Improper Integrals

To this point we have only considered integrals $\int_a^b f(x)\,dx$ with

- the limits of integration a and b finite and
- the integrand $f(x)$ bounded (and in fact continuous except possibly for finitely many jump discontinuities)

An integral having either an infinite limit of integration or an unbounded integrand is called improper. Here are two examples

$$\int_0^\infty \frac{dx}{1+x^2} \qquad \int_0^1 \frac{dx}{x}$$

The first has an infinite domain of integration and the integrand of the second tends to ∞ as x approaches the left end of the domain of integration. We'll start with an example that illustrates the traps that you can fall into if you treat such integrals sloppily. Then we'll see how to treat them carefully.

Example 1

The computation

$$\int_{-1}^1 \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^1 = \frac{1}{-1} - \frac{-1}{-1} = -2$$

is *wrong*. In fact, the answer is ridiculous. The integrand $\frac{1}{x^2} > 0$, so the integral has to be positive. The flaw in the argument is that the fundamental theorem of calculus, which says that if $F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$, is applicable only when $F'(x)$ exists and equals $f(x)$ for *all* $a \le x \le b$. In this case $F'(x) = \frac{1}{x^2}$ does not exist for $x = 0$. The given integral is improper. We'll see later that the correct answer is $+\infty$.

Example 1

  The careful way to treat an integral like $\int_a^\infty \frac{dx}{1+x^2}$ that has a bounded integrand but has a domain of integration that extends to $+\infty$ is to approximate the integral by one with a bounded domain of integration, like $\int_a^R \frac{dx}{1+x^2}$, and then take the limit $R \to \infty$. Here is the
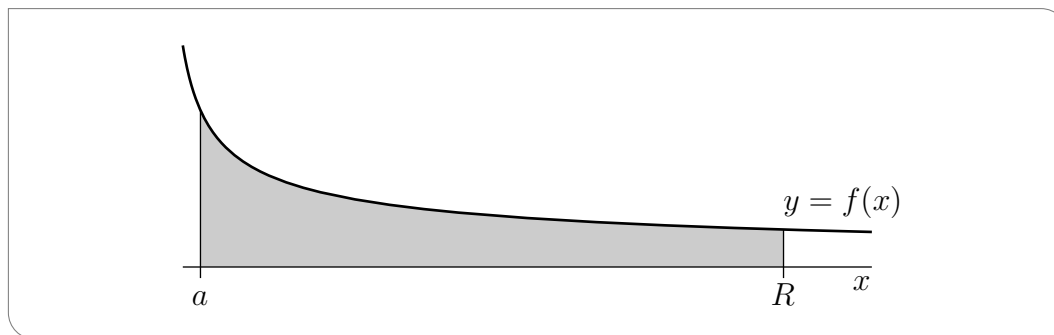
formal definition and some variations. We'll work through some examples in detail shortly.

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Definition 2.

(a) If the integral $\int_a^R f(x)\, dx$ exists for all $R > a$, then

$$\int_a^\infty f(x)\, dx = \lim_{R \to \infty} \int_a^R f(x)\, dx$$

when the limit exists (and is finite).

(b) If the integral $\int_r^b f(x)\, dx$ exists for all $r < b$, then

$$\int_{-\infty}^b f(x)\, dx = \lim_{r \to -\infty} \int_r^b f(x)\, dx$$
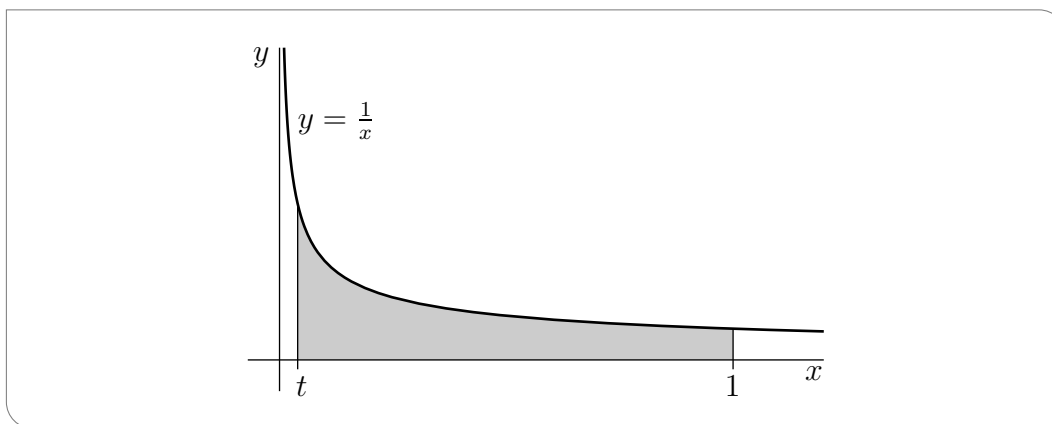
when the limit exists (and is finite).

(c) If the integral $\int_r^R f(x)\, dx$ exists for all $r < R$, then

$$\int_{-\infty}^\infty f(x)\, dx = \lim_{r \to -\infty} \int_r^c f(x)\, dx + \lim_{R \to \infty} \int_c^R f(x)\, dx$$

when both limits exist (and are finite). Any c can be used.

When the limit(s) exist, the integral is said to be convergent. Otherwise it is said to be divergent.

The careful way to treat an integral like $\int_0^1 \frac{dx}{x}$ that has a finite domain of integration but whose integrand is unbounded near one limit of integration is to approximate the integral by one with a domain of integration on which the integrand is bounded, like $\int_t^1 \frac{dx}{x}$, with $t > 0$, and then take the limit $t \to 0+$. Here is the formal definition and some variations.

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If an integral has more than one "source of impropriety", for example an infinite domain of integration and an integrand with an unbounded integrand or multiple infinite discontinuities, then you split it up into a sum of integrals with a single "source of impropriety" in each. For the integral, as a whole, to converge every term in that sum has to converge. For example, the integral $\int_{-\infty}^{\infty} \frac{dx}{(x-2)x^2}$ has a domain of integration that extends to both $+\infty$ and $-\infty$ and, in addition, has an integrand which is singular (i.e. becomes infinite) at $x = 2$ and at $x = 0$. So we would write

$$\int_{-\infty}^{\infty} \frac{dx}{(x-2)x^2} = \int_{-\infty}^{-1} \frac{dx}{(x-2)x^2} + \int_{-1}^{0} \frac{dx}{(x-2)x^2} + \int_0^1 \frac{dx}{(x-2)x^2} + \int_1^2 \frac{dx}{(x-2)x^2}$$

$$+ \int_2^3 \frac{dx}{(x-2)x^2} + \int_3^{\infty} \frac{dx}{(x-2)x^2}$$

(Here the "break point" -1 could be replaced by any number strictly less than 0; 1 could be replaced by any number strictly between 0 and 2; and 3 could be replaced by any number strictly bigger than 2.) On the right hand side

- the first integral has domain of integration extending to $-\infty$

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- the second integral has an integrand that becomes unbounded as $x \to 0-$,
- the third integral has an integrand that becomes unbounded as $x \to 0+$,
- the fourth integral has an integrand that becomes unbounded as $x \to 2-$,
- the fifth integral has an integrand that becomes unbounded as $x \to 2+$, and
- the last integral has domain of integration extending to $+\infty$.

We are now ready for some (important) examples.

Example 4 ($\int_1^\infty \frac{dx}{x^p}$)

Fix any $p > 0$. The domain of the integral $\int_1^\infty \frac{dx}{x^p}$ extends to $+\infty$ and the integrand $\frac{1}{x^p}$ is continuous and bounded on the whole domain. So the definition of this integral is

$$\int_1^\infty \frac{dx}{x^p} = \lim_{R \to \infty} \int_1^R \frac{dx}{x^p}$$

When $p \neq 1$, one antiderivative of x^{-p} is $\frac{x^{-p+1}}{-p+1}$, and when $p = 1$ and $x > 0$, one antiderivative of x^{-p} is $\ln x$. So

$$\int_1^\infty \frac{dx}{x^p} = \lim_{R \to \infty} \begin{cases} \frac{1}{1-p}\left[R^{1-p} - 1\right] & \text{if } p \neq 1 \\ \ln R & \text{if } p = 1 \end{cases}$$

As $R \to \infty$, $\ln R$ tends to ∞ and R^{1-p} tends to ∞ when $1 - p > 0$ (i.e. the exponent is positive) and tends to 0 when $1 - p < 0$ (i.e. the exponent is negative). So

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \text{divergent} & \text{if } p \leq 1 \\ \frac{1}{p-1} & \text{if } p > 1 \end{cases}$$

Example 4

Example 5 ($\int_0^1 \frac{dx}{x^p}$)

Again fix any $p > 0$. The domain of integration of the integral $\int_0^1 \frac{dx}{x^p}$ is finite, but the integrand $\frac{1}{x^p}$ becomes unbounded as x approaches the left end, 0, of the domain of integration. So the definition of this integral is

$$\int_0^1 \frac{dx}{x^p} = \lim_{t \to 0+} \int_t^1 \frac{dx}{x^p}$$

When $p \neq 1$, one antiderivative of x^{-p} is $\frac{x^{-p+1}}{-p+1}$, and when $p = 1$ and $x > 0$, one antiderivative of x^{-p} is $\ln x$. So

$$\int_0^1 \frac{dx}{x^p} = \lim_{t \to 0+} \begin{cases} \frac{1}{1-p}\left[1 - t^{1-p}\right] & \text{if } p \neq 1 \\ -\ln t & \text{if } p = 1 \end{cases}$$

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As $t \to 0+$, $\ln t$ tends to $-\infty$ and t^{1-p} tends to 0 when $1 - p > 0$ (i.e. the exponent is positive) and tends to ∞ when $1 - p < 0$ (i.e. the exponent is negative). So

$$\int_0^1 \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \text{divergent} & \text{if } p \geq 1 \end{cases}$$

Example 5

Example 6 ($\int_0^\infty \frac{dx}{x^p}$)

Yet again fix $p > 0$. This time the domain of integration of the integral $\int_0^\infty \frac{dx}{x^p}$ extends to $+\infty$, and in addition the integrand $\frac{1}{x^p}$ becomes unbounded as x approaches the left end, 0, of the domain of integration. So we have to split it up.
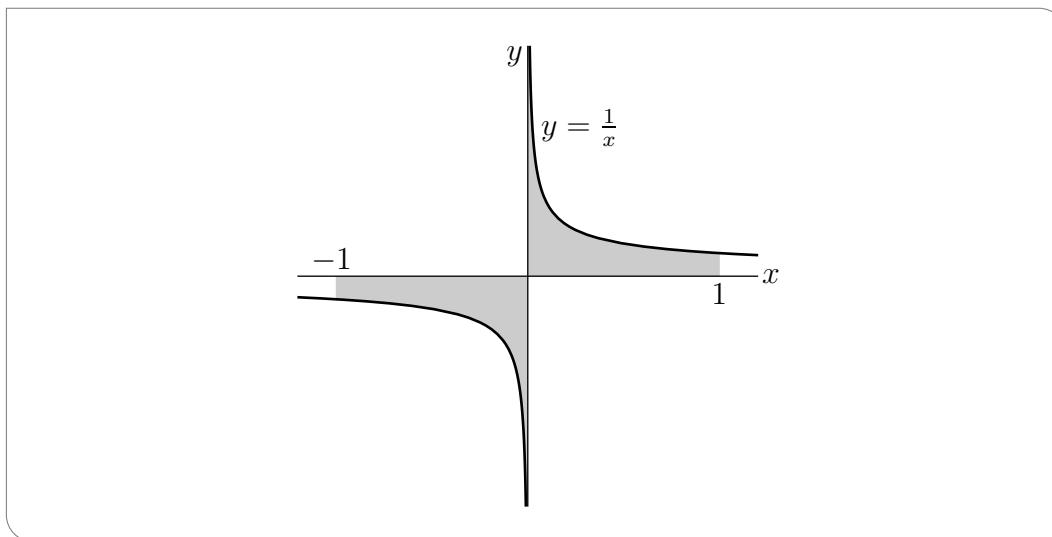
$$\int_0^\infty \frac{dx}{x^p} = \int_0^1 \frac{dx}{x^p} + \int_1^\infty \frac{dx}{x^p}$$

We saw, in Example 5, that the first integral diverged whenever $p \geq 1$, and we also saw, in Example 4, that the second integral diverged whenever $p \leq 1$. So the integral $\int_0^\infty \frac{dx}{x^p}$ diverges for all values of p.

Example 6

Example 7 ($\int_{-1}^1 \frac{dx}{x}$)

This is a pretty subtle example. It looks from the sketch below that the signed area to the

left of the y–axis should exactly cancel the area to the right of the y–axis making the value of the integral $\int_{-1}^1 \frac{dx}{x}$ exactly zero. But both of the integrals

$$\int_0^1 \frac{dx}{x} = \lim_{t \to 0+} \int_t^1 \frac{dx}{x} = \lim_{t \to 0+} \Big[\ln x \Big]_t^1 = \lim_{t \to 0+} \ln \frac{1}{t} = +\infty$$

$$\int_{-1}^0 \frac{dx}{x} = \lim_{T \to 0-} \int_{-1}^T \frac{dx}{x} = \lim_{T \to 0-} \Big[\ln |x| \Big]_{-1}^T = \lim_{T \to 0-} \ln |T| = -\infty$$

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diverge so $\int_{-1}^{1} \frac{dx}{x}$ diverges.

Don't make the mistake of thinking that $\infty - \infty = 0$. It is undefined — and for good reason. For example, we have just seen that the area to the right of the y–axis is

$$\lim_{t \to 0+} \int_t^1 \frac{dx}{x} = +\infty$$

and that the area to the left of the y–axis is (substitute $-7t$ for T above)

$$\lim_{t \to 0+} \int_{-1}^{-7t} \frac{dx}{x} = -\infty$$

If $\infty - \infty = 0$, the following limit should be 0.

$$\lim_{t \to 0+} \left[\int_t^1 \frac{dx}{x} + \int_{-1}^{-7t} \frac{dx}{x} \right] = \lim_{t \to 0+} \left[\ln \frac{1}{t} + \ln |-7t| \right] = \lim_{t \to 0+} \left[\ln \frac{1}{t} + \ln(7t) \right] = \lim_{t \to 0+} \ln 7$$
$$= \ln 7$$

This appears to give $\infty - \infty = \ln 7$. Of course the number 7 was picked at random. You can make $\infty - \infty$ be any number at all, by making a suitable replacement for 7.

Example 7

Example 8 (Example 1 revisited)

The careful computation of the integral of Example 1 is

$$\int_{-1}^1 \frac{1}{x^2} \, dx = \lim_{T \to 0-} \int_{-1}^T \frac{1}{x^2} \, dx + \lim_{t \to 0+} \int_t^1 \frac{1}{x^2} \, dx$$

$$= \lim_{T \to 0-} \left[-\frac{1}{x} \right]_{-1}^T + \lim_{t \to 0+} \left[-\frac{1}{x} \right]_t^1$$

$$= \infty + \infty$$

Example 8

Example 9 ($\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$)

Since

$$\lim_{R \to \infty} \int_0^R \frac{dx}{1+x^2} = \lim_{R \to \infty} \left[\arctan x \right]_0^R = \lim_{R \to \infty} \arctan R = \frac{\pi}{2}$$

$$\lim_{r \to -\infty} \int_r^0 \frac{dx}{1+x^2} = \lim_{r \to -\infty} \left[\arctan x \right]_r^0 = \lim_{r \to -\infty} - \arctan r = \frac{\pi}{2}$$

The integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ converges and takes the value π.

Example 9

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For what values of p does $\int_e^\infty \frac{dx}{x(\ln x)^p}$ converge?

Solution. For $x \geq e$, the denominator $x(\ln x)^p$ is never zero. So the integrand is bounded on the entire domain of integration and this integral is improper only because the domain of integration extends to $+\infty$ and we proceed as usual. We have

$$\int_e^\infty \frac{dx}{x(\ln x)^p} = \lim_{R\to\infty} \int_e^R \frac{dx}{x(\ln x)^p}$$

$$= \lim_{R\to\infty} \int_1^{\ln R} \frac{du}{u^p} \qquad \text{with } u = \ln x, \ du = \frac{dx}{x}$$

$$= \lim_{R\to\infty} \begin{cases} \frac{1}{1-p}\left[(\ln R)^{1-p} - 1\right] & \text{if } p \neq 1 \\ \ln(\ln R) & \text{if } p = 1 \end{cases}$$

$$= \begin{cases} \text{divergent} & \text{if } p \leq 1 \\ \frac{1}{p-1} & \text{if } p > 1 \end{cases}$$

just as in Example 4, but with R replaced by $\ln R$.

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