Substitution

Integrals with simple integrands are usually evaluated by using the fundamental theorem of calculus. There are a number of tools that are used to convert integrals with complicated integrands into integrals with simple integrands. The most important such tool is the substitution rule. The substitution rule is just the chain rule rewritten in terms of integrals. Suppose that $F(u)$ is a function whose derivative is $f(u)$. That is, $F(u)$ is an antiderivative for $f(u)$ so that

$$\int f(u)\,du = F(u) + C$$

Then the chain rule says that, for any function $u(x)$,

$$\frac{\mathrm{d}}{\mathrm{d}x} F\big(u(x)\big) = F'\big(u(x)\big)\,u'(x) = f\big(u(x)\big)\,u'(x)$$

So $F\big(u(x)\big)$ is one function with derivative $f\big(u(x)\big)\,u'(x)$ and $F\big(u(x)\big)$ is an antiderivative for $f\big(u(x)\big)\,u'(x)$. Thus $\int f\big(u(x)\big)\,u'(x)\,dx = F\big(u(x)\big) + C$ or

$$\int f\big(u(x)\big)\,u'(x)\,dx = \int f(u)\,du\,\Big|_{u=u(x)} \tag{1}$$

The notation on the right hand side means "evaluate $\int f(u)\,du$ and then replace every u by $u(x)$". This is the substitution rule for indefinite integrals. Note that, since $f\big(u(x)\big)\,u'(x)$, is a function of x, its indefinite integral must also be a function of x. On the right hand side, evaluating u at $u(x)$ ensures that we end up with a function of x.

Because $F\big(u(x)\big)$ is one antiderivative of $f\big(u(x)\big)\,u'(x)$,

$$\int_a^b f\big(u(x)\big)\,u'(x)\,dx = F\big(u(x)\big)\Big|_{x=a}^{x=b} = F\big(u(b)\big) - F\big(u(a)\big)$$

The right hand side is $F(u) = \int f(u)\,du$ evaluated at $u(b)$ minus the same function evaluated at $u(a)$. So

$$\int_a^b f\big(u(x)\big)\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du \tag{2}$$

This is the substitution rule for definite integrals. Notice that to get from the integral on the left hand side to the integral on the right hand side you

- substitute $u(x) \to u$ and $u'(x)\,dx \to du$ (which looks like $\frac{du}{dx} = u'(x)$ with the dx multiplied across)
- set the lower limit for the u integral to the value of u (namely $u(a)$) that corresponds to the lower limit of the x integral (namely $x = a$) and
- set the upper limit for the u integral to the value of u (namely $u(b)$) that corresponds to the upper limit of the x integral (namely $x = b$).

The substitution rule is used to simplify integrals, like $\int_0^\pi x^2 \sin\left(\frac{1}{3}x^3\right) dx$, in which the integrand

- has one factor $\left(\sin\left(\frac{1}{3}x^3\right)\text{ in this example}\right)$ which is some function $\left(\sin\text{ in this example}\right)$ evaluated at some complicated argument $\left(\frac{1}{3}x^3\text{ in this example}\right)$ and
- has a second factor $\left(x^2\text{ in this example}\right)$ which is the derivative of the complicated argument, or at least a constant times the derivative of the complicated argument.

In this case one chooses $u(x)$ to be the complicated argument $\left(\text{so } u(x) = \frac{1}{3}x^3\text{ in this example}\right)$.

Example 1

The integrand of

$$\int_0^1 e^x \sin\left(e^x\right)\,dx$$

is $e^x \sin\left(e^x\right)$. One factor of this integrand is $\sin\left(e^x\right)$, which is the function sin evaluated at e^x. The derivative of e^x is again e^x, which is the other factor in the integrand. Choose $u(x) = e^x$ and $f(u) = \sin u$. Then $f\big(u(x)\big) = \sin\left(e^x\right)$ and $u'(x) = e^x$ so

$$\int_0^1 e^x \sin\left(e^x\right)\,dx = \int_a^b f\big(u(x)\big)\,u'(x)\,dx$$

with $a = 0$ and $b = 1$. As $u(a) = u(0) = e^0 = 1$ and $u(b) = u(1) = e^1 = e$, the substitution rule gives

$$\begin{aligned}\int_0^1 e^x \sin\left(e^x\right)\,dx = \int_a^b f\big(u(x)\big)u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du = \int_1^e \sin u\,du = -\cos u\Big|_1^e \\ = -\cos e + \cos 1\end{aligned}$$

In conclusion

$$\int_0^1 e^x \sin\left(e^x\right)\,dx = \cos 1 - \cos e$$

Example 1

Example 2

The integrand of

$$\int_0^1 x^2 \sin\left(x^3+1\right)\,dx$$

is $x^2 \sin\left(x^3+1\right)$. One factor of this integrand is $\sin\left(x^3+1\right)$, which is the function sin evaluated at x^3+1. So set $u(x) = x^3+1$. The derivative $u'(x) = 3x^2$ is not quite the other factor, x^2, in the integrand. But we can arrange for $u'(x) = 3x^2$ to appear as a factor in the integrand just by multiplying and dividing by 3.

$$\int_0^1 x^2 \sin\left(x^3+1\right)\,dx = \int_0^1 \tfrac{1}{3}\sin\left(x^3+1\right)\,3x^2\,dx$$

The integrand $\frac{1}{3}\sin\big(x^3+1\big)\,3x^2$ now is of the form $f\big(u(x)\big)\,u'(x)$ with $u(x)=x^3+1$ and $f(u)=\frac{1}{3}\sin u$. The limits of integration are $x=0$ and $x=1$. So, choosing $u(x)=x^3+1$, $f(u)=\frac{1}{3}\sin u$, $a=0$ and $b=1$ we have

$$\int_0^1 \tfrac{1}{3}\sin\big(x^3+1\big)\,3x^2\,dx=\int_a^b f\big(u(x)\big)\,u'(x)\,dx=\int_{u(a)}^{u(b)} f(u)\,du=\int_1^2 \tfrac{1}{3}\sin u\,du=-\frac{1}{3}\cos u\Big|_1^2$$
$$=\frac{-\cos 2}{3}-\frac{-\cos 1}{3}$$

In conclusion

$$\int_0^1 \sin\big(x^3+1\big)\,x^2\ dx=\frac{\cos 1-\cos 2}{3}$$

Example 2

Once one has chosen $u(x)$, one can make the substitution without ever explicitly deciding what $f(u)$ is. One just has to note that the integrand on the right hand side of the substitution rule

$$\int_a^b f\big(u(x)\big)\,u'(x)\,dx=\int_{u(a)}^{u(b)} f(u)\,du$$

is constructed from the integrand on the left hand side by

- substituting u for $u(x)$ and
- substituting du for $u'(x)\,dx$

The substitution $du=u'(x)\,dx$ is easily remembered by *pretending* that $\frac{du}{dx}$ is an ordinary fraction. Then cross–multiplying $\frac{du}{dx}=u'(x)$ gives $du=u'(x)\,dx$.

Example 3 (*Example 2 revisited*)

Consider

$$\int_0^1 x^2\sin\big(x^3+1\big)\,dx$$

once again. We have observed that one factor of the integrand is $\sin\big(x^3+1\big)$, which is sin evaluated at x^3+1, and the other factor, x^2 is, aside from a factor of 3, the derivative of x^3+1. So we decide to try $u(x)=x^3+1$. Substitute u for x^3+1 and du for $3x^2\,dx$. That is $x^3+1=u$ and $du=3x^2\,dx$ or $x^2\,dx=\frac{du}{3}$. When $x=0$, $u=0^3+1=1$. When $x=1$, $u=1^3+1=2$. So

$$\int_0^1 \sin\big(x^3+1\big)\,x^2dx=\int_1^2 \sin u\,\tfrac{du}{3}$$

We ended up with exactly this integral in Example 2.

Example 3

Example 4

Consider $\int_0^{\pi/2} \cos(3x)\,dx$. Substitute for the argument of $\cos(3x)$. That, is $u(x) = 3x$. We are to substitute $u = 3x$ and $du = 3\,dx$ or $dx = \frac{du}{3}$. When $x = 0$, $u = 3 \times 0 = 0$ and when $x = \frac{\pi}{2}$, $u = \frac{3}{2}\pi$. So

$$\int_0^{\pi/2} \cos(3x)\,dx = \int_0^{3\pi/2} \cos(u)\,\frac{du}{3} = \frac{\sin u}{3}\Big|_0^{3\pi/2} = \frac{-1}{3} - \frac{0}{3} = -\frac{1}{3}$$

Example 4

Example 5

Consider $\int_0^1 \frac{1}{(2x+1)^3}\,dx$. Substitute for the argument, $2x+1$, of $[2x+1]^{-3}$. That is, $u = 2x+1$ and $du = 2\,dx$ or $dx = \frac{du}{2}$. When $x = 0$, $u = 2\times 0+1 = 1$ and when $x = 1$, $u = 2\times 1+1 = 3$. So

$$\int_0^1 \frac{1}{(2x+1)^3}\,dx = \int_1^3 \frac{1}{u^3}\,\frac{du}{2} = \frac{1}{2}\int_1^3 u^{-3}\,du = \frac{1}{2}\frac{u^{-2}}{-2}\Big|_1^3 = \frac{3^{-2}}{-4} - \frac{1^{-2}}{-4} = \frac{1}{4}\Big[1 - \frac{1}{9}\Big] = \frac{2}{9}$$

Example 5

Example 6

Consider $\int_0^1 \frac{x}{1+x^2}\,dx$. Think of the integrand as the product of $\frac{1}{1+x^2}$ and x. The first factor is the function "one over" evaluated at the argument $1 + x^2$. The derivative of the argument $1 + x^2$ is $2x$, which is, except for the 2, the second factor of the integrand. Substitute $u = 1 + x^2$, $du = 2x\,dx$ or $xdx = \frac{du}{2}$. When $x = 0$, $u = 1 + 0^2 = 1$ and when $x = 1$, $u = 1 + 1^2 = 2$. So

$$\int_0^1 \frac{x}{1+x^2}\,dx = \int_0^1 \frac{1}{1+x^2}\,xdx = \int_1^2 \frac{1}{u}\,\frac{du}{2} = \frac{1}{2}\ln|u|\Big|_1^2 = \frac{\ln 2}{2} - \frac{0}{2} = \frac{1}{2}\ln 2$$

Example 6

Example 7

Consider $\int x^3 \cos\big(x^4+2\big)\,dx$. The integrand is the product of cos evaluated at the argument $x^4 + 2$ times x^3, which aside from a factor of 4, is the derivative of the argument $x^4 + 2$. Substitute $u = x^4 + 2$, $du = 4x^3\,dx$ or $x^3dx = \frac{du}{4}$.

$$\int x^3 \cos\big(x^4+2\big)\,dx = \int \cos(u)\,\frac{du}{4} = \frac{1}{4}\sin u + C$$

Because we are dealing with indefinite integrals we need not worry about limits of integration. On the other hand, $x^3 \cos\big(x^4+2\big)$ is a function of x. So its indefinite integral $\big($which is defined to be a function whose derivative is $x^3\cos\big(x^4+2\big)$) must also be a function of x. We must substitute $u = u(x) = x^4+2$ in the answer too. That is what the substitution rule (1) says. The answer is $\frac{1}{4}\sin u(x) + C = \frac{1}{4}\sin\big(x^4+1\big) + C$.

Example 7

Example 8

Consider $\int\sqrt{1+x^2}\,x^3\,dx$. Substitute for the argument of the square root. That is, substitute $u = 1+x^2$, $du = 2x\,dx$ or $dx = \frac{du}{2x}$. You might think that this does not eliminate all of the x's from $\sqrt{1+x^2}\,x^3\,dx = \sqrt{u}x^3\,\frac{du}{2x} = \sqrt{u}x^2\,\frac{du}{2}$. But it does, provided you remember to substitute $x^2 = u-1$ for the remaining factor of x^2.

$$\begin{aligned}\int\sqrt{1+x^2}\,x^3\,dx &= \int\sqrt{u}(u-1)\,\frac{du}{2} = \frac{1}{2}\int\big(u^{3/2}-u^{1/2}\big)\,du = \frac{1}{2}\Big[\frac{u^{5/2}}{5/2}-\frac{u^{3/2}}{3/2}\Big]+C\\ &= \frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}+C\end{aligned}$$

Don't forget to express the final answer in terms of x using $u = 1+x^2$. Also, don't forget that you can always check that

$$\int\sqrt{1+x^2}\,x^3\,dx = \frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}+C$$

is correct — just differentiate the right hand side

$$\begin{aligned}\frac{d}{dx}\Big[\frac{1}{5}(1+x^2)^{5/2}-\frac{1}{3}(1+x^2)^{3/2}+C\Big] &= \frac{1}{5}\,\frac{5}{2}(1+x^2)^{3/2}(2x)-\frac{1}{3}\,\frac{3}{2}(1+x^2)^{1/2}(2x)\\ &= x(1+x^2)^{3/2}-x(1+x^2)^{1/2}\\ &= x\sqrt{1+x^2}\,\big[(1+x^2)-1\big]\\ &= x\sqrt{1+x^2}\,x^2 = x^3\sqrt{1+x^2}\end{aligned}$$

and verify that the derivative is the same as the original integrand.

Example 8

Example 9

Consider $\int\tan x\,dx$. The secret here is to write the integrand $\tan x = \frac{1}{\cos x}\sin x$. Think of the first factor as the function "one over" evaluated at the argument $\cos x$. The derivative of the argument $\cos x$ is, except for a -1, the same as the second factor $\sin x$. Substitute $u = \cos x$, $du = -\sin x\,dx$ or $\sin x\,dx = \frac{du}{-1}$.

$$\begin{aligned}\int\tan x\,dx &= \int\frac{1}{\cos x}\sin x\,dx = \int\frac{1}{u}\,\frac{du}{-1} = -\ln|u|+C = -\ln|\cos x|+C\\ &= \ln|\cos x|^{-1}+C = \ln|\sec x|+C\end{aligned}$$

Example 9