Introduction to Differential Equations

A differential equation is an equation for an unknown function that involves the derivative of the unknown function. For example

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 7y + \cos x$$

is a differential equation for an unknown function, $y(x)$. This particular equation is said to be a first order linear differential equation.

- It is a differential equation because it involves the derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ of the unknown function.
- It is a first order equation because the highest order derivative that appears is the first order derivative.
- It is a linear equation because y and its derivatives appear only to the power one. There are no y^2's or $y\frac{\mathrm{d}y}{\mathrm{d}x}$'s or $\cos(y)$'s. The most general first order linear differential equation is $a(x)\frac{\mathrm{d}y}{\mathrm{d}x} + b(x)y = c(x)$ where $a(x)$, $b(x)$, $c(x)$ are given functions.

Differential equations play a central role in modelling a huge number of different phenomena. Here is a table giving a bunch of named differential equations and what they are used for. It is far from complete.

Newton's Law of Motion	describes motion of particles
Maxwell's equations	describes electromagnetic radiation
Navier–Stokes equations	describes fluid motion
Heat equation	describes heat flow
Wave equation	describes wave motion
Schrödinger equation	describes atoms, molecules and crystals
Stress-strain equations	describes elastic materials
Black–Scholes models	used for pricing financial options
Predator–prey equations	describes ecosystem populations
Einstein's equations	connects gravity and geometry
Ludwig–Jones–Holling's equation	models spruce budworm/Balsam fir ecosystem
Zeeman's model	models heart beats and nerve impulses
Sherman–Rinzel–Keizer model	for electrical activity in Pancreatic β–cells
Hodgkin–Huxley equations	models nerve action potentials

We are just going to scratch the surface of the study of differential equations. Most universities offer half a dozen different undergraduate courses on various aspects of differential equations. We will just look at one special, but important, type of equation.

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Definition 1.

A *separable differential equation* is an equation for a function $y(x)$ of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = f(x)\ g\big(y(x)\big)$$

Usually one supresses the argument of y and writes the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\ g(y)$$

We'll start by developing a recipe for solving separable differential equations. Usually one supresses the argument of y and writes the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\ g(y)$$

and solves such an equation by cross multiplying/dividing to get all of the y's on one side of the equation and all of the x's, including the dx, on the other side of the equation.

$$\frac{dy}{g(y)} = f(x)\,dx$$

(We are of course assuming that $g(y)$ is nonzero.) Then you integrate both sides

$$\int \frac{dy}{g(y)} = \int f(x)\,dx \tag{1}$$

This looks illegal, and indeed is illegal — $\frac{\mathrm{d}y}{\mathrm{d}x}$ is not a fraction. But we'll now see that the answer is still correct. This procedure is simply a mnenomic device to help you remember that answer. Let $G(y)$ be an antiderivative of $\frac{1}{g(y)}$ (i.e. $G'(y) = \frac{1}{g(y)}$) and $F(x)$ be an antiderivative of $f(x)$ (i.e. $F'(x) = f(x)$. If we reinstate the argument of y, (1) is

$$G\big(y(x)\big) = F(x) + C \tag{2}$$

To check that a function $y(x)$ obeys $\frac{\mathrm{d}y}{\mathrm{d}x}(x) = f(x)\ g\big(y(x)\big)$ if and only if it obeys (2), just differentiate both sides of (2) with respect to x. By the chain rule

$$G\big(y(x)\big) = F(x) + C \iff G'\big(y(x)\big)\ y'(x) = F'(x) \iff \frac{y'(x)}{g(y(x))} = f(x)$$
$$\iff y'(x) = f(x)\ g(y(x))$$

(We have again assumed that $g(y)$ is nonzero.)

Observe that the solution (2) contains an arbitrary constant, C. The value of this arbitrary constant *can not* be determined by the differential equation. You need additional data to determine it. Often this data consists of the value of the unknown function for one value of x. That is, often the problem you have to solve is of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = f(x)\ g\big(y(x)\big) \qquad y(x_0) = y_0$$

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where $f(x)$ and $g(y)$ are given functions and x_0 and y_0 are given numbers. This type of problem is called an "initial value problem". It is solved by first using the method above to find the general solution to the differential equation, including the arbitrary constant C, and then using the "initial condition" $y(x_0) = y_0$ to determine the value of C. We'll see examples of this shortly.

Example 2

Let a and b be any two constants. We'll now solve the family of, first order, linear, differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(y - b)$$

using our mnemonic device.

$$\frac{dy}{y-b} = a\,dx \implies \int \frac{dy}{y-b} = \int a\,dx \implies \ln|y-b| = ax + c \implies |y-b| = e^{ax+c} = e^c e^{ax}$$
$$\implies y - b = Ce^{ax}$$

where C is either $+e^c$ or $-e^c$. We were a bit sloppy here. We implicitly assumed that $y - b$ was nonzero, so that we could divide it across. But the constant function $y = b$ is a perfectly good solution — when y is the constant function $y = b$, both $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $a(y - b)$ are zero. So the general solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = a(y - b)$ is $y(x) = Ce^{ax} + b$, where the constant C can be any real number. Note that when $y(x) = Ce^{ax} + b$ we have $y(0) = C + b$. So $C = y(0) - b$ and the general solution is

$$y(x) = \{y(0) - b\}\, e^{ax} + b$$

Example 2

Example 3

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$

Solution. When $y \neq 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \implies \frac{dy}{y^2} = dx \implies \frac{y^{-1}}{-1} = x + C \implies y = -\frac{1}{x + C}$$

When $y = 0$, this computation breaks down because $\frac{dy}{y^2}$ contains a division by 0. We can check if the function $y(x) = 0$ satisfies the differential equation by just subbing it in:

$$y(x) = 0 \implies y'(x) = 0,\ y(x)^2 = 0 \implies y'(x) = y(x)^2$$

So $y(x) = 0$ is a solution and the full solution is

$$y(x) = 0 \text{ or } y(x) = -\frac{1}{x + C}, \text{ for any constant } C$$

Example 3

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Example 4

When a raindrop falls it increases in size so that its mass $m(t)$, is a function of time t. The rate of growth of mass, i.e. $\frac{dm}{dt}$, is $km(t)$ for some positive constant k. According to Newton's law of motion, $\frac{\mathrm{d}}{\mathrm{d}t}(mv) = gm$, where v is the velocity of the raindrop (with v being positive for downward motion) and g is the acceleration due to gravity. Find the terminal velocity, $\lim_{t\to\infty} v(t)$, of a raindrop.

Solution. In this problem we have two unknown functions, $m(t)$ and $v(t)$, and two differential equations, $\frac{dm}{dt} = km$ and $\frac{\mathrm{d}}{\mathrm{d}t}(mv) = gm$. The first differential equation, $\frac{dm}{dt} = km$, involves only $m(t)$, not $v(t)$, so we use it to determine $m(t)$.

$$\frac{\mathrm{d}m}{\mathrm{d}t} = km \implies \frac{dm}{m} = k\,dt \implies \ln m = kt + c \implies m = de^{kt} \text{ where } d = e^c$$

for some positive constant d. Now that we know $m(t)$ (except for the value of the constant d), we can substitute it into the second differential equation, which we can then use to determine the remaining unknown function $v(t)$. Observe that the second equation, $\frac{\mathrm{d}}{\mathrm{d}t}(mv) = gm(t) = gde^{kt}$ tells that the derivative of the function $y(t) = m(t)v(t)$ is gde^{kt}. So $y(t)$ is just an antiderivative of gde^{kt}.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = gm(t) = gde^{kt} \implies dy = gde^{kt}\,dt \implies y(t) = \int gde^{kt}\,dt = gd\frac{e^{kt}}{k} + C$$

Now that we know $y(t) = m(t)v(t) = de^{kt}v(t)$, we can get $v(t)$ just by dividing out the de^{kt}.

$$y(t) = gd\frac{e^{kt}}{k} + C \implies de^{kt}v(t) = gd\frac{e^{kt}}{k} + C \implies v(t) = \frac{g}{k} + \frac{C}{de^{kt}}$$

Our solution, $v(t)$, contains two arbitrary constants, namely C and d. They will be determined by, for example, the mass and velocity at time $t = 0$. But since we are only interested in the terminal velocity $\lim_{t\to\infty} v(t)$, we don't need to know C and d. Since $k > 0$, $\lim_{t\to\infty} \frac{C}{e^{kt}} = 0$ and the terminal velocity $\lim_{t\to\infty} v(t) = \frac{g}{k}$.

Example 4

Example 5

A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances at a rate that is proportional to the concentration at that time. The concentration, $C(t)$, of the glucose in the bloodstream at time t obeys the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}t} = r - kC$$

where k is a positive constant of proportionality.

(a) Express $C(t)$ in terms of k and $C(0)$.

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(b) Find $\lim_{t\to\infty} C(t)$. Discuss.

Solution. (a) Since $r - kC = -k\big(C - \frac{r}{k}\big)$ this equation is of the form solved in Example 2 with $a = -k$ and $b = \frac{r}{k}$. So the solution is

$$C(t) = \frac{r}{k} + \Big(C(0) - \frac{r}{k}\Big)e^{-kt}$$

(b) For any $k > 0$, $\lim_{t\to\infty} e^{-kt} = 0$. Consequently, for any $C(0)$ and any $k > 0$, $\lim_{t\to\infty} C(t) = \frac{r}{k}$. We could have predicted this limit without solving for $C(t)$. If we assume that $C(t)$ approaches some equilibrium value C_e as t approaches infinity, then taking the limits of both sides of $\frac{dC}{dt} = r - kC$ as $t \to \infty$ gives

$$0 = r - kC_e \implies C_e = \frac{r}{k}$$

Example 5

Carbon Dating

Example 6

Scientists can determine the age of ancient objects by a method called *radiocarbon dating.* The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ${}^{14}C$, with a half–life of about 5730 years. Vegitation absorbs carbon dioxide from the atmosphere and animal life assimilates ${}^{14}C$ through the food chain. When a plant or animal dies, it stops replacing its carbon and the amount of ${}^{14}C$ begins to decrease through radioactive decay. Therefore the level of radioactivity also decreases. A parchment fragment was discovered that had about 74% as much ${}^{14}C$ radioactivity as does plant material on earth today. Estimate the age of the parchment.

Solution. Let $Q(t)$ denote the amount of ${}^{14}C$ in the parchment t years after it was first created. The number of radioactive decays per unit time, at time t, is proportional to the amount $Q(t)$ present at time t. Thus

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = KQ(t)$$

for some constant of proportionality K. This is a separable differential equation. We solve it in the usual way.

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = KQ \implies \frac{dQ}{Q} = K\ dt \implies \ln Q = Kt + C \implies Q(t) = e^C e^{Kt}$$

At time 0, $Q(0) = e^C$. So

$$Q(t) = Q(0)e^{Kt}$$

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To finish the problem, we still have to determine (a) the value of the constant of proportionality K and (b) the time, call it t_p, for which $Q(t_p) = 0.74\,Q(0)$. In the statement of the problem, we are also given two constants — the half–life of 5730 years and the "0.74" in $Q(t_p) = 0.74\,Q(0)$. Not surprisingly, the first will determine K and then the second will determine t_p.

(a) By definition, the half–life of ^{14}C is the length of time that it takes for half of the ^{14}C to decay. That is, the half–life $t_{1/2}$ is determined by

$$Q(t_{1/2}) = \tfrac{1}{2}Q(0) \iff Q(0)e^{Kt_{1/2}} = \tfrac{1}{2}Q(0) \iff e^{Kt_{1/2}} = \tfrac{1}{2}$$

Taking the logarithm of both sides gives

$$Kt_{1/2} = \ln\frac{1}{2} = -\ln 2 \implies K = -\frac{\ln 2}{t_{1/2}}$$

We are told that, for ^{14}C, the half–life $t_{1/2} = 5730$, so

$$K = -\frac{\ln 2}{5730} = -0.000121$$

Note that K is negative. We should have known that K would be negative — $Q(t)$ is positive and $\frac{\mathrm{d}Q}{\mathrm{d}t}$ is negative, since $Q(t)$ decreases as t increases. So $\frac{\mathrm{d}Q}{\mathrm{d}t} = KQ$ forces K to be negative.

(b) Finally, the time $t = t_p$ at which $Q(t)$ reaches $0.74\,Q(0)$ is determined by

$$\begin{aligned} Q(t_p) = 0.74\,Q(0) &\implies Q(0)\,e^{Kt_p} = 0.74\,Q(0) \implies e^{Kt_p} = 0.74 \\ &\implies Kt_p = \ln 0.74 \qquad\implies t_p = \frac{\ln 0.74}{K} = 2490 \end{aligned}$$

The parchment is about 25 centuries old.

Example 6

Newton's Law of Cooling

Newton's law of cooling says:

> The rate of change of temperature of an object is proportional to the difference in temperature between the object and its surroundings. The temperature of the surroundings is sometimes called the ambient temperature.

If we denote by $T(t)$ the temperature of the object at time t and by A the temperature of its surroundings, Newton's law of cooling says that there is some constant of proportionality, K, such that

$$\frac{\mathrm{d}T}{\mathrm{d}t}(t) = K\big[T(t) - A\big] \tag{3}$$

Let's start by thinking a little about the sign of the constant of proportionality. At any time t, there are three possibilities.

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- If $T(t) > A$, that is, if the body is warmer than its surroundings, we would expect heat to flow from the body into its surroundings and so we would expect the body to cool off so that $\frac{dT}{dt}(t) < 0$. For this expectation to be consistent with (3), we need $K < 0$.
- If $T(t) < A$, that is the body is cooler than its surroundings, we would expect heat to flow from the surroundings into the body and so we would expect the body to warm up so that $\frac{dT}{dt}(t) > 0$. For this expectation to be consistent with (3), we again need $K < 0$.
- Finally if $T(t) = A$, that is the body and its environment have the same temperature, we would not expect any heat to flow between the two and so we would expect that $\frac{dT}{dt}(t) = 0$. This does not impose any condition on K.

In conclusion, we would expect $K < 0$. Of course, we could have chosen to call the constant of proportionality $-k$, rather than K. Then the differential equation would be $\frac{dT}{dt} = -k\big(T - A\big)$ and we would expect $k > 0$.

Example 7

The temperature of a glass of iced tea is initially 5°. After 5 minutes, the tea has heated to 10° in a room where the air temperature is 30°.

(a) Determine the temperature as a function of time.

(b) What is the temperature after 10 minutes?

(c) Determine when the tea will reach a temperature of 20°.

Solution. (a) Denote by $T(t)$ the temperature of the tea t minutes after it was removed from the fridge. By Newton's law of cooling,

$$\frac{dT}{dt} = K(T - A) = K(T - 30)$$

for some, as yet unknown, constant of proportionality K, since, in this problem, the ambient temperature $A = 30°$. By Example 2 with $a = K$ and $b = 30$,

$$T(t) = [T(0) - 30]\, e^{Kt} + 30 = 30 - 25e^{Kt}$$

since the initial temperature $T(0) = 5$. This solution is not complete because it still contains an unknown constant, namely K. We have not yet used the given data that $T(5) = 10$. We can use it to determine K. At $t = 5$,

$$\begin{aligned} T(5) = 30 - 25e^{5K} = 10 \implies e^{5K} = \frac{20}{25} \implies 5K = \ln\frac{20}{25} \\ \implies K = \frac{1}{5}\ln\frac{4}{5} = -0.044629 \end{aligned}$$

(b) At $t = 10$,

$$T(10) = 30 - 25e^{10K} = 30 - 25e^{-10\times 0.044629} = 30 - 16 = 14°$$

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to the nearest degree.
(c) The temperature is 20° when

$$30 - 25e^{Kt} = 20 \implies e^{Kt} = \frac{10}{25} \implies Kt = \ln\frac{10}{25}$$
$$\implies t = \frac{1}{K}\ln\frac{2}{5} = 20.5 \text{ min}$$

to one decimal place.

Example 7

Example 8

A dead body is discovered at 3:45pm in a room where the temperature is 20°C. At that time the temperature of the body 1s 27°C. Two hours later, at 5:45pm, the temperature of the body is 25.3 °C. What was the time of death?

Solution. Denote by $T(t)$ the temperature of the body at time t, with $t = 0$ corresponding to 3:45pm. If we call the time of death t_d, we have been told that

(1) $\frac{\mathrm{d}T}{\mathrm{d}t} = K(T - A)$ where K is an unknown constant and A is the ambient temerature. This is Newton's law of cooling.
(2) $A = 20$
(3) $T(0) = 27$
(4) $T(2) = 25.3$
(5) $T(t_d) = 37$. That's the normal body temperature.

By Example 2,
$$T(t) = [T(0) - A]\, e^{Kt} + A = 20 + 7e^{Kt}$$

Two unknowns remain, K and t_d. The first, K, is determined by condition (4).

$$25.3 = T(2) = 20 + 7e^{2K} \implies 7e^{2K} = 5.3 \implies 2K = \ln\left(\tfrac{5.3}{7}\right) \implies K = \tfrac{1}{2}\ln\left(\tfrac{5.3}{7}\right) = -0.139$$

Finally, t_d is determined by (5).

$$37 = T(t_d) = 20 + 7e^{-0.139t_d} \implies e^{-0.139t_d} = \tfrac{17}{7} \implies -0.139t_d = \ln\left(\tfrac{17}{7}\right)$$
$$\implies t_d = -\tfrac{1}{0.139}\ln\left(\tfrac{17}{7}\right) = -6.38$$

Now 6.38 hours is 6 hours and $0.38 \times 60 = 23$ minutes. So the time of death was 6 hours and 23 minutes before 3:45pm, which is 9:22am.

Example 8

Example 9

On a hot day, a thermometer is taken outside from an air–conditioned room where the temperature is 21°C. After one minute, it reads 27°C and after two minutes, it reads 30°C. What is the outdoor temperature?

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Solution. Let A be the outdoor temperature and $T(t)$ be the temperature of the thermometer t minutes after it is taken outside. Then the temperature of the thermometer obeys, by Newton's law of cooling,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = K(T-A) \implies T(t) = A + \big(T(0) - A\big)e^{Kt}$$

by Example 2. We are told $T(0) = 21$, so $T(t) = A + \big(21 - A\big)e^{Kt}$. We are also told $T(1) = 27$, which gives

$$27 = A + \big(21 - A\big)e^{K} \implies e^K = \frac{27-A}{21-A}$$

and $T(2) = 30$, which gives

$$30 = A + \big(21 - A\big)e^{2K} = A + \big(21 - A\big)\Big(\frac{27-A}{21-A}\Big)^2 = A + \frac{(27-A)^2}{21-A}$$

or

$$\begin{aligned} 30 - A = \frac{(27-A)^2}{21-A} &\Rightarrow (30-A)(21-A) = (27-A)^2 \\ &\Rightarrow 630 - 51A + A^2 = 729 - 54A + A^2 \\ &\Rightarrow 3A = 99 \Rightarrow A = 33°\mathrm{C} \end{aligned}$$

Example 9

Logistic Growth

Logistic growth is a simple model for predicting the size $P(t)$ of a population as a function of the time t.

In the most naive model of population growth, each couple produces β offspring (for some constant β) and then dies. Thus over the course of one generation $\beta\frac{P(t)}{2}$ children are produced and $P(t)$ parents die so that the size of the population grows from $P(t)$ to

$$P(t+t_g) = P(t) + \beta\frac{P(t)}{2} - P(t) = \frac{\beta}{2}P(t)$$

where t_g denotes the lifespan of one generation. The rate of change of the size of the population per unit time is

$$\frac{P(t+t_g) - P(t)}{t_g} = \frac{1}{t_g}\Big[\frac{\beta}{2}P(t) - P(t)\Big] = bP(t)$$

where $b = \frac{\beta-2}{2t_g}$ is the net birthrate per member of the population per unit time. If we approximate $\frac{P(t+t_g)-P(t)}{t_g} \approx \frac{\mathrm{d}P}{\mathrm{d}t}(t)$ we get the differential equation

$$P'(t) = bP(t)$$

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Logistic growth adds one more wrinkle to this model. It assumes that the population only has access to limited resources. As the size of the population grows the amount of food available to each member decreases. This in turn causes the net birth rate b to decrease. In the logistic growth model $b = b_0\left(1 - \frac{P}{K}\right)$, where K is called the carrying capacity of the environment, so that

$$P'(t) = b_0\left(1 - \frac{P(t)}{K}\right)P(t)$$

This is a separable differential equation and we can solve it explicitly. We shall do so shortly. See Example 10, below. But, before doing that, we'll see what we can learn about the behaviour of solutions to differential equations like this without finding formulae for the solutions. It turns out that we can learn a lot just by watching the sign of $P'(t)$. For concreteness, we'll look at solutions of the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t}(t) = \big(6000 - 3P(t)\big)\,P(t)$$

We'll sketch the graphs of four functions $P(t)$ that obey this equation.

- For the first function, $P(0) = 0$.
- For the second function, $P(0) = 1000$.
- For the third function, $P(0) = 2000$.
- For the fourth function, $P(0) = 3000$.

The sketchs will be based on the observation that $(6000 - 3P)\,P = 3(2000 - P)\,P$

- is zero for $P = 0,\ 2000$,
- is strictly positive for $0 < P < 2000$ and
- is strictly negative for $P > 2000$.

Consequently

$$\frac{\mathrm{d}P}{\mathrm{d}t}(t)\begin{cases} = 0 & \text{if } P(t) = 0 \\ > 0 & \text{if } 0 < P(t) < 2000 \\ = 0 & \text{if } P(t) = 2000 \\ < 0 & \text{if } P(t) > 2000 \end{cases}$$

Thus if $P(t)$ is some function that obeys $\frac{\mathrm{d}P}{\mathrm{d}t}(t) = \big(6000 - 3P(t)\big)P(t)$, then as the graph of $P(t)$ passes through the point $\big(t, P(t)\big)$

$$\text{the graph has}\begin{cases} \text{slope zero,} & \text{i.e. is horizontal,} & \text{if } P(t) = 0 \\ \text{positive slope,} & \text{i.e. is increasing,} & \text{if } 0 < P(t) < 2000 \\ \text{slope zero,} & \text{i.e. is horizontal,} & \text{if } P(t) = 2000 \\ \text{negative slope,} & \text{i.e. is decreasing,} & \text{if } 0 < P(t) < 2000 \end{cases}$$

as illustrated in the figure

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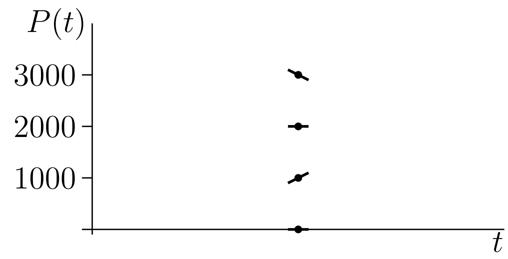


As a result,

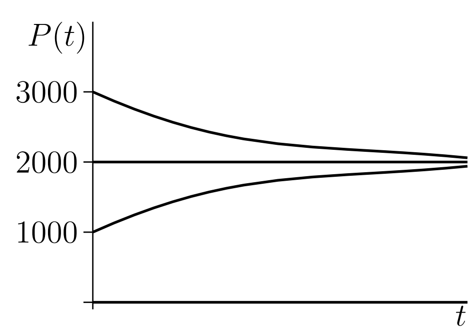
- if $P(0) = 0$, the graph starts out horizontally. In other words, as t starts to increase, $P(t)$ remains at zero, so the slope of the graph remains at zero. The population size remains zero for all time. As a check, observe that the function $P(t) = 0$ obeys $\frac{\mathrm{d}P}{\mathrm{d}t}(t) = \big(6000 - 3P(t)\big)P(t)$ for all t.

- Similarly, if $P(0) = 2000$, the graph again starts out horizontally. So $P(t)$ remains at 2000 and the slope remains at zero. The population size remains 2000 for all time. Again, the function $P(t) = 2000$ obeys $\frac{\mathrm{d}P}{\mathrm{d}t}(t) = \big(6000 - 3P(t)\big)P(t)$ for all t.

- If $P(0) = 1000$, the graph starts out with positive slope. So $P(t)$ increases with t. As $P(t)$ increases towards 2000, the slope $(6000 - 3P(t)\big)P(t)$, while remaining positive, gets closer and closer to zero. As the graph approachs height 2000, it becomes more and more horizontal. The graph cannot actually cross from below 2000 to above 2000, because to do so it would have to have strictly positive slope for some value of P above 2000, which is not allowed.

- If $P(0) = 3000$, the graph starts out with negative slope. So $P(t)$ decreases with t. As $P(t)$ decreases towards 2000, the slope $(6000 - 3P(t)\big)P(t)$, while remaining negative, gets closer and closer to zero. As the graph approachs height 2000, it becomes more and more horizontal. The graph cannot actually cross from above 2000 to below 2000, because to do so it would have to have negative slope for some value of P below 2000, which is not allowed.

These curves are sketched in the figure below. We conclude that for any initial population size $P(0)$, except $P(0) = 0$, the population size approachs 2000 as $t \to \infty$.



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Now we'll do an example in which we explictly solve the logistic growth equation.

Example 10

In 1986, the population of the world was 5 billion and was increasing at a rate of 2% per year. Using the logistic growth model with an assumed maximum population of 100 billion, predict the population of the world in the years 2000, 2100 and 2500.

Solution. Let $y(t)$ be the population of the world, in billions of people, at time $1986 + t$. The logistic growth model assumes

$$y' = ay(K - y)$$

where K is the carrying capacity and $a = \frac{b_0}{K}$.

First we'll determine the values of the constants a and K from the given data.

- We know that, if at time zero the population is below K, then as time increases the population increases, approaching the limit K as t tends to infinity. So in this problem K is the maximum population. That is, $K = 100$.
- We are also told that, at time zero, the percentage rate of change of population, $100\frac{y'}{y}$, is 2, so that, at time zero, $\frac{y'}{y} = 0.02$. But, from the differential equation, $\frac{y'}{y} = a(K-y)$. Hence at time zero, $0.02 = a(100 - 5)$, so that $a = \frac{2}{9500}$.

We now know a and K and can solve the (separable) differential equation

$$\begin{aligned}\frac{dy}{dt} = ay(K-y) &\implies \frac{dy}{y(K-y)} = a\,dt \implies \int \frac{1}{K}\Big[\frac{1}{y} - \frac{1}{y-K}\Big]dy = \int a\,dt \\ &\implies \frac{1}{K}[\ln|y| - \ln|y-K|] = at + C \\ &\implies \ln\frac{|y|}{|y-K|} = aKt + CK \implies \Big|\frac{y}{y-K}\Big| = De^{aKt}\end{aligned}$$

with $D = e^{CK}$. We know that y remains between 0 and K, so that $\left|\frac{y}{y-K}\right| = \frac{y}{K-y}$ and our solution obeys

$$\frac{y}{K-y} = De^{aKt}$$

At this stage, we know the values of the constants a and K, but not the value of the constant D. We are given that at $t = 0$, $y = 5$. Subbing in this, and the values of K and a,

$$\frac{5}{100-5} = De^0 \implies D = \frac{5}{95}$$

So the solution obeys the algebraic equation

$$\frac{y}{100-y} = \frac{5}{95}e^{2t/95}$$

which we can solve to get y as a function of t.

$$\begin{aligned}y = (100-y)\frac{5}{95}e^{2t/95} &\implies 95y = (500-5y)e^{2t/95} \\ &\implies \big(95 + 5e^{2t/95}\big)y = 500e^{2t/95} \\ &\implies y = \frac{500e^{2t/95}}{95 + 5e^{2t/95}} = \frac{100e^{2t/95}}{19 + e^{2t/95}} = \frac{100}{1 + 19e^{-2t/95}}\end{aligned}$$

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Finally,

- In the year 2000, $t = 14$ and $y = \frac{100}{1+19e^{-28/95}} \approx 6.6$ billion.
- In the year 2100, $t = 114$ and $y = \frac{100}{1+19e^{-228/95}} \approx 36.7$ billion.
- In the year 2200, $t = 514$ and $y = \frac{100}{1+19e^{-1028/95}} \approx 100$ billion.

Example 10

Mixing Problems

Example 11

At time $t = 0$, where t is measured in minutes, a tank with a 5–litre capacity contains 3 litres of water in which 1 kg of salt is dissolved. Fresh water enters the tank at a rate of 2 litres per minute and the fully mixed solution leaks out of the tank at the *varying* rate of $2t$ litres per minute.

(a) Determine the volume of solution $V(t)$ in the tank at time t.

(b) Determine the amount of salt $Q(t)$ in solution when the amount of water in the tank is at maximum.

Solution. (a) The rate of change of the volume in the tank, at time t, is $2 - 2t$, because water is entering at a rate 2 and solution is leaking out at a rate $2t$. Thus

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 2 - 2t \implies dV = (2-2t)\,dt \implies V = \int (2-2t)\,dt = 2t - t^2 + C$$

at least until $V(t)$ reaches either the capacity of the tank or zero. When $t = 0$, $V = 3$ so $C = 3$ and $V(t) = 3 + 2t - t^2$. Observe that $V(t)$ is at a maximum when $\frac{\mathrm{d}V}{\mathrm{d}t} = 2 - 2t = 0$, or $t = 1$.

(b) In the very short time interval from time t to time $t + dt$, $2t\,dt$ litres of brine leaves the tank. That is, the fraction $\frac{2t\,dt}{V(t)}$ of the total salt in the tank, namely $Q(t)\frac{2t\,dt}{V(t)}$ kilograms, leaves. Thus salt is leaving the tank at the rate

$$\frac{Q(t)\frac{2t\,dt}{V(t)}}{dt} = \frac{2tQ(t)}{V(t)} = \frac{2tQ(t)}{3+2t-t^2} \text{ kilograms per minute}$$

so

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{2tQ(t)}{3+2t-t^2} \implies \frac{dQ}{Q} = -\frac{2t}{3+2t-t^2} = -\frac{2t}{(3-t)(1+t)} = \frac{3/2}{t-3} + \frac{1/2}{t+1}$$
$$\implies \ln Q = \frac{3}{2}\ln|t-3| + \frac{1}{2}\ln|t+1| + C$$

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We are interested in the time interval $0 \le t \le 1$. In this time interval $|t-3| = 3-t$ and $|t+1| = t+1$ so

$$\ln Q = \frac{3}{2}\ln(3-t) + \frac{1}{2}\ln(t+1) + C$$

At $t = 0$, Q is 1 so

$$\ln 1 = \frac{3}{2}\ln(3-0) + \frac{1}{2}\ln(0+1) + C \implies C = \ln 1 - \frac{3}{2}\ln 3 - \frac{1}{2}\ln 1 = -\frac{3}{2}\ln 3$$

At $t = 1$

$$\ln Q = \frac{3}{2}\ln(3-1) + \frac{1}{2}\ln(1+1) - \frac{3}{2}\ln 3 = 2\ln 2 - \frac{3}{2}\ln 3 = \ln 4 - \ln 3^{3/2}$$

so $Q = \frac{4}{3^{3/2}}$.

Example 11

Example 12

A tank contains 1500 liters of brine with a concentration of 0.3 kg of salt per liter. Another brine solution, this with a concentration of 0.1 kg of salt per liter is poured into the tank at a rate of 20 li/min. At the same time, 20 li/min of the solution in the tank, which is stirred continuously, is drained from the tank.

(a) How many kilograms of salt will remain in the tank after half an hour?

(b) How long will it take to reduce the concentration to 0.2 kg/li?

Solution. Denote by $Q(t)$ the amount of salt in the tank at time t. In a very short time interval dt, the incoming solution adds $20\,dt$ liters of a solution carrying 0.1 kg/li. So the incoming solution adds $0.1 \times 20\,dt = 2\,dt$ kg of salt. In the same time interval $20\,dt$ liters is drained from the tank. The concentration of the drained brine is $\frac{Q(t)}{1500}$. So $\frac{Q(t)}{1500}20\,dt$ kg were removed. All together, the change in the salt content of the tank during the short time interval is

$$dQ = 2\,dt - \frac{Q(t)}{1500}20\,dt = \Big(2 - \frac{Q(t)}{75}\Big)dt$$

The rate of change of salt content per unit time is

$$\frac{dQ}{dt} = 2 - \frac{Q(t)}{75} = -\frac{1}{75}\big(Q(t) - 150\big)$$

The solution of this equation is

$$Q(t) = \big\{Q(0) - 150\big\}e^{-t/75} + 150$$

by Example 2, with $a = -\frac{1}{75}$ and $b = 150$. At time 0, $Q(0) = 1500 \times 0.3 = 450$. So

$$Q(t) = 150 + 300e^{-t/75}$$

(a) At $t = 30$

$$Q(30) = 150 + 300e^{-30/75} = 351.1 \text{ kg}$$

(b) $Q(t) = 0.2 \times 1500 = 300$ kg is achieved when

$$150 + 300e^{-t/75} = 300 \implies 300e^{-t/75} = 150 \implies e^{-t/75} = 0.5$$
$$\implies -\frac{t}{75} = \ln(0.5) \implies t = -75\ln(0.5) = 51.99 \text{ min}$$

Example 12

Interest on Investments

Suppose that you deposit $\$P$ in a bank account at time $t = 0$. The account pays $r\%$ interest per year compounded n times per year.

- The first interest payment is made at time $t = \frac{1}{n}$. Because the balance in the account during the time interval $0 < t < \frac{1}{n}$ is $\$P$ and interest is being paid for $\left(\frac{1}{n}\right)^{\text{th}}$ of a year, that first interest payment is $\frac{1}{n} \times \frac{r}{100} \times P$. After the first interest payment, the balance in the account is $P + \frac{1}{n} \times \frac{r}{100} \times P = \left(1 + \frac{r}{100n}\right)P$.
- The second interest payment is made at time $t = \frac{2}{n}$. Because the balance in the account during the time interval $\frac{1}{n} < t < \frac{2}{n}$ is $\left(1 + \frac{r}{100n}\right)P$ and interest is being paid for $\left(\frac{1}{n}\right)^{\text{th}}$ of a year, the second interest payment is $\frac{1}{n} \times \frac{r}{100} \times \left(1 + \frac{r}{100n}\right)P$. After the second interest payment, the balance in the account is $\left(1+\frac{r}{100n}\right)P + \frac{1}{n} \times \frac{r}{100} \times \left(1+\frac{r}{100n}\right)P = \left(1+\frac{r}{100n}\right)^2 P$.
- And so on.

In general, at time $t = \frac{m}{n}$ (just after the m^{th} interest payment), the balance in the account is

$$B(t) = \left(1 + \frac{r}{100n}\right)^m P = \left(1 + \frac{r}{100n}\right)^{nt} P \tag{4}$$

Three common values of n are 1 (interest is paid once a year), 12 (i.e. interest is paid once a month) and 365 (i.e. interest is paid daily). The limit $n \to \infty$ is called continuous compounding[1]. Under continuous compounding, the balance at time t is

$$B(t) = \lim_{n\to\infty} \left(1 + \frac{r}{100n}\right)^{nt} P$$

You may have already seen the limit

$$\lim_{x\to 0}(1 + x)^{a/x} = e^a \tag{5}$$

[1]There are banks that advertise continuous compounding. You can find some by googling "interest is compounded continuously and paid"

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If so, you can evaluate $B(t)$ by applying (5) with $x = \frac{r}{100n}$ and $a = \frac{rt}{100}$ (so that $nt = \frac{a}{x}$). As $n \to \infty$, $x \to 0$ so that

$$B(t) = \lim_{n\to\infty} \left(1 + \frac{r}{100n}\right)^{nt} P = \lim_{x\to 0}(1+x)^{a/x} P = e^a P = e^{rt/100} P \qquad (6)$$

If you haven't seen (5) before, that's OK. In the following example, we rederive (6) using a differential equation instead of (5).

Example 13

Suppose, again, that you deposit $\$P$ in a bank account at time $t = 0$, and that the account pays $r\%$ interest per year compounded n times per year, and denote by $B(t)$ the balance at time t. Suppose that you have just received an interest payment at time t. Then the next interest payment will be made at time $t + \frac{1}{n}$ and will be $\frac{1}{n} \times \frac{r}{100} \times B(t) = \frac{r}{100n} B(t)$. So, calling $\frac{1}{n} = h$,

$$B(t+h) = B(t) + \frac{r}{100} B(t) h \qquad \text{or} \qquad \frac{B(t+h) - B(t)}{h} = \frac{r}{100} B(t)$$

To get continuous compounding we take the limit $n \to \infty$ or, equivalently, $h \to 0$. This gives

$$\lim_{h\to 0} \frac{B(t+h) - B(t)}{h} = \frac{r}{100} B(t) \qquad \text{or} \qquad \frac{\mathrm{d}B}{\mathrm{d}t}(t) = \frac{r}{100} B(t)$$

By Example 2, with $a = \frac{r}{100}$ and $b = 0$, $B(t) = e^{rt/100} B(0) = e^{rt/100} P$, once again.

Example 13

Example 14

(a) A bank advertises that it compounds interest continuously and that it will double your money in ten years. What is the annual interest rate?

(b) A bank advertises that it compounds monthly and that it will double your money in ten years. What is the annual interest rate?

Solution. (a) Let the interest rate be $r\%$ per year. If you start with $\$P$, then after t years, you have $Pe^{rt/100}$, under continuous compounding. This was (6). After 10 years you have $Pe^{r/10}$. This is supposed to be $2P$, so

$$Pe^{r/10} = 2P \quad \Longrightarrow \quad e^{r/10} = 2 \quad \Longrightarrow \quad \frac{r}{10} = \ln 2 \quad \Longrightarrow \quad r = 10 \ln 2 = 6.93\%$$

(b) Let the interest rate be $r\%$ per year. If you start with $\$P$, then after t years, you have $P\big(1 + \frac{r}{100\times 12}\big)^{12t}$, under monthly compounding. This was (4). After 10 years you have $P\big(1 + \frac{r}{100\times 12}\big)^{120}$. This is supposed to be $2P$, so

$$P\Big(1 + \frac{r}{100\times 12}\Big)^{120} = 2P \quad \Longrightarrow \quad \Big(1 + \frac{r}{1200}\Big)^{120} = 2 \quad \Longrightarrow \quad 1 + \frac{r}{1200} = 2^{1/120}$$
$$\Longrightarrow \quad \frac{r}{1200} = 2^{1/120} - 1 \quad \Longrightarrow \quad r = 1200\big(2^{1/120} - 1\big) = 6.95\%$$

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Example 14

Example 15

A 25 year old graduate of UBC is given \$50,000 which is invested at 5% per year compounded continuously. The graduate also intends to deposit money continuously at the rate of \$2000 per year.

(a) Find a differential equation that $A(t)$ obeys, assuming that the interest rate remains 5%.

(b) Determine the amount of money in the account when the graduate is 65.

(c) At age 65, the graduate will withdraw money continuously at the rate of W dollars per year. If the money must last until the person is 85, what is the largest possible value of W?

Solution. (a) Let's consider what happens to A over a very short time interval from time t to time $t+\Delta t$. At time t the account balance is $A(t)$. During the (really short) specified time interval the balance remains very close to $A(t)$ and so earns interst of $\frac{5}{100}\times\Delta t\times A(t)$. During the same time interval, the graduate also deposits an additional $\$2000\Delta t$. So

$$A(t+\Delta t)\approx A(t)+0.05A(t)\Delta t+2000\Delta t \implies \frac{A(t+\Delta t)-A(t)}{\Delta t}\approx 0.05A(t)+2000$$

In the limit $\Delta t\to 0$, the approximation becomes exact and we get

$$\frac{\mathrm{d}A}{\mathrm{d}t}=0.05A+2000$$

(b) The amount of money at time t obeys

$$\frac{\mathrm{d}A}{\mathrm{d}t}=0.05A(t)+2{,}000=0.05\big(A(t)+40{,}000\big)$$

So by Example 2 (with $a=0.05$ and $b=-40{,}000$),

$$A(t)=\big(A(0)+40{,}000\big)e^{0.05t}-40{,}000$$

At time 0 (when the graduate is 25), $A(0)=50{,}000$, so the amount of money at time t is

$$A(t)=90{,}000\,e^{0.05t}-40,000$$

In particular, when the graduate is 65 years old, $t=40$ and

$$A(40)=90{,}000\,e^{0.05\times 40}-40,000=\$625{,}015.05$$

(c) When the graduate stops depositing money and instead starts withdrawing money at a rate W, the equation for A becomes

$$\frac{\mathrm{d}A}{\mathrm{d}t}=0.05A-W=0.05(A-20W)$$

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assuming that the interest rate remains 5%. This time, Example 2 (with $a = 0.05$ and $b = 20W$) gives

$$A(t) = \big(A(0) - 20W\big)e^{0.05t} + 20W$$

If we now reset our clock so that $t = 0$ when the graduate is 65, $A(0) = 625,015.05$. So the amount of money at time t is

$$A(t) = 20W + e^{0.05t}(625,015.05 - 20W)$$

We want the account to be depleted when the graduate is 85. So, we want $A(20) = 0$. This is the case if

$$\begin{aligned} 20W + e^{0.05\times 20}(625,015.05 - 20W) = 0 &\implies 20W + e(625,015.05 - 20W) = 0 \\ &\implies 20(e-1)W = 625,015.05e \\ &\implies W = \frac{625,015.05e}{20(e-1)} = \$49,437.96 \end{aligned}$$

Example 15

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