

Integration by Parts

“Integration by parts” is just the product rule translated into the language of integrals. Recall that the product rule says

$$\frac{d}{dx}u(x)v(x) = u'(x)v(x) + u(x)v'(x)$$

This tells us that

$$\begin{aligned}\int [u'(x)v(x) + u(x)v'(x)] dx &= [\text{a function with derivative } u'v + uv'] + C \\ &= u(x)v(x) + C\end{aligned}$$

Theorem 1.

Let $u(x)$ and $v(x)$ be continuously differentiable. Then

$$\int u(x)v'(x) dx = u(x)v(x) - \int v(x)u'(x) dx$$

or, writing dv for $v'(x) dx$ and du for $u'(x) dx$

$$\int u dv = uv - \int v du$$

The corresponding statement for definite integrals is

$$\int_a^b u(x)v'(x) dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x) dx$$

Integration by parts is often used

- to eliminate a $\ln x$ from an integrand by using that $\frac{d}{dx} \ln x = \frac{1}{x}$ and
- to eliminate factors of x from an integrand like xe^x by using that $\frac{d}{dx} x = 1$ and
- to eliminate inverse trig functions, like $\tan^{-1} x$, from an integrand by using that, for example, $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$.

Example 2 ($\int \ln x dx$)

We don't know an antiderivative for $\ln x$, i.e. a function whose derivative is $\ln x$. So we want to eliminate $\ln x$ from the integrand. We may do so by integrating by parts with $u = \ln x$. The point of doing so is that the integrand on the right hand side, $uv - \int v du$, of the integration by parts formula contains $du = u'(x) dx = \frac{dx}{x}$, instead of $u(x) = \ln x$.

The first step in implementing this strategy is to write the integral of interest in the form, $\int u dv$, of the left hand side of the integration by parts formula.

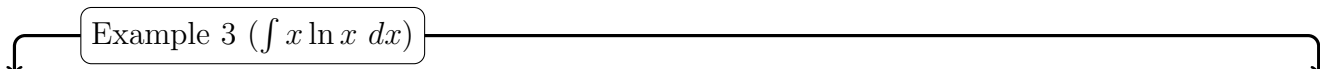
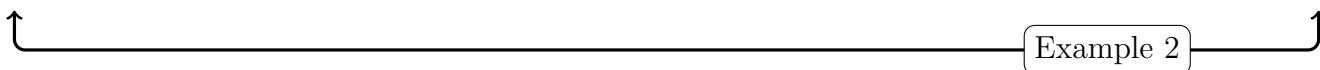
$$\int \ln x dx = \int u dv \quad \text{with } u = \ln x, \quad dv = dx$$

Next we need a function $v(x)$ obeying $dv = v'(x) dx = dx$, i.e. obeying $v'(x) = 1$. Any function will do. We'll chose the simplest one, namely $v(x) = x$. We're now ready to evaluate the integral.

$$\begin{aligned} \int \ln x dx &= \int u dv && \text{with } u = \ln x, \quad dv = dx \\ &= uv - \int v du && \text{with } v = x, \quad du = \frac{1}{x} dx \\ &= x \ln x - \int x \frac{1}{x} dx \\ &= x \ln x - x + C \end{aligned}$$

As always, it is a good idea to check our result by verifying that the derivative of the answer really is the integrand.

$$\frac{d}{dx} [x \ln x - x + C] = \ln x + x \frac{1}{x} - 1 + 0 = \ln x$$



Once again, we want to eliminate $\ln x$ from the integrand. So we again integrate by parts with $u = \ln x$.

$$\begin{aligned} \int x \ln x dx &= \int u dv && \text{with } u = \ln x, \quad dv = x dx \\ &= uv - \int v du && \text{with } v = \frac{x^2}{2}, \quad du = \frac{1}{x} dx \\ &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} dx \\ &= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C \end{aligned}$$

Checking:

$$\frac{d}{dx} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4} + C \right] = x \ln x + \frac{x^2}{2} \frac{1}{x} - \frac{x}{2} + 0 = x \ln x$$



Example 4 ($\int xe^x dx$)

We not not know an indefinite integral for xe^x , but we do know one for e^x . So it would be nice to eliminate the x from the integrand. We may do so by integrating by parts with $u = x$, since then $du = dx$.

$$\begin{aligned}\int xe^x dx &= \int u dv && \text{with } u = x, \quad dv = e^x dx \\ &= uv - \int v du && \text{with } v = e^x, \quad du = dx \\ &= xe^x - \int e^x dx \\ &= xe^x - e^x + C\end{aligned}$$

Checking:

$$\frac{d}{dx} [xe^x - e^x + C] = e^x + xe^x - e^x + 0 = xe^x$$

Example 4

Example 5 ($\int x^2 e^x dx$)

Integrating by parts with $u = x^2$ does not eliminate the x^2 completely, but at least it reduces the power of x , leaving us with an integral that we can handle.

$$\begin{aligned}\int x^2 e^x dx &= \int u dv && \text{with } u = x^2, \quad dv = e^x dx \\ &= uv - \int v du && \text{with } v = e^x, \quad du = 2x dx \\ &= x^2 e^x - \int 2x e^x dx \\ &= x^2 e^x - 2x e^x + 2e^x + C && \text{by Example 4}\end{aligned}$$

Example 5

Example 6 ($\int \tan^{-1} x dx$)

This time, we want to eliminate $\tan^{-1} x$ from the integrand, because we don't know how to integrate it. So we integrate by parts with $u = \tan^{-1} x$.

$$\begin{aligned}\int \tan^{-1} x dx &= \int u dv && \text{with } u = \tan^{-1} x, \quad dv = dx \\ &= uv - \int v du && \text{with } v = x, \quad du = \frac{1}{1+x^2} dx\end{aligned}$$

$$\begin{aligned}
&= x \tan^{-1} x - \int \frac{x}{1+x^2} dx \\
&= x \tan^{-1} x - \int \frac{1}{y} \frac{dy}{2} \quad \text{with } y = 1+x^2, \quad dy = 2x dx \\
&= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C
\end{aligned}$$

Checking:

$$\frac{d}{dx} \left[x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C \right] = \tan^{-1} x + \frac{x}{1+x^2} - \frac{1}{2} \frac{2x}{1+x^2} + 0 = \tan^{-1} x$$

Example 6

Example 7 ($\int_a^b e^x \sin x dx$ and $\int_a^b e^x \cos x dx$)

This time we're going to do the two integrals

$$I_1 = \int_a^b e^x \sin x dx \quad I_2 = \int_a^b e^x \cos x dx$$

at more or less the same time. First

$$\begin{aligned}
I_1 &= \int_a^b e^x \sin x dx = \int_a^b u dv && \text{with } u = e^x, \quad dv = \sin x dx \\
&= uv \Big|_a^b - \int_a^b v du && \text{with } v = -\cos x, \quad du = e^x dx \\
&= \left[-e^x \cos x \right]_a^b + \int_a^b e^x \cos x dx
\end{aligned}$$

We have not found I_1 but we have related it to I_2 .

$$I_1 = \left[-e^x \cos x \right]_a^b + I_2 \tag{1}$$

Now start over with I_2 .

$$\begin{aligned}
I_2 &= \int_a^b e^x \cos x dx = \int_a^b u dv && \text{with } u = e^x, \quad dv = \cos x dx \\
&= uv \Big|_a^b - \int_a^b v du && \text{with } v = \sin x, \quad du = e^x dx \\
&= \left[e^x \sin x \right]_a^b - \int_a^b e^x \sin x dx
\end{aligned}$$

Once again, we have not found I_2 but we have related it back to I_1 .

$$I_2 = \left[e^x \sin x \right]_a^b - I_1 \tag{2}$$

If we now substitute (2) into (1) we get

$$I_1 = \left[-e^x \cos x + e^x \sin x \right]_a^b - I_1 \implies I_1 = \frac{1}{2} \left[e^x (\sin x - \cos x) \right]_a^b \quad (3)$$

and if we substitute (1) into (2) we get

$$I_2 = \left[e^x \sin x + e^x \cos x \right]_a^b - I_2 \implies I_2 = \frac{1}{2} \left[e^x (\sin x + \cos x) \right]_a^b \quad (4)$$

That is,

$$\int_a^b e^x \sin x \, dx = \frac{1}{2} \left[e^x (\sin x - \cos x) \right]_a^b \quad \int_a^b e^x \cos x \, dx = \frac{1}{2} \left[e^x (\sin x + \cos x) \right]_a^b$$

This also says, for example, that $\frac{1}{2}e^x(\sin x - \cos x)$ is an antiderivative of $e^x \sin x$ so that

$$\int e^x \sin x \, dx = \frac{1}{2}e^x(\sin x - \cos x) + C$$

Note that we can always check whether or not this is correct. It is correct if and only if the derivative of the right hand side is $e^x \sin x$. Here goes. By the product rule

$$\frac{d}{dx} \left[\frac{1}{2}e^x(\sin x - \cos x) + C \right] = \frac{1}{2} \left[e^x(\sin x - \cos x) + e^x(\cos x + \sin x) \right] = e^x \sin x$$

which is the desired derivative.

There is another way to find $\int e^x \sin x \, dx$ and $\int e^x \cos x \, dx$ that, in contrast to the above computations, doesn't involve any trickery. But it does require the use of complex numbers and so is beyond the scope of this course. The secret is to use that $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$ and $\cos x = \frac{e^{ix} + e^{-ix}}{2}$, where i is the square root of -1 of the complex number system.

Example 7