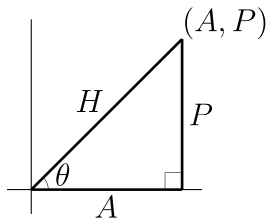
Trigonometric Functions

Here is a review the basic definitions and properties of the trigonometric functions. We provide a list of trig identities at the end.

1. Definitions

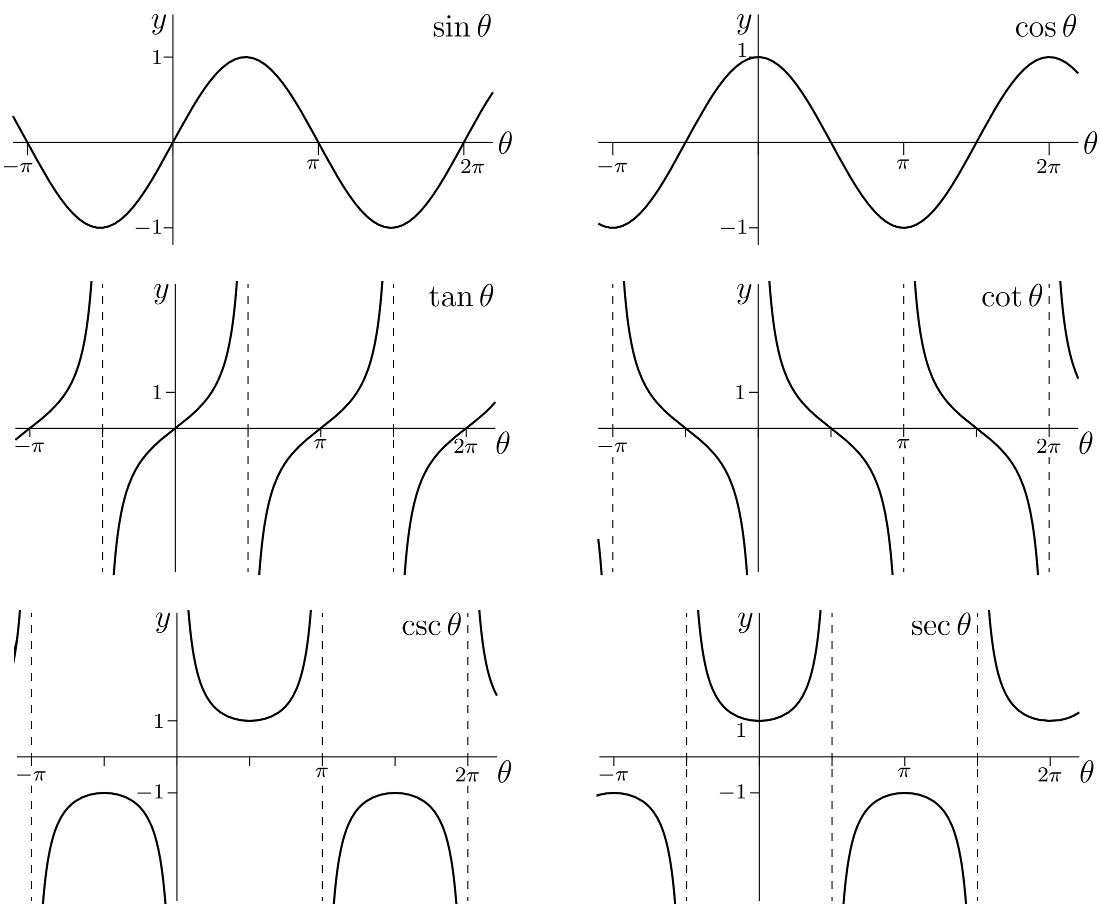
The trigonometric functions are defined as ratios of the lengths of the sides of a right angle triangle as shown below. In the figure, "H" stands for "Hypotenuse", "A" stands for "Adjacent side" and P stands for "oPposite side".



$$\sin\theta = \frac{P}{H} \qquad \cos\theta = \frac{A}{H} \qquad \tan\theta = \frac{P}{A} \tag{1a}$$

$$\csc\theta = \frac{H}{P} \qquad \sec\theta = \frac{H}{A} \qquad \cot\theta = \frac{A}{P} \tag{1b}$$

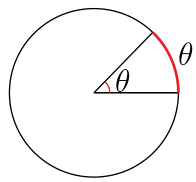
Here are the graphs of the trigonometric functions.



2. Radians

For use in calculus, angles are best measured in units called radians. By definition, an arc of length θ on a circle of radius one subtends an angle of θ radians at the center of the circle.

Because the circumference of a circle of radius one is 2π, we have

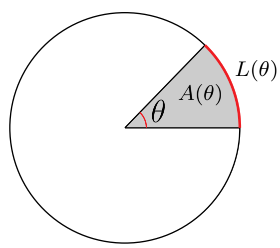


$$2\pi \text{ radians } = 360^\circ \qquad \pi \text{ radians } = 180^\circ \qquad \tfrac{\pi}{2} \text{ radians } = 90^\circ$$

$$\tfrac{\pi}{3} \text{ radians } = 60^\circ \qquad \tfrac{\pi}{4} \text{ radians } = 45^\circ \qquad \tfrac{\pi}{6} \text{ radians } = 30^\circ$$

3. Arclength and Area

Consider a circle of radius r. Denote by $L(\theta)$ the length of an arc of that circle that subtends an angle of θ radians. The angle subtended, θ, is the fraction $\frac{\theta}{2\pi}$ of the full circle. So $L(\theta)$ is the fraction $\frac{\theta}{2\pi}$ of the circumference of the full circle, which is $2\pi r$. Similarly, denote by $A(\theta)$ the area of a wedge of that circle that subtends an angle of θ radians. The area $A(\theta)$ is the fraction $\frac{\theta}{2\pi}$ of the area of the full circle, which is πr^2. So

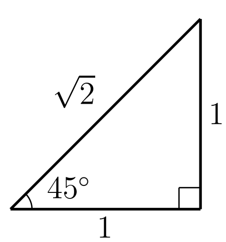


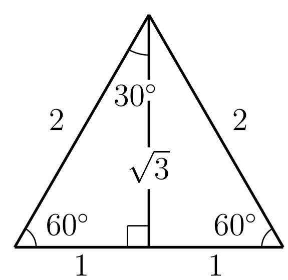
$$L(\theta) = \tfrac{\theta}{2\pi} \cdot 2\pi r = \theta\, r$$

$$A(\theta) = \tfrac{\theta}{2\pi} \cdot \pi r^2 = \tfrac{1}{2}\theta r^2$$

4. Special Triangles

From the triangles





we can read off the values of all of the trigonometric functions for the angles $\theta = \frac{\pi}{6} = 30^\circ$, $\theta = \frac{\pi}{4} = 45^\circ$ and $\theta = \frac{\pi}{3} = 60^\circ$.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$0^\circ = 0$ rad	0	1	0		1	
$30^\circ = \frac{\pi}{6}$ rad	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$45^\circ = \frac{\pi}{4}$ rad	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60^\circ = \frac{\pi}{3}$ rad	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
$90^\circ = \frac{\pi}{2}$ rad	1	0		1		0
$180^\circ = \pi$ rad	0	-1	0		-1	

The empty boxes mean that the trig function is undefined (i.e. $\pm\infty$) for that angle.

5. Trig Identities – Basic

The following identities are all immediate consequences of the definitions in (1).

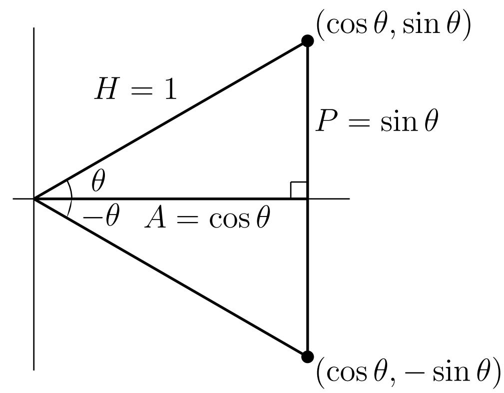
$$\csc\theta = \tfrac{1}{\sin\theta} \qquad \sec\theta = \tfrac{1}{\cos\theta} \qquad \tan\theta = \tfrac{\sin\theta}{\cos\theta} \qquad \cot\theta = \tfrac{1}{\tan\theta} = \tfrac{\cos\theta}{\sin\theta}$$

Because 2π radians is $360°$, the angles θ and $\theta + 2\pi$ are really the same, so

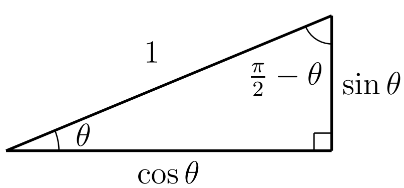
$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$$

The following trig identities are consequences of the figure to their right and Pythagoras' theorem, $A^2 + P^2 = H^2$.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta)$$



The following trig identities are consequences of the figure to their left.



$$\sin\left(\tfrac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta$$

6. Trig Identities – Addition and Double Angle Formulae

The following trig identities are derived below in §9.

$$\begin{aligned}
\sin(x+y) &= \sin x\cos y + \cos x\sin y \\
\sin(x-y) &= \sin x\cos y - \cos x\sin y \\
\cos(x+y) &= \cos x\cos y - \sin x\sin y \\
\cos(x-y) &= \cos x\cos y + \sin x\sin y
\end{aligned} \tag{2}$$

Setting $y = x$ gives

$$\begin{aligned}
\sin(2x) &= 2\sin x\cos x \\
\cos(2x) &= \cos^2 x - \sin^2 x \\
&= 2\cos^2 x - 1 \qquad \text{since } \sin^2 x = 1 - \cos^2 x \\
&= 1 - 2\sin^2 x \qquad \text{since } \cos^2 x = 1 - \sin^2 x
\end{aligned}$$

Solving $\cos(2x) = 2\cos^2 x - 1$ for $\cos^2 x$ and $\cos(2x) = 1 - 2\sin^2 x$ for $\sin^2 x$ gives

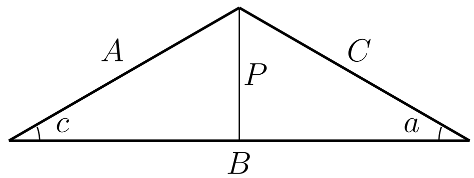
$$\begin{aligned}
\cos^2 x &= \tfrac{1+\cos(2x)}{2} \\
\sin^2 x &= \tfrac{1-\cos(2x)}{2}
\end{aligned}$$

7. Trig Identities – the Sine Law

The sine law says that, if a triangle has sides of length A, B and C and the angles opposite those sides are a, b and c, then

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$$

To derive $\frac{\sin a}{A} = \frac{\sin c}{C}$, drop a perpendicular to the side of length B from the vertex opposite it, as in the figure below.



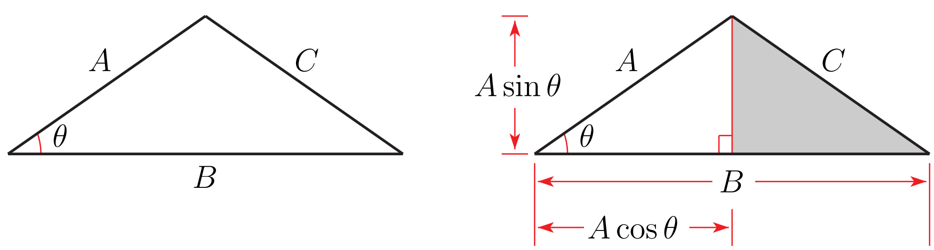
The perpedicular has length $P = A\,\sin c = C\,\sin a$. Dividing by $A \cdot C$ gives $\frac{\sin a}{A} = \frac{\sin c}{C}$.

8. Trig Identities – the Cosine Law

The cosine law says that, if a triangle has sides of length A, B and C and the angle opposite the side of length C is θ, then

$$C^2 = A^2 + B^2 - 2AB\cos\theta$$

Observe that, when $\theta = \frac{\pi}{2}$, this reduces to, (surpise!) Pythagoras' theorem $C^2 = A^2 + B^2$. To derive the Cosine law, apply Pythagoras to the shaded triangle in the right hand figure of



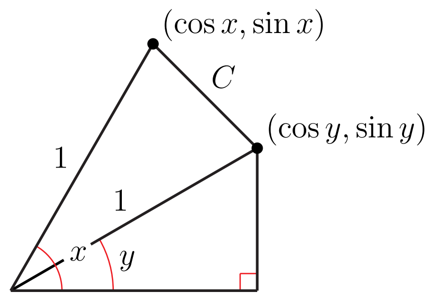
That triangle is a right triangle whose hypotenuse has length C and whose other two sides have lengths $\big(B - A\cos\theta\big)$ and $A\sin\theta$. So Pythagoras gives

$$\begin{aligned} C^2 &= \big(B - A\cos\theta\big)^2 + \big(A\sin\theta\big)^2 \\ &= B^2 - 2A\,B\,\cos\theta + A^2\cos^2\theta + A^2\sin^2\theta \\ &= B^2 - 2A\,B\,\cos\theta + A^2 \qquad\qquad (\text{since } \sin^2\theta + \cos^2\theta = 1) \end{aligned}$$

as desired.

9. Trig Identities – Derivation of the Addition Formulae

We now derive the addition formulae (2). The first step is to prove the fourth addition formula, $\cos(x-y) = \cos x \cos y + \sin x \sin y$. The angle, of the upper triangle in the figure,



that is opposite the side of length C is $x-y$. So, by the cosine law,

$$C^2 = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot \cos(x-y) = 2 - 2\cos(x-y)$$

But the side of length C joins the points $(\cos y, \sin y)$ and $(\cos x, \sin x)$ and so we also have, by Pythagoras,

$$\begin{aligned} C^2 &= (\cos y - \cos x)^2 + (\sin y - \sin x)^2 \\ &= \cos^2 y - 2\cos x \cos y + \cos^2 x + \sin^2 y - 2\sin x \sin y + \sin^2 x \\ &= 2 - 2\cos x \cos y - 2\sin x \sin y \qquad (\text{using } \cos^2 z + \sin^2 z = 1 \text{ for } z = x \text{ and } z = y) \end{aligned}$$

Setting the two formulae for C^2 equal to each other gives

$$\begin{aligned} 2 - 2\cos(x-y) &= 2 - 2\cos x \cos y - 2\sin x \sin y \\ \Longrightarrow \qquad -2\cos(x-y) &= -2\cos x \cos y - 2\sin x \sin y \\ \Longrightarrow \qquad \cos(x-y) &= \cos x \cos y + \sin x \sin y \end{aligned}$$

which is the fourth addition formula. Replacing y by $-y$ gives

$$\cos(x+y) = \cos x \cos(-y) + \sin x \sin(-y) = \cos x \cos y - \sin x \sin y$$

which is the third addition formula. Now, replacing x by $\frac{\pi}{2} - x$ gives

$$\cos\left(\tfrac{\pi}{2} - x + y\right) = \cos\left(\tfrac{\pi}{2} - x\right)\cos y - \sin\left(\tfrac{\pi}{2} - x\right)\sin y$$

Recalling that $\sin\left(\frac{\pi}{2} - z\right) = \cos z$ and $\cos\left(\frac{\pi}{2} - z\right) = \sin z$,

$$\sin\left(x - y\right) = \sin x \cos y - \cos x \sin y$$

which is the second addition formula. Finally, replacing y by $-y$ gives the first addition formula.

10. Trig Identities – Summary

The basic trig identities are

$$\tan\theta = \tfrac{\sin\theta}{\cos\theta} \qquad \csc\theta = \tfrac{1}{\sin\theta} \qquad \sec\theta = \tfrac{1}{\cos\theta} \qquad \cot\theta = \tfrac{1}{\tan\theta} = \tfrac{\cos\theta}{\sin\theta} \tag{T1}$$

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \tag{T2}$$

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta \tag{T3a}$$

$$\sin(\theta + \pi) = -\sin\theta \qquad \cos(\theta + \pi) = -\cos\theta \tag{T3b}$$

$$\sin(\tfrac{\pi}{2} - \theta) = \cos\theta \qquad \cos(\tfrac{\pi}{2} - \theta) = \sin\theta \tag{T3c}$$

$$\sin^2\theta + \cos^2\theta = 1 \tag{T4}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{T5}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \tag{T6}$$

$$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi \tag{T7a}$$

$$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi \tag{T7b}$$

The following trig identities are easily derived from the basic identities above. We use the "code" that the identities in (T4') are easily derived from the identity (T4), that the identity in (T5',T6') is easily derived by dividing the identities in (T5) and (T6) and that the identities in (T7") are easily derived from the identities in (T7) and (T7').

$$\tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta \tag{T4'}$$

$$\tan(2\theta) = \tfrac{2\tan\theta}{1-\tan^2\theta} \tag{T5',T6'}$$

$$\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \tag{T6'}$$

$$\cos^2\theta = \tfrac{1+\cos(2\theta)}{2} \qquad \sin^2\theta = \tfrac{1-\cos(2\theta)}{2} \qquad \tan^2\theta = \tfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi \tag{T7'}$$

$$\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$$

$$\tan(\theta + \varphi) = \tfrac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi} \qquad \tan(\theta - \varphi) = \tfrac{\tan\theta - \tan\varphi}{1 + \tan\theta\tan\varphi}$$

$$\sin\theta\cos\varphi = \tfrac{1}{2}\big\{\sin(\theta + \varphi) + \sin(\theta - \varphi)\big\} \tag{T7''}$$

$$\sin\theta\sin\varphi = \tfrac{1}{2}\big\{\cos(\theta - \varphi) - \cos(\theta + \varphi)\big\}$$

$$\cos\theta\cos\varphi = \tfrac{1}{2}\big\{\cos(\theta + \varphi) + \cos(\theta - \varphi)\big\}$$

$$\sin\alpha + \sin\beta = 2\sin\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2} \qquad \sin\alpha - \sin\beta = 2\cos\tfrac{\alpha+\beta}{2}\sin\tfrac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\tfrac{\alpha+\beta}{2}\cos\tfrac{\alpha-\beta}{2} \qquad \cos\alpha - \cos\beta = -2\sin\tfrac{\alpha+\beta}{2}\sin\tfrac{\alpha-\beta}{2}$$