + \ln|\sec x + \tan x| + C \\ \Rightarrow \int \sec^3 x\;dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C\end{aligned}$$

for a new arbitrary constant C.

$\int \sec^3 x\;dx$ — by partial fractions

Another method for integrating $\int \sec^3 x\;dx$, that is more tedious, but less dependent on trickery, is to convert $\int \sec^3 x\,dx$ into the integral of a rational function using the substitution

$y = \sin x$, $dy = \cos x\, dx$ and then use partial fractions.

$$\int \sec^3 x\ dx = \int \frac{1}{\cos^3 x}\ dx = \int \frac{\cos x}{\cos^4 x}\ dx = \int \frac{\cos x}{[1-\sin^2 x]^2}\ dx = \int \frac{1}{[1-y^2]^2}\ dy$$

$$= \int \frac{1}{[y^2-1]^2}\ dy = \int \Big[\frac{1}{2}\Big(\frac{1}{y-1} - \frac{1}{y+1}\Big)\Big]^2\ dy$$

$$= \frac{1}{4}\int \Big[\frac{1}{(y-1)^2} - \frac{2}{(y-1)(y+1)} + \frac{1}{(y+1)^2}\Big]\ dy$$

$$= \frac{1}{4}\int \Big[\frac{1}{(y-1)^2} - \frac{1}{y-1} + \frac{1}{y+1} + \frac{1}{(y+1)^2}\Big]\ dy$$

$$= \frac{1}{4}\Big[-\frac{1}{y-1} - \ln|y-1| + \ln|y+1| - \frac{1}{y+1}\Big] + C$$

$$= -\frac{1}{4}\frac{2y}{y^2-1} + \frac{1}{4}\ln\Big|\frac{y+1}{y-1}\Big| + C = \frac{1}{2}\frac{y}{1-y^2} + \frac{1}{4}\ln\Big|\frac{y+1}{y-1}\Big| + C$$

$$= \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{4}\ln\Big|\frac{\sin x + 1}{\sin x - 1}\Big| + C$$

$\int \csc^3\theta\ d\theta$ — by the $x = \tan\frac{\theta}{2}$ substitution

As another example of the

$$x = \tan\frac{\theta}{2} \qquad d\theta = \frac{2}{1+x^2}dx \qquad \sin\theta = \frac{2x}{1+x^2} \qquad \cos\theta = \frac{1-x^2}{1+x^2}$$

substitution, we evaluate

$$\int \csc^3\theta\ d\theta = \int \frac{1}{\sin^3\theta}\ d\theta = \int \Big(\frac{1+x^2}{2x}\Big)^3\ \frac{2}{1+x^2}dx = \frac{1}{4}\int \frac{1+2x^2+x^4}{x^3}\ dx$$

$$= \frac{1}{4}\Big[\frac{x^{-2}}{-2} + 2\ln|x| + \frac{x^2}{2}\Big] + C$$

$$= \frac{1}{8}\Big[-\cot^2\frac{\theta}{2} + 4\ln\Big|\tan\frac{\theta}{2}\Big| + \tan^2\frac{\theta}{2}\Big] + C$$

By the usual double angle formulae

$$\tan^2\frac{\theta}{2} - \cot^2\frac{\theta}{2} = \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} - \frac{\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} = \frac{\sin^4\frac{\theta}{2} - \cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}$$

$$= \frac{\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}} \qquad \text{since } \sin^2\tfrac{\theta}{2} + \cos^2\tfrac{\theta}{2} = 1$$

$$= \frac{-\cos\theta}{\frac{1}{4}\sin^2\theta}$$

so we may also write

$$\int \csc^3\theta\ d\theta = -\frac{1}{2}\cot\theta\csc\theta + \frac{1}{2}\ln|\cot\theta - \csc\theta| + C$$

by (1).

