Trignometric Substitution

Trigonometric substitution refers simply to substitutions of the form

$$x = a\sin u \quad \text{or} \quad x = a\tan u \quad \text{or} \quad x = a\sec u$$

It is generally used in conjunction with the trignometric identities

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{and} \quad 1 + \tan^2\theta = \sec^2\theta$$

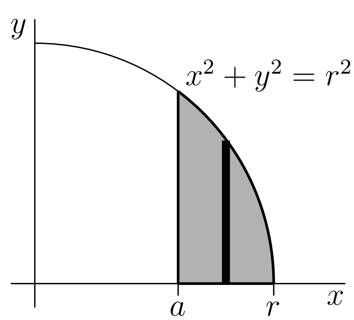
to

- eliminate $\sqrt{a^2-x^2}$ from an integrand by substituting $x = a\sin u$ to give $\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2 u} = \sqrt{a^2\cos^2 u} = |a\cos u|$ or to
- eliminate $\sqrt{a^2+x^2}$ from an integrand by substituting $x = a\tan u$ to give $\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2 u} = \sqrt{a^2\sec^2 u} = |a\sec u|$ or to
- eliminate $\sqrt{x^2-a^2}$ from an integrand by substituting $x = a\sec u$ to give $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2 u - a^2} = \sqrt{a^2\tan^2 u} = |a\tan u|$.

When we have used substitutions before, we usually gave the new integration variable, u, as a function of the old integration variable x. Here we are giving the old integration variable, x, in terms of the new integration variable u. We may do so, as long as we may invert to get u as a function of x. For example, with $x = a\sin u$, we may take $u = \arcsin\frac{x}{a}$. This is a good time for you to review the definitions of $\arcsin\theta$, $\arctan\theta$ and $\operatorname{arcsec}\theta$. See the notes "Inverse Functions".

Example 1 ($\int_a^r \sqrt{r^2-x^2}\,dx$)

Let's find the area of the shaded region in the sketch below.



We'll set up the integral using vertical strips. The strip in the figure has width dx and height $\sqrt{r^2-x^2}$. So the area is $\int_a^r \sqrt{r^2-x^2}\,dx$. To evaluate the integral we substitute

$$x = r\sin u \qquad dx = r\cos u\,du$$

because then we will be able to use

$$r^2 - x^2 = r^2 - r^2\sin^2 u = r^2\big(1-\sin^2 u\big) = r^2\cos^2 u$$

to eliminate the square root from the integrand. Let's think about the limits of integration. Our integral has x running from $x = a$ to $x = r$. The value of u that corresponds to $x = r$ is $u = {}^{\pi}\!/_2$ (which solves $x = r = r\sin u$, i.e. which solves $\sin u = 1$) and the value of u that corresponds to $x = a$ is $u = \arcsin {}^a\!/_r$ (which solves $x = a = r\sin u$. i.e. which solves $\sin u = \frac{a}{r}$). As u runs from $u = \arcsin {}^a\!/_r$ to $u = \frac{\pi}{2}$, $x = r\sin u$ runs from $x = a$ to $x = r$ covering exactly the domain of integration. So we'll make the domain of integration, in the u integral, $\arcsin {}^a\!/_r \le u \le \frac{\pi}{2}$. We are now ready to do the integral.

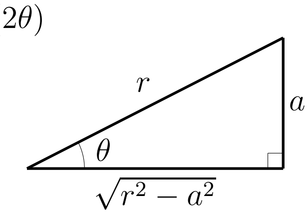
$$\begin{aligned}\int_a^r \sqrt{r^2 - x^2}\ dx &= \int_{\arcsin a/r}^{\pi/2} \sqrt{r^2 - r^2\sin^2 u}\ r\cos u\ du \qquad \text{with } x = r\sin u,\ dx = r\cos u\ du\\ &= \int_{\arcsin a/r}^{\pi/2} \sqrt{r^2\cos^2 u}\ r\cos u\ du\\ &= \int_{\arcsin a/r}^{\pi/2} r^2\cos^2 u\ du\end{aligned}$$

Be careful about taking the square root in the last step. Because $\sqrt{r^2 - x^2}$ denotes the *positive* square root of $a^2 - x^2$, $\sqrt{r^2\cos^2 u}$ denotes the positive square root of $r^2\cos^2 u$. Fortunately, the domain of integration is contained in $0 \le u \le \frac{\pi}{2}$ and $\cos u \ge 0$ there. So $r\cos u$ really is the *positive* square root of $r^2\cos^2 u$ in our integral. If our domain of integration had contained u's between $\frac{\pi}{2}$ and π, for example, we would have needed to write $\sqrt{r^2\cos^2 u} = r|\cos u|$. Now back to evaluating the integral.

$$\begin{aligned}\int_a^r \sqrt{r^2 - x^2}\ dx &= \int_{\arcsin a/r}^{\pi/2} r^2\cos^2 u\ du\\ &= \frac{r^2}{2}\int_{\arcsin a/r}^{\pi/2} \big[1 + \cos(2u)\big]\ du \qquad \text{since } \cos^2 u = \frac{1 + \cos(2u)}{2}\\ &= \frac{r^2}{2}\Big[u + \frac{\sin(2u)}{2}\Big]_{\arcsin a/r}^{\pi/2}\\ &= \frac{r^2}{2}\Big[\frac{\pi}{2} - \arcsin\frac{a}{r} - \frac{\sin(2\arcsin a/r)}{2}\Big]\end{aligned}$$

To simplify $\frac{\sin(2\arcsin a/r)}{2}$, let's write $\arcsin {}^a\!/_r = \theta$. Then θ is the angle in the triangle on the right below. By the double angle formula for $\sin(2\theta)$

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\ \cos\theta\\ &= 2\ \frac{a}{r}\ \frac{\sqrt{r^2 - a^2}}{r}\end{aligned}$$



So our final answer is

$$\text{Area} = \int_a^r \sqrt{r^2 - x^2}\ dx = \frac{\pi r^2}{4} - \frac{r^2}{2}\arcsin\frac{a}{r} - \frac{1}{2}a\sqrt{r^2 - a^2} \tag{1}$$

This is a relatively complicated formula, but we can make some "reasonableness" checks, by looking at special values of a. If $a = 0$ the shaded region, in the figure at the beginning of this example, is exactly one quarter of a disk of radius r and so has area $\frac{1}{4}\pi r^2$. Subbing $a = 0$ into (1) does indeed give $\frac{1}{4}\pi r^2$. At the other extreme, if $a = r$, the shaded region disappears completely and so has area 0. Subbing $a = r$ into (1) does indeed give 0, since $\arcsin 1 = \frac{\pi}{2}$.

Example 1

Example 2 ($\int_a^r x\sqrt{r^2-x^2}\ dx$)

The integral $\int_a^r x\sqrt{r^2-x^2}\ dx$ looks a lot like the integral we just did in Example 1. It can also be evaluated using the trigonometric substitution $x = r\sin u$. But just because you have now learned how to use trig substitution doesn't mean that you should forget everything you learned before. This integral is *much* more easily evaluated using the simple substitution $u = r^2 - x^2$.

$$\begin{aligned}\int_a^r x\sqrt{r^2-x^2}\ dx &= \int_{r^2-a^2}^0 \sqrt{u}\ \frac{du}{-2} \qquad \text{with } u = r^2-x^2,\ du = -2x\ dx\\ &= -\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_{r^2-a^2}^0\\ &= \frac{1}{3}\left[r^2-a^2\right]^{3/2}\end{aligned}$$

Example 2

Example 3 ($\int \frac{dx}{x^2\sqrt{9+x^2}}$)

This time we'll substitute

$$x = 3\tan u \qquad dx = 3\sec^2 u\ du$$

because then we will be able to use

$$\sqrt{9+x^2} = \sqrt{9+9\tan^2 u} = 3\sqrt{1+\tan^2 u} = 3\sqrt{\sec^2 u} = 3|\sec u|$$

to eliminate the square root from the integral. Note that, to satisfy $x = 3\tan u$, we can take $u = \arctan\frac{x}{3}$, with "arctan" being the "standard" arctangent that always takes values between $-\pi/2$ and $+\pi/2$. So u will always take values between $-\pi/2$ and $+\pi/2$ and $\cos u$ will always be positive, so that $|\sec u| = \sec u$. So our integral

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{9+x^2}} &= \int \frac{3\sec^2 u\ du}{9\tan^2 u\ 3\sec u} \qquad \text{with } x = 3\tan u,\ dx = 3\sec^2 u\ du\\ &= \frac{1}{9}\int \frac{\sec u}{\tan^2 u}\ du\end{aligned}$$

$$= \frac{1}{9}\int \frac{\cos u}{\sin^2 u}\, du \qquad \text{since } \sec u = \frac{1}{\cos u} \text{ and } \frac{1}{\tan^2 u} = \frac{\cos^2 u}{\sin^2 u}$$

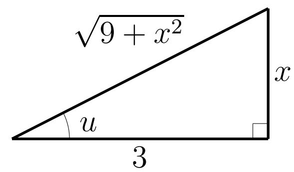
$$= \frac{1}{9}\int \frac{dy}{y^2} \qquad \text{with } y = \sin u,\ dy = \cos u\ du$$

$$= -\frac{1}{9y} + C$$

$$= -\frac{1}{9\sin u} + C$$

The original integral was a function of x, so we still have to rewrite $\sin u$ in terms of x. Remember that $x = 3\tan u$ or $u = \arctan\frac{x}{3}$. So u is the angle shown in the triangle below and we can read off the triangle that

$$\sin u = \frac{x}{\sqrt{9+x^2}}$$



$$\implies \int \frac{dx}{x^2\sqrt{9+x^2}} = -\frac{\sqrt{9+x^2}}{9x} + C$$

Example 3

Example 4 ($\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\, dx$)

This time we have an integral with a square root in the integrand, but the argument of the square root, while a quadratic function of x, is not in one of the standard forms $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$. The reason that it is not in one of those forms is that the argument, x^2-2x-3, contains a term , namely, $-2x$ that is of degree one on x. So we try manipulate it into one of the standard forms by completing the square, which means that we try to express x^2-2x-3 in the form $(x-a)^2+b$ for some constants a and b. Observe that if we square out $(x-a)^2+b$ we get $x^2-2ax+a^2+b$, which will be exactly x^2-2x-3 if we choose a and b so that $-2a=-2$ (to give the correct coefficient of x) and $a^2+b=-3$ (to give the correct constant term). So $a=1$, $b=-4$ works and we now know that

$$x^2-2x-3 = (x-1)^2-4$$

Then to convert the square root of the integrand into a standard form, we just make the simple substitution $y = x-1$. Here goes

$$\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\, dx = \int_3^5 \frac{\sqrt{(x-1)^2-4}}{x-1}\, dx$$

$$= \int_2^4 \frac{\sqrt{y^2-4}}{y}\, dy \qquad \text{with } y = x-1,\ dy = dx$$

$$= \int_0^{\pi/3} \frac{\sqrt{4\sec^2 u-4}}{2\sec u}\, 2\sec u\tan u\ du \qquad \text{with } y = 2\sec u,\ dy = 2\sec u\tan u\ du$$

To get the limits of integration we used that

- the value of u that corresponds to $y = 2$ obeys $2 = y = 2\sec u = \frac{2}{\cos u}$ or $\cos u = 1$, so that $u = 0$ works and
- the value of u that corresponds to $y = 4$ obeys $4 = y = 2\sec u = \frac{2}{\cos u}$ or $\cos u = \frac{1}{2}$, so that $u = {}^{\pi}\!/_{3}$ works.

Now returning to the evaluation of the integral, we simplify and continue.

$$\begin{aligned}
\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\,dx &= \int_0^{\pi/3} 2\sqrt{\sec^2 u - 1}\ \tan u\ du \\
&= 2\int_0^{\pi/3} \tan^2 u\ du && \text{since } \sec^2 u = 1 + \tan^2 u \\
&= 2\int_0^{\pi/3} \big[\sec^2 u - 1\big]\ du && \text{since } \sec^2 u = 1 + \tan^2 u\text{, again} \\
&= 2\Big[\tan u - u\Big]_0^{\pi/3} \\
&= 2\big[\sqrt{3} - {}^{\pi}\!/_{3}\big]
\end{aligned}$$

Example 4