The Fundamental Theorem of Calculus

The single most important tool used to evaluate integrals is called "The Fundamental Theorem of Calculus". It converts any table of derivatives into a table of integrals and vice versa. Here it is

Theorem 1 (Fundamental Theorem of Calculus)**.**

Let $f(x)$ be a function which is defined and continuous for $a \le x \le b$.

Part 1: Define, for $a \le x \le b$, $F(x) = \int_a^x f(t)\ dt$. Then $F(x)$ is differentiable and

$$F'(x) = f(x)$$

Part 2: Let $G(x)$ be any function which is defined and continuous on $[a, b]$ and which is also differentiable and obeys $G'(x) = f(x)$ for all $a < x < b$. Then

$$\int_a^b f(x)\ dx = G(b) - G(a) \qquad \text{or} \qquad \int_a^b G'(x)\ dx = G(b) - G(a)$$

A function $G(x)$ that obeys $G'(x) = f(x)$ is called an antiderivative of f. The form $\int_a^b G'(x)\ dx = G(b) - G(a)$ of the Fundamental Theorem is occasionally called the "net change theorem".

"Proof" of Part 1. By definition

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

For notational simplicity, let's only consider the case that f is alway nonnegative. Then

$$\begin{aligned} F(x+h) &= \text{the area of the region } \big\{ (t, y) \bigm| a \le t \le x+h,\ 0 \le y \le f(t) \big\} \\ F(x) &= \text{the area of the region } \big\{ (t, y) \bigm| a \le t \le x, \qquad 0 \le y \le f(t) \big\} \end{aligned}$$

So

$$F(x+h) - F(x) = \text{the area of the region } \big\{ (t, y) \bigm| x \le t \le x+h,\ 0 \le y \le f(t) \big\}$$

That's the more darkly shaded region in the figure

As t runs from x to $x = h$, $f(t)$ runs only over a very range of values, all close to $f(x)$. So the darkly shaded region is almost a rectangle of width h and height $f(x)$ and so has an area which is very close to $f(x)h$. Thus $\frac{F(x+h)-F(x)}{h}$ is very close to $f(x)$. In the limit $h \to 0$, $\frac{F(x+h)-F(x)}{h}$ becomes exactly $f(x)$, which is exactly what we want. We won't justify this limiting argument on a mathematically rigorous level (which is why we put quotation marks around "Proof", above), but it should at least look very reasonable to you. □

"Proof" of Part 2. We want to show that $\int_a^b f(t)\ dt = G(b) - G(a)$, or equivalently that $\int_a^b f(t)\ dt - G(b) + G(a) = 0$. We'll just rename b to x and show that

$$H(x) = \int_a^x f(t)\ dt - G(x) + G(a)$$

is always zero. This will imply, in particular, that $H(b) = \int_a^b f(t)\ dt - G(b) + G(a)$ is zero.

First we'll check if $H(x)$ is at least a constant, by computing the derivative

$$\begin{aligned} H'(x) &= \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\ dt - G'(x) \\ &= f(x) - f(x) \qquad \text{(by Part 1 and the hypothesis } G'(x) = f(x)) \\ &= 0 \end{aligned}$$

So $H(x)$ must be a constant function and the value of the constant is

$$H(a) = \int_a^a f(t)\ dt - G(a) + G(a) = 0$$

as we want. □

We'll first do some examples illustrating the use of part 1 of the Fundamental Theorem of Calculus. Then we'll move on to part 2.

Example 2 ($\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x e^{-t^2}\ dt$)

Find $\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x e^{-t^2}\ dt$.

Solution. We don't know how to evaluate the integral $\int_0^x e^{-t^2}\ dt$. In fact $\int_0^x e^{-t^2}\ dt$ cannot be expressed in terms of standard functions like polynomials, exponentials, trig functions and so on. Even so, we can find its derivative by just applying the first part of the Fundamental Theorem of Calculus with $f(t) = e^{-t^2}$ and $a = 0$. That gives

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x e^{-t^2}\ dt = e^{-x^2}$$

Example 2

Example 3 ($\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} e^{-t^2}\ dt$)

Find $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} e^{-t^2}\ dt$.

Solution. Once again, we will apply part 1 of the Fundamental Theorem of Calculus. But we must do so with some care. The Fundamental Theorem tells us how to compute the derivative of functions of the form $\int_a^x f(t)\ dt$. The integral $\int_0^{x^2} e^{-t^2}\ dt$ is *not* of the specified form because the upper limit of $\int_0^{x^2} e^{-t^2}\ dt$ is x^2 while the upper limit of $\int_a^x f(t)\ dt$ is x. The trick for getting around this obstacle is to define the auxiliary function

$$E(x) = \int_0^x e^{-t^2}\ dt$$

The Fundamental Theorem tells us that $E'(x) = e^{-x^2}$. (We found that in Example 2, above.) The integral of interest is

$$\int_0^{x^2} e^{-t^2}\ dt = E(x^2)$$

So by the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} e^{-t^2}\ dt = \frac{\mathrm{d}}{\mathrm{d}x}E(x^2) = 2x\,E'(x^2) = 2xe^{-x^4}$$

Example 3

Example 4 ($\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2} e^{-t^2}\ dt$)

Find $\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2} e^{-t^2}\ dt$.

Solution. Yet again, we can't just blindly apply the Fundamental Theorem. This time, not only is the upper limit of integration x^2 rather than x, but the lower limit of integration also depends on x, unlike the lower limit of the integral $\int_a^x f(t)\ dt$ of the Fundamental Theorem. Fortunately we can use the basic properties of integrals to split $\int_x^{x^2} e^{-t^2}\ dt$ into pieces whose derivatives we already know.

$$\int_x^{x^2} e^{-t^2}\ dt = \int_x^0 e^{-t^2}\ dt + \int_0^{x^2} e^{-t^2}\ dt = -\int_0^x e^{-t^2}\ dt + \int_0^{x^2} e^{-t^2}\ dt$$

So, by the previous two examples,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2} e^{-t^2}\ dt &= -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{-t^2}\ dt + \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} e^{-t^2}\ dt \\ &= -e^{-x^2} + 2xe^{-x^4}\end{aligned}$$

Example 4

We're almost ready for examples using part 2 of the Fundamental Theorem. We just need a little terminology and notation.

Definition 5.

(a) A function $F(x)$ whose derivative $F'(x) = f(x)$ is called an antiderivative of $f(x)$.

(b) The symbol $\int f(x)\, dx$ is read "the indefinite integral of $f(x)$". It stands for *all* functions having derivative $f(x)$. If $F(x)$ is any antiderivative of $f(x)$, and C is any constant, then the derivative of $F(x)+C$ is again $f(x)$, so that $F(x)+C$ is also an antiderivative of $f(x)$. Conversely, the difference between any two antiderivatives of $f(x)$ must be a constant, because a function has derivative zero if and only if it is a constant. So $\int f(x)\, dx = F(x) + C$, with the consant C called an "arbitrary constant" or "constant of integration".

(c) The symbol $\int f(x)\, dx\Big|_a^b$ means

- take any function whose derivative is $f(x)$. Call the function you have chosen $F(x)$.
- Then $\int f(x)\, dx\Big|_a^b$ means $F(b) - F(a)$.

We'll later develop some strategies for computing more complicated integrals. But for now, we'll stick to integrals that are simple enough that we can just guess the answer.

Example 6

Find $\int_1^2 x\, dx$.

Solution. The main step in evaluating an integral like this is finding the indefinite integral of x. That is, finding a function whose derivative is x. So we have to think back and try and remember a function whose derivative is something like x. We recall that

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$$

We want the derivative to be x to the power one, so we should take $n = 2$. So far, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} x^2 = 2x$$

This derivative is just a factor of 2 larger than we want. So we divide the whole equation by 2. We now have

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{1}{2}x^2\Big) = x$$

which says that $\frac{1}{2}x^2$ is an antiderivative for x. Once one has an antiderivative, it is easy to compute the definite integral

$$\int_1^2 x\, dx = \overbrace{\frac{1}{2}x^2}^{\text{a function with derivative } x.}\Big|_1^2 = \frac{1}{2}2^2 - \frac{1}{2}1^2 = \frac{3}{2}$$

as well as the indefinite integral

$$\int x\ dx = \frac{1}{2}x^2 + C$$

Example 6

Example 7

Find $\int_0^{\pi/2} \sin x\ dx$.

Solution. Once again, the crux of the solution is guessing a function whose derivative is $\sin x$. The standard derivative that comes closest to $\sin x$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$

which is the derivative we want, multiplied by a factor of -1. So we multiply the whole equation by -1.

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(-\cos x\big) = \sin x$$

This tells us that the indefinite integral $\int \sin x dx = -\cos x + C$. To answer the question, we don't need the whole indefinite integral. We just need one function whose derivative is $\sin x$, that is, one antiderivative of $\sin x$. We'll use the simplest one, namely $-\cos x$. The prescribed integral is

$$\int_0^{\pi/2} \sin x\ dx = \overbrace{-\cos x}^{\substack{\text{a function with}\\ \text{derivative } \sin x.}}\Big|_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = -0 + 1 = 1$$

Example 7

Example 8

Find $\int_1^2 \frac{1}{x}\ dx$.

Solution. Once again, the crux of the solution is guessing a function whose derivative is $\frac{1}{x}$. Our standard way to get derivatives that are powers of x is

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$

That is not going to work this time, since to get $\frac{1}{x}$ on the right hand side we need to take $n = 0$, which gives a right hand side of 0. Fortunately, we also have

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$$

which is exactly the derivative we want. We're now ready to compute the prescribed integral.

$$\int_1^2 \frac{1}{x}\ dx = \overbrace{\ln x}^{\substack{\text{a function with}\\ \text{derivative } 1/x.}}\Big|_1^2 = \ln 2 - \ln 1 = \ln 2$$

Example 8

Example 9

Find $\int_{-2}^{-1} \frac{1}{x}\, dx$.

Solution. As we saw in the last example,

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$$

But we cannot use $\ln x$ in this example because, here, x runs from -2 to -1, and in particular is negative, and $\ln x$ is not defined when x is negative. A variant of $\ln x$ which is defined when x is negative is $\ln(-x) = \ln|x|$, so let's compute

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$$

by the chain rule. Fortunately, this is exactly the derivative we want, so we're now ready to compute the prescribed integral.

$$\int_{-2}^{-1} \frac{1}{x}\, dx = \overbrace{\ln(-x)}^{\substack{\text{a function with} \\ \text{derivative } 1/x.}} \Big|_{-2}^{-1} = \ln 1 - \ln 2 = \ln \frac{1}{2}$$

The statements

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x} \qquad \text{for } x > 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln(-x) = \frac{1}{x} \qquad \text{for } x < 0$$

are often combined into

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln|x| = \frac{1}{x}$$

Example 9

Example 10

Find $\int_{-1}^{1} \frac{1}{x^2}\, dx$.

Solution. Beware that this is a particularly nasty example, which illustrates a booby trap hidden in the Fundamental Theorem of Calculus. The booby trap explodes when the theorem is applied sloppily. The sloppy solution starts, as normal, with the observation that

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x} = -\frac{1}{x^2}$$

so that

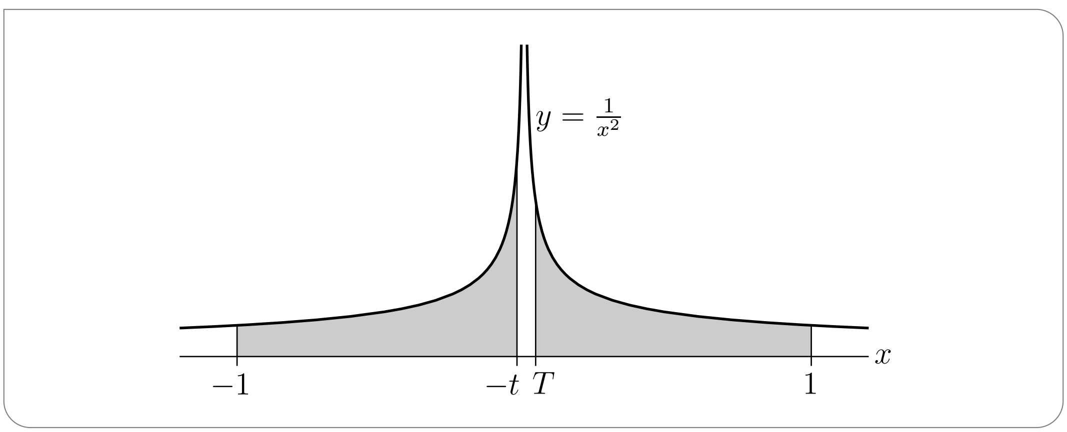
$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(-\frac{1}{x}\Big) = \frac{1}{x^2}$$

and it appears that

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \Big[\overbrace{-\frac{1}{x}}^{\substack{\text{a function with}\\ \text{derivative } 1/x^2.}}\Big]_{-1}^{1} = -\frac{1}{1} - \Big(-\frac{1}{-1}\Big) = -2$$

Unfortunately, this answer cannot be correct. In fact it is ridiculous. The integrand $\frac{1}{x^2} > 0$, so the integral has to be positive. The flaw in the argument is that the Fundamental Theorem of Calculus, which says that if $F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$, is applicable only when $F'(x)$ exists and equals $f(x)$ for **all** x between a and b. In this case $F'(x) = \frac{1}{x^2}$ does not exist for $x = 0$. So we cannot apply the Fundamental Theorem of Calculus as we tried to above.

An integral, like $\int_{-1}^{1} \frac{1}{x^2}\,dx$, whose integrand is undefined somewhere in the domain of integration is called improper. We'll later give a more thorough treatment of improper integrals. For now, we'll just say that the correct way to define improper integrals is as a limit of well–defined approximating integrals. The approximating integrals have restricted domains of integration that exclude the "bad" points where the integrand is undefined. In the current example, the original domain of integration is $-1 \le x \le 1$. The domains of integration of the approximating integrals exclude from $[-1, 1]$ small intervals around $x = 0$. The shaded area in the figure below illustrates a typical approximating integral, whose domain of integration consists of the original domain of integration, $[-1, 1]$, but with the interval $[-t, T]$ excluded.



The full domain of integration is only recovered in the limit $t, T \to 0$.

For this example, the correct computation is

$$\begin{aligned}\int_{-1}^{1} \frac{1}{x^2}\,dx &= \lim_{t\to 0+} \int_{-1}^{-t} \frac{1}{x^2}\,dx \ + \lim_{T\to 0+} \int_{T}^{1} \frac{1}{x^2}\,dx \\ &= \lim_{t\to 0+} \Big[-\frac{1}{x}\Big]_{-1}^{-t} + \lim_{T\to 0+} \Big[-\frac{1}{x}\Big]_{T}^{1}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to 0+}\Big[\Big(-\frac{1}{-t}\Big)-\Big(-\frac{1}{-1}\Big)\Big] + \lim_{T\to 0+}\Big[\Big(-\frac{1}{1}\Big)-\Big(-\frac{1}{T}\Big)\Big] \\
&= \lim_{t\to 0+}\frac{1}{t} + \lim_{T\to 0+}\frac{1}{T} - 2 \\
&= +\infty
\end{aligned}
$$

Example 10

The above examples have illustrated how we can use the Fundamental Theorem of Calculus to convert knowledge of derivatives into knowledge of integrals. We are now in a position to easily built a table of integrals. Here is a short table of the most important derivatives that we know.

$F(x)$	1	x^p	$\sin x$	$\cos x$	$\tan x$	e^x	$\ln x$	$\arcsin x$	$\arctan x$
$f(x) = F'(x)$	0	px^{p-1}	$\cos x$	$-\sin x$	$\sec^2 x$	e^x	$\frac{1}{x}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$

And here is the corresponding short table of integrals.

$f(x)$	$F(x) = \int f(x)\ dx$
1	$x + C$
x^p	$\frac{x^{p+1}}{p+1} + C$ if $p \neq -1$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
e^x	$e^x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\frac{1}{1+x^2}$	$\arctan x + C$