Techniques of Integration — Partial Fractions

Partial fractions is the name given to a technique of integration that may be used to integrate any ratio of polynomials. A ratio of polynomials is called a rational function. Suppose that $N(x)$ and $D(x)$ are polynomials. The basic strategy is to write $\frac{N(x)}{D(x)}$ as a sum of very simple, easy to integrate rational functions, namely

- polynomials (which are needed only if the degree[1] of $N(x)$ is equal to or strictly bigger than the degree of $D(x)$) and
- rational functions of the particularly simple form $\frac{A}{(ax+b)^n}$ and
- rational functions of the form $\frac{Ax+B}{(ax^2+bx+c)^m}$. (We will not cover this case.)

That is, to find the integral of the rational function on the far right hand side of

$$x + \frac{1}{x+1} + \frac{1}{x-1} = \frac{x(x+1)(x-1) + (x-1) + (x+1)}{(x+1)(x-1)} = \frac{x^3+x}{x^2-1} \tag{1}$$

you rewrite it as the left hand side and then integrate x and $\frac{1}{x+1}$ and $\frac{1}{x-1}$. So the main problem is to write a complicated rational function as a sum of simple pieces. The techique that will be used is based on two observations about (1).

- The denominators on the left hand side of (1) are the factors of the denominator $x^2-1 = (x-1)(x+1)$ on the right hand side of (1).
- Use $P(x)$ to denote the polynomial on the left hand side (i.e. $P(x) = x$) and $N(x)$ to denote the numerator of the right hand side (i.e. $N(x) = x^3+x$) and $D(x)$ to denote the denominator of the right hand side (i.e. $D(x) = x^2-1$). Then highest degree term in $N(x)$ is x^3. It came from multiplying $P(x)$ by $D(x)$. In particular the degree of $N(x)$ is the sum of the degree $P(x)$ and the degree of $D(x)$. The presence of a polynomial on the left hand side is signalled on the right hand side by the fact that the degree of the numerator is at least as large as the degree of the denominator.

We'll introduce the technique through some examples.

Example 1 ($\int \frac{x-3}{x^2-3x+2}\, dx$)

In this example, we integrate $\frac{N(x)}{D(x)} = \frac{x-3}{x^2-3x+2}$.

Step 1. We first check to see if the degree of the numerator, $N(x)$, is strictly smaller than the degree of the denominator $D(x)$. In this example, the numerator, $x-3$, has degree one and that is indeed strictly smaller than the degree of the denominator, x^2-3x+2, which is two. In this case, the first step is not needed and we move on to step 2.

Step 2. The second step is to factor the denominator

$$x^2 - 3x + 2 = (x-1)(x-2)$$

[1] The degree of a polynomial is the largest power of x. For example, the degree of $2x^3+4x^2+6x+8$ is three.

Step 3. The third step is to write $\frac{x-3}{x^2-3x+2}$ in the form

$$\frac{x-3}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2}$$

for some constants A and B. To determine the values of the constants A, B, we put the right hand side back over the common denominator $(x-1)(x-2)$.

$$\frac{x-3}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.

$$x-3 = A(x-2) + B(x-1)$$

There are a couple of different ways to determine the values of A and B from this equation.

The conceptually clearest procedure is to write the right hand side as a polynomial in standard form (i.e. collect up all x terms and all constant terms)

$$x-3 = (A+B)x + (-2A-B)$$

For these two polynomials to be the same, the coefficient of x on the left hand side and the coefficient of x on the right hand side must be the same. Similarly the coefficients of x^0 (i.e. the constant terms) must match. This gives us a system of two equations.

$$A+B=1 \qquad -2A-B=-3$$

in the two unknowns A, B. We can solve this system by using the first equation, namely $A+B=1$, to determine A in terms of B: $\quad A=1-B$. Substituting this into the remaining equation eliminates the A from second equation, leaving one equation in the one unknown B.

$$\begin{aligned}
& A = 1-B & -2A-B &= -3 \\
\Rightarrow & & -2(1-B)-B &= -3 \\
\Rightarrow & & B &= -1 \quad A = 1-B = 1-(-1) = 2
\end{aligned}$$

There is also a second, more efficient, procedure for determining A and B from

$$x-3 = A(x-2) + B(x-1)$$

This equation must be true for all values of x. In particular, it must be true for $x=1$. When $x=1$, the factor $(x-1)$ multiplying B is exactly zero. So B disappears from the equation, leaving us with an easy equation to solve for A:

$$x-3\Big|_{x=1} = A(x-2)\Big|_{x=1} + B(x-1)\Big|_{x=1} \Longrightarrow -2 = -A \Longrightarrow A = 2$$

Similarly, when $x=2$, the factor $(x-2)$ multiplying A is exactly zero. So A disappears from the equation, leaving us with an easy equation to solve for B:

$$x-3\Big|_{x=2} = A(x-2)\Big|_{x=2} + B(x-1)\Big|_{x=2} \Longrightarrow -1 = B \Longrightarrow B = -1$$

Step 4. The final step is to integrate.

$$\int \frac{x-3}{x^2-3x+2}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{-1}{x-2}\,dx = 2\ln|x-1| - \ln|x-2| + C$$

Example 1

Example 2 ($\int \frac{3x^3-8x^2+4x-1}{x^2-3x+2}\,dx$)

In this example, we integrate $\frac{N(x)}{D(x)} = \frac{3x^3-8x^2+4x-1}{x^2-3x+2}$.

Step 1. We first check to see if the degree of the numerator $N(x)$ is strictly smaller than the degree of the denominator $D(x)$. In this example, the numerator, $3x^3-8x^2+4x-1$, has degree three and the denominator, x^2-3x+2, has degree two. As $3 > 2$, we have to implement the first step. The goal of the first step is to write $\frac{N(x)}{D(x)}$ in the form

$$\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$$

with $P(x)$ being a polynomial and $R(x)$ being a polynomial of degree strictly smaller than the degree of $D(x)$. The right hand side is $\frac{P(x)D(x)+R(x)}{D(x)}$, so we have to express the numerator in the form $N(x) = P(x)D(x) + R(x)$, with $P(x)$ and $R(x)$ being polynomials and with the degree of R being strictly smaller than the degree of D. $P(x)D(x)$ is a sum of expressions of the form $ax^nD(x)$. We want to pull as many expressions of this form as possible out of the numerator $N(x)$, leaving only a low degree remainder $R(x)$.

This step is accomplished by long division — the same long division you learned in public school, but with the base 10 replaced by x. We start by observing that to get from the highest degree term in the denominator (x^2) to the highest degree term in the numerator ($3x^3$), we have to multiply by $3x$. So we write,

$$\begin{array}{r|l} & 3x \\ \cline{2-2} x^2-3x+2 & 3x^3 - 8x^2 + 4x - 1 \end{array}$$

(The denominator is on the left, the numerator is on the right and $3x$ is written above the highest order term of the numerator. Alway put lower powers of x to the right of higher powers of x.) Now we subtract $3x$ times the denominator, x^2-3x+2, which is $3x^3-9x^2+6x$, from the numerator.

$$\begin{array}{r|l} & 3x \\ \cline{2-2} x^2-3x+2 & 3x^3 - 8x^2 + 4x - 1 \\ & 3x^3 - 9x^2 + 6x \qquad \longleftarrow 3x(x^2-3x+2) \\ \cline{2-2} & \qquad\; x^2 - 2x - 1 \end{array}$$

This has left a remainder of x^2-2x-1. To get from the highest degree term in the denominator (x^2) to the highest degree term in the remainder (x^2), we have to multiply by 1. So we write,

$$\begin{array}{r|l}
 & 3x+1 \\ \hline
x^2-3x+2 & 3x^3-8x^2+4x-1 \\
 & 3x^3-9x^2+6x \\ \hline
 & \phantom{3x^3-{}}x^2-2x-1
\end{array}$$

Now we subtract 1 times the denominator, x^2-3x+2, which is x^2-3x+2, from the remainder.

$$\begin{array}{r|ll}
 & 3x+1 & \\ \hline
x^2-3x+2 & 3x^3-8x^2+4x-1 & \\
 & 3x^3-9x^2+6x & \longleftarrow 3x(x^2-3x+2) \\ \hline
 & \phantom{3x^3-{}}x^2-2x-1 & \\
 & \phantom{3x^3-{}}x^2-3x+2 & \longleftarrow 1\,(x^2-3x+2) \\ \hline
 & \phantom{3x^3-x^2-{}}x-3 &
\end{array}$$

This leaves a remainder of $x-3$. Because the remainder has degree 1, which is smaller than the degree of the denominator, which is 2, we stop.

In this example, when we subtracted $3x(x^2-3x+2)$ and $1(x^2-3x+2)$ from $3x^3-8x^2+4x-1$ we ended up with $x-3$. That is,

$$3x^3-8x^2+4x-1 \ - \ 3x(x^2-3x+2) \ - \ 1(x^2-3x+2) \ = \ x-3$$

or, collecting the two terms proportional to (x^2-3x+2)

$$3x^3-8x^2+4x-1 \ - \ (3x+1)(x^2-3x+2) \ = \ x-3$$

Moving the $(3x+1)(x^2-3x+2)$ to the right hand side and dividing the whole equation by x^2-3x+2 gives

$$\frac{3x^3-8x^2+4x-1}{x^2-3x+2} \ = \ 3x+1 \ + \ \frac{x-3}{x^2-3x+2}$$

This is of the form $\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$, with the degree of $R(x)$ strictly smaller than the degree of $D(x)$, which is what we wanted. Observe that $R(x)$ is the final remainder of the long division procedure and $P(x)$ is at the top of the long division computation.

$$\begin{array}{r|ll}
 & 3x+1 & \longleftarrow P(x) \\ \hline
D(x)\longrightarrow x^2-3x+2 & 3x^3-8x^2+4x-1 & \longleftarrow N(x) \\
 & 3x^3-9x^2+6x & \longleftarrow 3x\cdot D(x) \\ \hline
 & \phantom{3x^3-{}}x^2-2x-1 & \longleftarrow N(x)-3x\cdot D(x) \\
 & \phantom{3x^3-{}}x^2-3x+2 & \longleftarrow 1\cdot D(x) \\ \hline
 & \phantom{3x^3-x^2-{}}x-3 & \longleftarrow R(x)=N(x)-3x\cdot D(x)-1\cdot D(x)
\end{array}$$

This is the end of Step 1.

Step 2. The second step is to factor the denominator

$$x^2-3x+2=(x-1)(x-2)$$

We already did this in Example 1.

Step 3. The third step is to write $\frac{x-3}{x^2-3x+2}$ in the form

$$\frac{x-3}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2}$$

for some constants A and B. We already did this in Example 1. We found $A = 2$ and $B = -1$.

Step 4. The final step is to integrate.

$$\begin{aligned}\int \frac{3x^3-8x^2+4x-1}{x^2-3x+2}\,dx &= \int \big[3x+1\big]\,dx + \int \frac{2}{x-1}\,dx + \int \frac{-1}{x-2}\,dx \\ &= \frac{3}{2}x^2 + x + 2\ln|x-1| - \ln|x-2| + C\end{aligned}$$

Example 2

Example 3 ($\int \frac{x^4+9x^3+31x^2+49x+27}{x^3+5x^2+8x+4}\,dx$)

In this example, we integrate $\frac{N(x)}{D(x)} = \frac{x^4+9x^3+31x^2+49x+27}{x^3+5x^2+8x+4}$.

Step 1. The degree of the numerator $N(x)$ is greater than the degree of the denominator $D(x)$, so the first step to write $\frac{N(x)}{D(x)}$ in the form

$$\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$$

with $P(x)$ being a polynomial and $R(x)$ being a polynomial of degree strictly smaller than the degree of $D(x)$. By long division

$$\begin{array}{r|l} & \quad x+\ \ 4 \\ \hline x^3+5x^2+8x+4 & x^4+9x^3+31x^2+49x+27 \\ & \underline{x^4+5x^3+\ 8x^2+\ 4x\qquad} \\ & \qquad 4x^3+23x^2+45x+27 \\ & \qquad \underline{4x^3+20x^2+32x+16} \\ & \qquad\qquad\ \ 3x^2+13x+11 \end{array}$$

so

$$\frac{x^4+9x^3+31x^2+49x+27}{x^3+5x^2+8x+4} = x+4+\frac{3x^2+13x+11}{x^3+5x^2+8x+4}$$

Step 2. The second step is to factorize $D(x) = x^3+5x^2+8x+4$. In the "real world" factorization of polynomials is often very hard. Fortunately, this is not the "real world" and there is a trick available to help us find this factorization. The trick exploits the fact that most polynomials that appear in homework assignments and on tests have integer coefficients and some integer roots. Any integer root of a polynomial that has integer coefficients, like $D(x) = x^3+5x^2+8x+4$, must divide the constant term of the polynomial exactly. (Why this is true is explained in the notes "Roots of Polynomials".) So any integer root of x^3+5x^2+8x+4 must divide 4 exactly. The only integers which can be roots of $D(x)$ are ±1, ±2 and ±4. To test if ±1 are roots, we sub them into $D(x)$:

$$\begin{aligned} D(1) &= (1)^3+5(1)^2+8(1)+4 \neq 0 && \Rightarrow \quad x=1 \text{ is not a root} \\ D(-1) &= (-1)^3+5(-1)^2+8(-1)+4 = 0 && \Rightarrow \quad x=-1 \text{ is a root} \end{aligned}$$

So $(x+1)$ must divide x^3+5x^2+8x+4 exactly. By long division

$$\begin{array}{r|l} & x^2+4x+4 \\ \hline x+1 & x^3+5x^2+8x+4 \\ & \underline{x^3+\ \ x^2} \\ & \quad 4x^2+8x+4 \\ & \quad \underline{4x^2+4x} \\ & \qquad\quad 4x+4 \\ & \qquad\quad \underline{4x+4} \\ & \qquad\qquad\ \ 0 \end{array}$$

so

$$x^3+5x^2+8x+4 = (x+1)(x^2+4x+4) = (x+1)(x+2)(x+2)$$

This is the end of step 2. We now know

$$\frac{x^4+9x^3+31x^2+49x+27}{x^3+5x^2+8x+4} = x+4+\frac{3x^2+13x+11}{(x+1)(x+2)^2}$$

Step 3. The third step is to write $\frac{3x^2+13x+11}{(x+1)(x+2)^2}$ in the form

$$\frac{3x^2+13x+11}{(x+1)(x+2)^2} = \frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$

for some constants A, B and C. To determine the values of the constants A, B, C, we put the right hand side back over the common denominator $(x+1)(x+2)^2$.

$$\begin{aligned} \frac{3x^2+13x+11}{(x+1)(x+2)^2} &= \frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2} \\ &= \frac{A(x+2)^2+B(x+1)(x+2)+C(x+1)}{(x+1)(x+2)^2} \end{aligned}$$

The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.

$$3x^2+13x+11 = A(x+2)^2+B(x+1)(x+2)+C(x+1)$$

As in Example 1, there are a couple of different ways to determine the values of A, B and C from this equation.

The conceptually clearest procedure is to write the right hand side as a polynomial in standard form (i.e. collect up all x^2 terms, all x terms and all constant terms)

$$3x^2+13x+11 = (A+B)x^2+(4A+3B+C)x+(4A+2B+C)$$

For these two polynomials to be the same, the coefficient of x^2 on the left hand side and the coefficient of x^2 on the right hand side must be the same. Similarly the coefficients of x^1 and the coefficients of x^0 (i.e. the constant terms) must match. This gives us a system of three equations,

$$A + B = 3 \qquad 4A + 3B + C = 13 \qquad 4A + 2B + C = 11$$

in the three unknowns A, B, C. We can solve this system by using the first equation, namely $A+B = 3$, to determine A in terms of B: $\quad A = 3 - B$. Substituting this into the remaining equations eliminates the A, leaving two equations in the two unknown B, C.

$$4(3 - B) + 3B + C = 13 \qquad 4(3 - B) + 2B + C = 11$$

or

$$-B + C = 1 \qquad -2B + C = -1$$

We can now solve the first of these equations, namely $-B + C = 1$, for B in terms of C, giving $B = C - 1$. Substituting this into the last equation, namely $-2B + C = -1$, gives $-2(C - 1) + C = -1$ which is easily solved to give $C = 3$, and then $B = C - 1 = 2$ and then $A = 3 - B = 1$.

The second, sneakier, method for finding A, B and C exploits the fact that $3x^2+13x+11 = (A+B)x^2 + (4A+3B+C)x + (4A+2B+C)$ must be true for all values of x. In particular, it must be true for $x = -1$. When $x = -1$, the factor $(x+1)$ multiplying B and C is exactly zero. So B and C disappear from the equation, leaving us with an easy equation to solve for A:

$$3x^2 + 13x + 11\Big|_{x=-1} = A(x+2)^2\Big|_{x=-1} + B(x+1)(x+2)\Big|_{x=-1} + C(x+1)\Big|_{x=-1}$$
$$\implies 1 = A$$

Sub this value of A back in and simplify.

$$3x^2 + 13x + 11 = (1)(x+2)^2 + B(x+1)(x+2) + C(x+1)$$
$$2x^2 + 9x + 7 = B(x+1)(x+2) + C(x+1) = (xB + 2B + C)(x+1)$$

Since $(x+1)$ is a factor on the right hand side, it must also be a factor on the left hand side.

$$(2x+7)(x+1) = (xB + 2B + C)(x+1) \quad \Rightarrow \quad (2x+7) = (xB + 2B + C)$$

For the coefficients of x to match, B must be 2. For the constant terms to match, $2B + C$ must be 7, so C must be 3. Subbing into $\frac{3x^2+13x+11}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$, we now have

$$\frac{x^4 + 9x^3 + 31x^2 + 49x + 27}{x^3 + 5x^2 + 8x + 4} = x + 4 + \frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{(x+2)^2}$$

Step 4. The final step is to integrate

$$\int \frac{x^4 + 9x^3 + 31x^2 + 49x + 27}{x^3 + 5x^2 + 8x + 4}\,dx = \int (x+4)\,dx + \int \frac{1}{x+1}\,dx + \int \frac{2}{x+2}\,dx + \int \frac{3}{(x+2)^2}\,dx$$
$$= \tfrac{1}{2}x^2 + 4x + \ln|x+1| + 2\ln|x+2| - \frac{3}{x+2} + C$$

Example 3

Example 4 ($\int \sec x\ dx$)

In this example, we integrate $\sec x$. It is not yet clear what this integral has to do with partial fractions. To get to a partial fractions computation, we first make one of our old substitutions.

$$\begin{aligned}\int \sec x\ dx &= \int \frac{1}{\cos x}\ dx = \int \frac{\cos x}{\cos^2 x}\ dx \\ &= -\int \frac{du}{u^2-1} \qquad \text{with } u=\sin x,\ du=\cos x\ dx,\ \cos^2 x = 1-\sin^2 x = 1-u^2\end{aligned}$$

So we now have to integrate $\frac{1}{u^2-1}$, which is a rational function of u, and so is perfect for partial fractions.

Step 1. The degree of the numerator, 1, is zero, which is strictly smaller than the degree of the denominator, u^2-1, which is two. So the first step is skipped.

Step 2. The second step is to factor the denominator

$$u^2-1 = (u-1)(u+1)$$

Step 3. The third step is to write $\frac{1}{u^2-1}$ in the form

$$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$$

for some constants A and B. To determine the values of the constants A, B, we put the right hand side back over the common denominator $(u-1)(u+1)$.

$$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1} = \frac{A(u+1)+B(u-1)}{(u-1)(u+1)}$$

The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.

$$1 = A(u+1) + B(u-1)$$

The fast way to find A and B is to remember that this equation must be true for all values of u. In particular, it must be true for $u=1$. When $u=1$, the factor $(u-1)$ multiplying B is exactly zero. So B disappears from the equation, leaving us with an easy equation to solve for A:

$$1\Big|_{u=1} = A(u+1)\Big|_{u=1} + B(u-1)\Big|_{u=1} \Longrightarrow 1 = 2A \Longrightarrow A = \frac{1}{2}$$

Similarly, when $u=-1$, the factor $(u+1)$ multiplying A is exactly zero. So A disappears from the equation, leaving us with an easy equation to solve for B:

$$1\Big|_{u=-1} = A(u+1)\Big|_{u=-1} + B(u-1)\Big|_{u=-1} \Longrightarrow 1 = -2B \Longrightarrow B = -\frac{1}{2}$$

So we have now found that $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. It is a good idea to check that

$$1 = A(u+1) + B(u-1) = \frac{1}{2}(u+1) - \frac{1}{2}(u-1)$$

is really true. It is. So we now know that

$$\frac{1}{u^2-1} = \frac{1}{2}\Big[\frac{1}{u-1} - \frac{1}{u+1}\Big] \tag{2}$$

Step 4. The final step is to integrate.

$$\begin{aligned}
\int \sec x \ dx &= -\int \frac{du}{u^2-1} \\
&= -\frac{1}{2}\int \frac{1}{u-1}\ du + \frac{1}{2}\int \frac{1}{u+1}\ du \\
&= -\frac{1}{2}\ln|u-1| + \frac{1}{2}\ln|u+1| + C \\
&= -\frac{1}{2}\ln|\sin x - 1| + \frac{1}{2}\ln|\sin x + 1| + C \\
&= \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + C
\end{aligned}$$

Example 4

Example 5 ($\int \sec^3 x \ dx$)

We'll now do another example that is similar in spirit to, but harder than, Example 5, namely $\int \sec^3 x\, dx$. We'll start by converting it into the integral of a rational function using the substitution $u = \sin x$, $du = \cos x\, dx$.

$$\int \sec^3 x \ dx = \int \frac{1}{\cos^3 x}\ dx = \int \frac{\cos x}{\cos^4 x}\ dx = \int \frac{\cos x \ dx}{\big[1-\sin^2 x\big]^2} = \int \frac{du}{\big[1-u^2\big]^2}$$

We could now find the partial fractions expansion of the integrand $\frac{1}{[1-u^2]^2}$ by executing the usual four steps. But it is easier to use that we already know, from (2), that

$$\frac{1}{u^2-1} = \frac{1}{2}\Big[\frac{1}{u-1} - \frac{1}{u+1}\Big]$$

Squaring this gives

$$\begin{aligned}
\frac{1}{\big[1-u^2\big]^2} &= \frac{1}{4}\Big[\frac{1}{u-1} - \frac{1}{u+1}\Big]^2 \\
&= \frac{1}{4}\Big[\frac{1}{(u-1)^2} - \frac{2}{(u-1)(u+1)} + \frac{1}{(u+1)^2}\Big] \\
&= \frac{1}{4}\Big[\frac{1}{(u-1)^2} - \frac{1}{u-1} + \frac{1}{u+1} + \frac{1}{(u+1)^2}\Big] \qquad \text{(by (2), again)}
\end{aligned}$$

It only remains to do the integrals and simplify.

$$\begin{aligned}\int \sec^3 x\ dx &= \frac{1}{4}\int \Big[\frac{1}{(u-1)^2} - \frac{1}{u-1} + \frac{1}{u+1} + \frac{1}{(u+1)^2}\Big]du \\ &= \frac{1}{4}\Big[-\frac{1}{u-1} - \ln|u-1| + \ln|u+1| - \frac{1}{u+1}\Big] + C \\ &= -\frac{1}{4}\frac{2u}{u^2-1} + \frac{1}{4}\ln\Big|\frac{u+1}{u-1}\Big| + C \ = \ \frac{1}{2}\frac{u}{1-u^2} + \frac{1}{4}\ln\Big|\frac{u+1}{u-1}\Big| + C \\ &= \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{2}\ln\Big|\frac{\sin x+1}{\sin x-1}\Big| + C\end{aligned}$$

Example 5

The Form of Partial Fractions Decompositions

In Step 3 of the partial fractions algorithm we decompose a rational function $\frac{N(x)}{D(x}$ (or $\frac{R(x)}{D(x)}$), for which the degree of the numerator is strictly smaller than the degree of the denominator, into a sum of particularly simple rational functions, like $\frac{A}{x-a}$. We seen examples of this in Examples 1–4. But we have not yet seen what the form of the decomposition is, in general. We fill this gap now, by stating, without justification, what the form is. The justification is discussed in the next (optional) section. In the following it is assumed that

- $N(x)$ and $D(x)$ are polynomials with the degree of $N(x)$ strictly smaller than the degree of $D(x)$.
- The denominator is a product of linear factors.
- K is a constant.
- a_1, a_2, $\cdots$, a_j are all different numbers.
- m_1, m_2, $\cdots$, m_j, n_1, n_2, $\cdots$, n_k are all strictly positive integers.

Simple Linear Factor Case

If the denominator $D(x) = K(x-a_1)(x-a_2)\cdots(x-a_j)$ is a product of j different linear factors, then

$$\frac{N(x)}{D(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_j}{x-a_j} \tag{3}$$

General Linear Factor Case

If the denominator $D(x) = K(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_j)^{m_j}$ then

$$\begin{aligned}\frac{N(x)}{D(x)} &= \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}} \\ &\quad + \frac{A_{2,1}}{x-a_2} + \frac{A_{2,2}}{(x-a_2)^2} + \cdots + \frac{A_{2,m_2}}{(x-a_2)^{m_2}} + \cdots \\ &\quad + \frac{A_{j,1}}{x-a_j} + \frac{A_{j,2}}{(x-a_j)^2} + \cdots + \frac{A_{j,m_j}}{(x-a_j)^{m_j}}\end{aligned} \tag{4}$$

Each line could be rewritten

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_m}{(x-a)^m}=\frac{A_1(x-a)^{m-1}+A_2(x-a)^{m-2}+\cdots+A_m}{(x-a)^m}$$
$$=\frac{B_1x^{m-1}+B_2x^{m-2}+\cdots+B_m}{(x-a)^m}$$

which is a polynomial whose degree, $m-1$, is strictly smaller than that of the denominator $(x-a)^m$. But the form of (4) is preferable because it is easier to integrate.

Justification of The Partial Fraction Decompositions (Optional)

We will now see the justification for the form of the partial fraction decompositions. We will only consider the case in which the denominator has only linear factors. The arguments when there are quadratic factors too are similar. (Better still, allow complex numbers. Then there are only linear factors.)

Simple Linear Factor Case

In the most common partial fraction decomposition, we split up

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)}$$

into a sum of the form

$$\frac{A_1}{x-a_1}+\cdots+\frac{A_d}{x-a_d}$$

We now show that this decomposition can always be achieved, under the assumptions that the a_i's are all different and $N(x)$ is a polynomial of degree at most $d-1$. To do so, we shall repeatedly apply the following Lemma. (The word Lemma just signifies that the result is not that important – it is only used as a tool to prove a more important result.)

Lemma 6. *Let $N(x)$ and $D(x)$ be polynomials of degree n and d respectively, with $n \le d$. Suppose that a is NOT a zero of $D(x)$. Then there is a polynomial $P(x)$ of degree $p<d$ and a numbers A such that*

$$\frac{N(x)}{D(x)\,(x-a)}=\frac{P(x)}{D(x)}+\frac{A}{x-a}$$

Proof. To save writing, let $z=x-a$. Then $\tilde{N}(z)=N(z+a)$ and $\tilde{D}(z)=D(z+a)$ are again polynomials of degree n and d respectively, $\tilde{D}(0)=D(a)\neq 0$ and we have to find a polynomial $\tilde{P}(z)$ of degree $p<d$ and a number A such that

$$\frac{\tilde{N}(z)}{\tilde{D}(z)\,z}=\frac{\tilde{P}(z)}{\tilde{D}(z)}+\frac{A}{z}=\frac{\tilde{P}(z)z+A\tilde{D}(z)}{\tilde{D}(z)\,z}$$

or equivalently, such that

$$\tilde{P}(z)z+A\tilde{D}(z)=\tilde{N}(z)$$

Now look at the polynomial on the left hand side. Every term in $\tilde{P}(z)z$, has at least one power of z. So the constant term on the left hand side is exactly the constant term in $A\tilde{D}(z)$, which is $A\tilde{D}(0)$. The constant term on the right hand side is $\tilde{N}(0)$. So the constant terms on the left and right hand sides are the same if we choose $A = \frac{\tilde{N}(0)}{\tilde{D}(0)}$. Recall that $\tilde{D}(0)$ cannot be zero. Now move $A\tilde{D}(z)$ to the right hand side.

$$\tilde{P}(z)z = \tilde{N}(z) - A\tilde{D}(z)$$

The constant terms in $\tilde{N}(z)$ and $A\tilde{D}(z)$ are the same, so the right hand side contains no constant term and the right hand side is of the form $\tilde{N}_1(z)z$. Since $\tilde{N}(z)$ is of degree at most d and $A\tilde{D}(z)$ is of degree exactly d, $\tilde{N}_1$ is a polynomial of degree $d-1$. It now suffices to choose $\tilde{P}(z) = \tilde{N}_1(z)$. □

Now back to

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)}$$

Apply Lemma 6, with $D(x) = (x-a_2)\times\cdots\times(x-a_d)$ and $a = a_1$. It says

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)} = \frac{A_1}{x-a_1} + \frac{P(x)}{(x-a_2)\times\cdots\times(x-a_d)}$$

for some polynomial P of degree at most $d-2$ and some number A_1. Apply Lemma 6 a second time, with $D(x) = (x-a_3)\times\cdots\times(x-a_d)$, $N(x) = P(x)$ and $a = a_2$. It says

$$\frac{P(x)}{(x-a_2)\times\cdots\times(x-a_d)} = \frac{A_2}{x-a_2} + \frac{Q(x)}{(x-a_3)\times\cdots\times(x-a_d)}$$

for some polynomial Q of degree at most $d-3$ and some number A_2. At this stage, we know that

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \frac{Q(x)}{(x-a_3)\times\cdots\times(x-a_d)}$$

If we just keep going, repeatedly applying Lemma 1, we eventually end up with

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_d}{x-a_d}$$

The general case with linear factors

Now consider splitting

$$\frac{N(x)}{(x-a_1)^{n_1}\times\cdots\times(x-a_d)^{n_d}}$$

into a sum of the form

$$\Big[\frac{A_{1,1}}{x-a_1} + \cdots + \frac{A_{1,n_1}}{(x-a_1)^{n_1}}\Big] + \cdots + \Big[\frac{A_{d,1}}{x-a_d} + \cdots + \frac{A_{d,n_d}}{(x-a_d)^{n_d}}\Big]$$

(If we allow ourselves to use complex numbers as roots, this is the general case.) We now show that this decomposition can always be achieved, under the assumptions that the a_i's are all different and $N(x)$ is a polynomial of degree at most $n_1 + \cdots + n_d - 1$. To do so, we shall repeatedly apply the following Lemma.

Lemma 7. *Let $N(x)$ and $D(x)$ be polynomials of degree n and d respectively, with $n < d+m$. Suppose that a is NOT a zero of $D(x)$. Then there is a polynomial $P(x)$ of degree $p < d$ and numbers $A_1, \cdots, A_m$ such that*

$$\frac{N(x)}{D(x)\,(x-a)^m} = \frac{P(x)}{D(x)} + \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m}$$

Proof. To save writing, let $z = x - a$. Then $\tilde{N}(z) = N(z+a)$ and $\tilde{D}(z) = D(z+a)$ are polynomials of degree n and d respectively, $\tilde{D}(0) = D(a) \neq 0$ and we have to find a polynomial $\tilde{P}(z)$ of degree $p < d$ and numbers $A_1, \cdots, A_m$ such that

$$\begin{aligned}\frac{\tilde{N}(z)}{\tilde{D}(z)\,z^m} &= \frac{\tilde{P}(z)}{\tilde{D}(z)} + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_m}{z^m} \\ &= \frac{\tilde{P}(z)z^m + A_1 z^{m-1}\tilde{D}(z) + A_2 z^{m-2}\tilde{D}(z) + \cdots + A_m\tilde{D}(z)}{\tilde{D}(z)\,z^m}\end{aligned}$$

or equivalently, such that

$$\tilde{P}(z)z^m + A_1 z^{m-1}\tilde{D}(z) + A_2 z^{m-2}\tilde{D}(z) + \cdots + A_{m-1} z\tilde{D}(z) + A_m\tilde{D}(z) = \tilde{N}(z)$$

Now look at the polynomial on the left hand side. Every single term on the left hand side, except for the very last one, $A_m\tilde{D}(z)$, has at least one power of z. So the constant term on the left hand side is exactly the constant term in $A_m\tilde{D}(z)$, which is $A_m\tilde{D}(0)$. The constant term on the right hand side is $\tilde{N}(0)$. So the constant terms on the left and right hand sides are the same if we choose $A_m = \frac{\tilde{N}(0)}{\tilde{D}(0)}$. Recall that $\tilde{D}(0) \neq 0$. Now move $A_m\tilde{D}(z)$ to the right hand side.

$$\tilde{P}(z)z^m + A_1 z^{m-1}\tilde{D}(z) + A_2 z^{m-2}\tilde{D}(z) + \cdots + A_{m-1} z\tilde{D}(z) = \tilde{N}(z) - A_m\tilde{D}(z)$$

The constant terms in $\tilde{N}(z)$ and $A_m\tilde{D}(z)$ are the same, so the right hand side contains no constant term and the right hand side is of the form $\tilde{N}_1(z)z$ with $\tilde{N}_1$ a polynomial of degree at most $d+m-2$. (Recall that $\tilde{N}$ is of degree at most $d+m-1$ and $\tilde{D}$ is of degree at most d.) Divide the whole equation by z.

$$\tilde{P}(z)z^{m-1} + A_1 z^{m-2}\tilde{D}(z) + A_2 z^{m-3}\tilde{D}(z) + \cdots + A_{m-1}\tilde{D}(z) = \tilde{N}_1(z)$$

Now, we can repeat the previous argument. The constant term on the left hand side, which is exactly $A_{m-1}\tilde{D}(0)$ matchs the constant term on the right hand side, which is $\tilde{N}_1(0)$ if we choose $A_{m-1} = \frac{\tilde{N}_1(0)}{\tilde{D}(0)}$. With this choice of A_{m-1}

$$\tilde{P}(z)z^{m-1} + A_1 z^{m-2}\tilde{D}(z) + A_2 z^{m-3}\tilde{D}(z) + \cdots + A_{m-2} z\tilde{D}(z) = \tilde{N}_1(z) - A_{m-1}\tilde{D}(z) = \tilde{N}_2(z)z$$

with $\tilde{N}_2$ a polynomial of degree at most $d+m-3$. Divide by z and continue. After m steps like this, we end up with

$$\tilde{P}(z)z = \tilde{N}_{m-1}(z) - A_1\tilde{D}(z)$$

after having chosen $A_1 = \frac{\tilde{N}_{m-1}(0)}{\tilde{D}(0)}$. There is no constant term on the right side so that $\tilde{N}_{m-1}(z) - A_1\tilde{D}(z)$ is of the form $\tilde{N}_m(z)z$ with $\tilde{N}_m$ a polynomial of degree $d-1$. Choosing $\tilde{P}(z) = \tilde{N}_m(z)$ completes the proof. □

Now back to

$$\frac{N(x)}{(x-a_1)^{n_1}\times\cdots\times(x-a_d)^{n_d}}$$

Apply Lemma 7, with $D(x)=(x-a_2)^{n_2}\times\cdots\times(x-a_d)^{n_d}$, $m=n_1$ and $a=a_1$. It says

$$\frac{N(x)}{(x-a_1)^{n_1}\times\cdots\times(x-a_d)^{n_d}}=\frac{A_{1,1}}{x-a_1}+\frac{A_{1,2}}{(x-a_1)^2}+\cdots+\frac{A_{1,n_1}}{(x-a)^{n_1}}+\frac{P(x)}{(x-a_2)^{n_2}\times\cdots\times(x-a_d)^{n_d}}$$

Apply Lemma 7 a second time, with $D(x)=(x-a_3)^{n_3}\times\cdots\times(x-a_d)^{n_d}$, $N(x)=P(x)$, $m=n_2$ and $a=a_2$. And so on. Eventually, we end up with

$$\Big[\frac{A_{1,1}}{x-a_1}+\cdots+\frac{A_{1,n_1}}{(x-a_1)^{n_1}}\Big]+\cdots+\Big[\frac{A_{d,1}}{x-a_d}+\cdots+\frac{A_{d,n_d}}{(x-a_d)^{n_d}}\Big]$$

as desired.