Inverse Functions

We are now going to consider the class of problems in which

- we have a given function, that we'll call f, and
- for each number X
- we wish to find a number Y obeying

$$f(Y) = X \tag{1}$$

If we're lucky, then for each real number X there is exactly one real number Y, that we'll call $f^{-1}(X)$, obeying (1). Then f^{-1} is called the inverse function of f. A (trivial) example in which this happens is given in Example 1, below.

If we're a little less lucky, there is a set of real numbers $\mathcal{D}$ (that does not contain all of $\mathbb{R}$) such that

- for each real number X in $\mathcal{D}$ there is exactly one real number Y, that we'll again call $f^{-1}(X)$, obeying (1) but
- for each real number X that is *not* in $\mathcal{D}$ there is *no* Y obeying (1).

Then f^{-1} is again called the inverse function of f and $\mathcal{D}$ is called the domain of f^{-1}. One example of this is $f(x) = e^x$. We'll review this example in Example 2, below.

If we're still little less lucky, there is at least one real number X for which there is more than one real number Y obeying (1). The trigonometric functions are like this. We'll take a first quick look at this in Example 3, below and take a more thorough look in the last half of these notes.

Example 1

Let $f(x) = 2x$. For this $f(x)$, equation (1) becomes

$$2Y = X$$

For each real number X, there is exactly one Y, namely $Y = \frac{X}{2}$, that obeys $2Y = X$. So, the function $f(x) = 2x$ has inverse function $f^{-1}(X) = \frac{X}{2}$.

Example 1

Example 2

Let $f(x) = e^x$. For this $f(x)$, equation (1) becomes

$$e^Y = X$$

For concreteness, let's pick a specific value of X, say $X = 2$. The graph of e^Y, as a function of Y, is sketched below. In that sketch, the x–axis has been renamed the Y–axis, because we are interested in e^Y as a function of Y. (Be careful to distinguish the upper case Y from the lower case y.) The number of Y's obeying $e^Y = 2$ is exactly the number of times the

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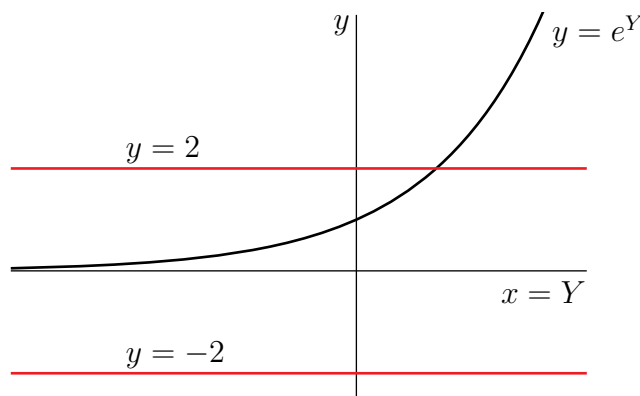
horizontal straight line $y = 2$ intersects the graph $y = e^Y$, which is one. So for $X = 2$, there is exactly one Y obeying $e^Y = X$. On the other hand, for $X = -2$, the number of Y's obeying $e^Y = -2$ is exactly the number of times the horizontal straight line $y = -2$ intersects the graph $y = e^Y$, which is zero. So for $X = -2$, no Y's obey $e^Y = X$.

As Y runs from $-\infty$ to $+\infty$, e^Y takes each strictly positive value exactly once and never takes any value zero or smaller. So the domain of $\ln x$, the inverse function of e^x, is exactly the interval $(0, \infty)$.
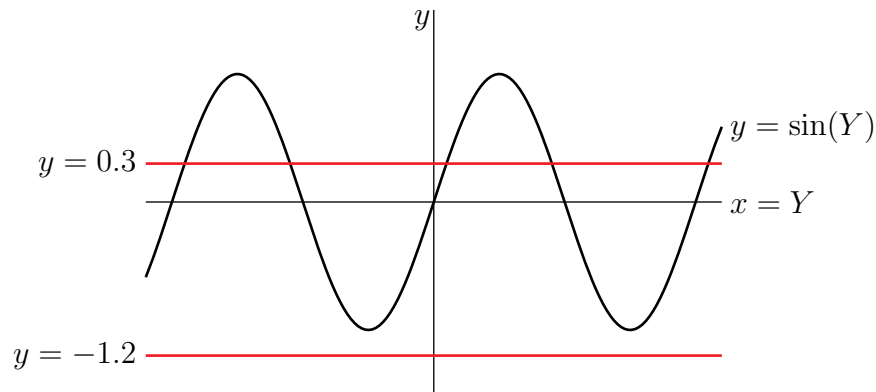
Example 2

Example 3

Let $f(x) = \sin(x)$. For this $f(x)$, equation (1) becomes

$$\sin(Y) = X$$

For each fixed real number X, the number of Y's that obey $\sin(Y) = X$, is exactly the number of times the horizontal straight line $y = X$ intersects the graph $y = \sin(Y)$. When $-1 \le X \le 1$, the line $y = X$ intersects the graph $y = \sin(Y)$ infinitely many times. This is illustrated in the figure below by the line $y = 0.3$. On the other hand, when $X < -1$ or $X > 1$, the line $y = X$ never intersects the graph $y = \sin(Y)$. This is illustrated in the figure below by the line $y = -1.2$. We'll see what is normally done about below.
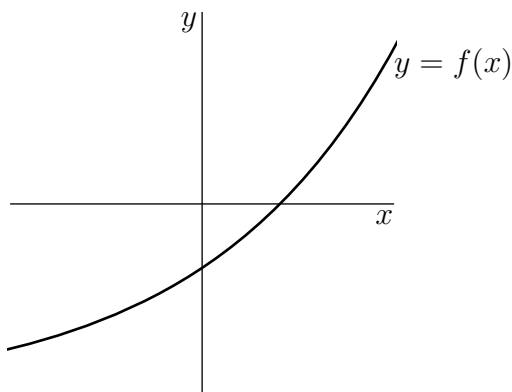

Example 3

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It is an easy matter to construct the graph of an inverse function from the graph of the original function. We just need to remember that
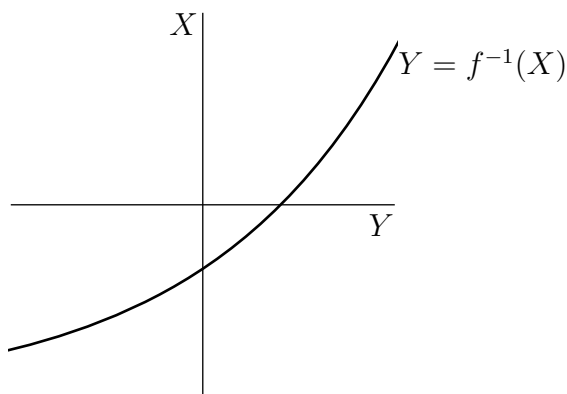
$$Y = f^{-1}(X) \iff f(Y) = X$$

which is $y = f(x)$ with x renamed to Y and y renamed to X.

Start by drawing the graph of f, labelling the x– and y–axes and labelling the curve $y = f(x)$.

Now replace each x by Y and each y by X abd replace the resulting label $X = f(Y)$ on the curve by the equivalent $Y = f^{-1}(X)$.

Finally we just need to redraw the sketch with the Y axis running vertically (with Y increasing upwards) and the X axis running horizontally (with X increasing to the right). To do so, pretend that the sketch was on a transparency or on a very thin piece of paper that you can see through. Lift the sketch up and flip it over so that the Y axis runs vertically and the X axis runs horizontally. If you want can also convert the upper case X into a lower case x and the upper case Y into a lower case y.

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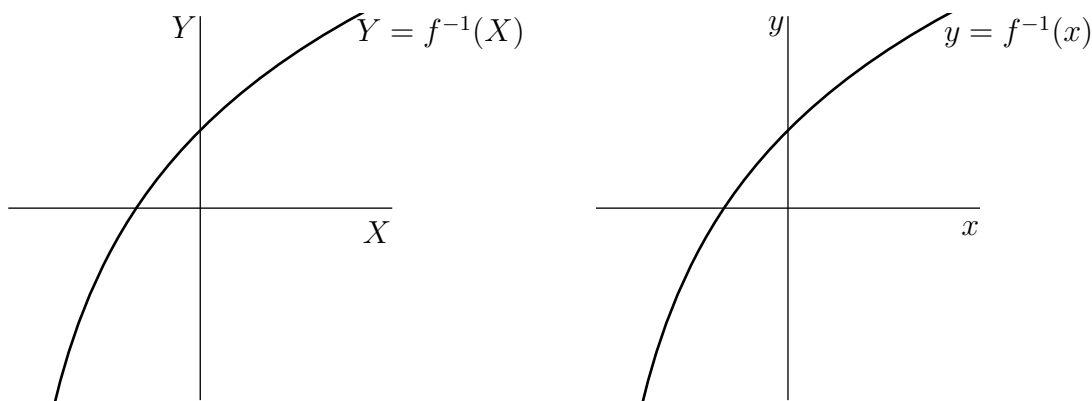
It is also an easy matter to use implicit differentiation to find a formula for the derivative[1] of f^{-1} in terms of the derivative of f. Substitute $Y = f^{-1}(X)$ into $f(Y) = X$ to give

$$f\big(f^{-1}(X)\big) = X$$

Rename X to x and apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides.

$$\frac{\mathrm{d}}{\mathrm{d}x} f\big(f^{-1}(x)\big) = \frac{\mathrm{d}}{\mathrm{d}x} x = 1$$

By the chain rule

$$f'\big(f^{-1}(x)\big) \cdot \frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = 1 \implies \frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'\big(f^{-1}(x)\big)} \tag{2}$$

Example 4

The inverse function of $f(x) = e^x$ is $f^{-1}(x) = \ln x$. Since $f'(x) = e^x$, (2) gives

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

which is the conclusion you undoubtedly expected.

Example 4

Inverse Trigonometric Functions

We now return to the question of finding inverse functions for the trigonometric functions, starting with $\sin x$. So set $f(x) = \sin x$. As we saw in Example 3, when $|X| > 1$ no Y obeys $\sin(Y) = X$ and, for each $-1 \le X \le 1$, the line $y = X$ (illustrated in the figure below with $X = 0.3$) crosses the curve $y = \sin(Y)$ infinitely many times, so that there are infinitely many Y's that obey $f(Y) = \sin Y = X$. However exactly one of those crossings (the dot in the figure) has $-\pi/2 \le Y \le \pi/2$. That is, for each $-1 \le X \le 1$, there is exactly one Y that obeys both

$$\sin Y = X \qquad \text{and} \qquad -\frac{\pi}{2} \le Y \le \frac{\pi}{2}$$

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[1] There is a theorem called the Inverse Function Theorem, which we will not prove, that says that, under reasonable hypotheses on $f(x)$, $f^{-1}(x)$ is differentiable.

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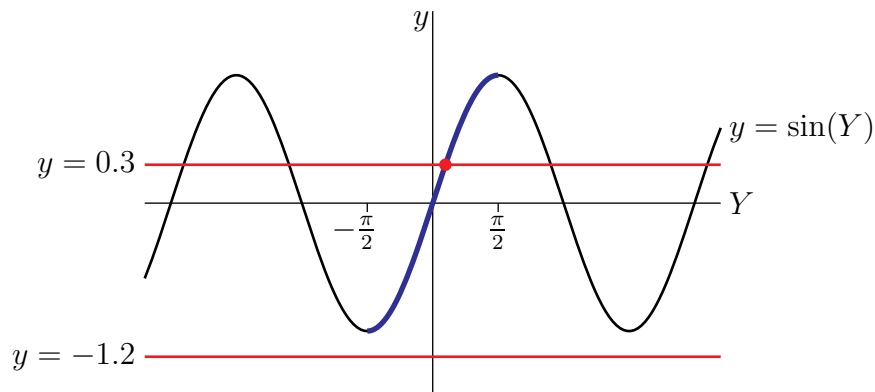
That unique Y is denoted[2] $\sin^{-1} X$ or $\arcsin(X)$. That is (renaming X to x), $\arcsin(x)$ is defined for all $-1 \le x \le 1$ and is determined by

$$\sin\big(\arcsin(x)\big) = x \qquad \text{and} \qquad -\frac{\pi}{2} \le \arcsin(x) \le \frac{\pi}{2} \qquad (3)$$

Example 5

Since

$$\sin\frac{\pi}{2} = 1 \qquad \sin\frac{\pi}{6} = \frac{1}{2}$$

and $-\pi/2 \le \pi/6, \pi/2 \le \pi/2$, we have

$$\arcsin 1 = \frac{\pi}{2} \qquad \arcsin\frac{1}{2} = \frac{\pi}{6}$$

Even though

$$\sin(2\pi) = 0$$

it is **not** true that $\arcsin 0 = 2\pi$, and it is **not** true that $\arcsin\big(\sin(2\pi)\big) = 2\pi$, because 2π is not between $-\pi/2$ and $\pi/2$. More generally

$$\arcsin\big(\sin(x)\big) = \text{ the unique angle } Y \text{ between } -\pi/2 \text{ and } \pi/2 \text{ obeying } \sin Y = \sin x$$
$$= x \quad \text{if and only if } -\pi/2 \le x \le \pi/2$$

So, for example, $\arcsin\big(\sin\left(^{11\pi}/_{16}\right)\big)$ cannot be $^{11\pi}/_{16}$ because $^{11\pi}/_{16}$ is bigger than $\pi/2$. So how do we find the correct answer? Start by sketching the graph of sin.

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[2]The notations $\mathrm{asin}(X)$ and $\mathrm{Arcsin}(X)$ are also used. Sometimes $\arcsin(X)$ is used for the "multivalued" function which gives all Y's obeying $\sin(Y) = X$ with $\mathrm{Arcsin}(X)$ being reserved for the Y that also obeys $-\pi/2 \le Y \le \pi/2$.

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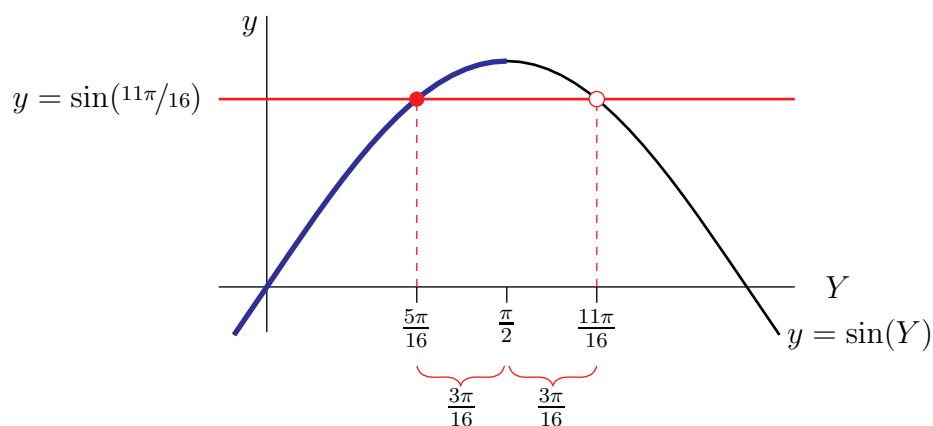
It looks like the graph of $\sin Y$ is symmetric about $Y = \pi/2$. The mathematical way to say that "graph of $\sin Y$ is symmetric about $Y = \pi/2$" is "$\sin(\pi/2 - \theta) = \sin(\pi/2 + \theta)$" for all θ. That is indeed true — there are trig identities saying that they are equal to $\cos\theta$. Now $11\pi/16 = \pi/2 + 3\pi/16$ so

$$\sin\left(\frac{11\pi}{16}\right) = \sin\left(\frac{\pi}{2} + \frac{3\pi}{16}\right) = \sin\left(\frac{\pi}{2} - \frac{3\pi}{16}\right) = \sin\left(\frac{5\pi}{16}\right)$$

and, since $5\pi/16$ is indeed between $-\pi/2$ and $\pi/2$,

$$\arcsin\left(\sin\left(\frac{11\pi}{16}\right)\right) = \frac{5\pi}{16} \qquad \left(\text{and } \textbf{not } \frac{11\pi}{16}\right)$$

Example 5

To find $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x)$, we'll use (2) with $f(x) = \sin x$ and $f'(x) = \cos x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\cos\left(\arcsin(x)\right)}$$

This answer is not very satisfying because the right hand side is expressed in terms of $\arcsin(x)$ and we do not have an explicit formula for $\arcsin(x)$. However even without an explicit formula for $\arcsin(x)$, it is a simple matter to get an explicit formula for $\cos\left(\arcsin(x)\right)$, which is all we need. Just draw a right–angled with one angle being $\arcsin(x)$. This is done in the figure below.[3] To save writing, we are using θ to stand for $\arcsin(x)$ in the figure.
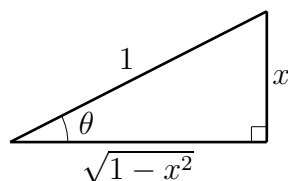
[3]The figure is drawn for the case that $0 \le \arcsin(x) \le \pi/2$. Virtually the same argument works for the case $-\pi/2 \le \arcsin(x) \le 0$
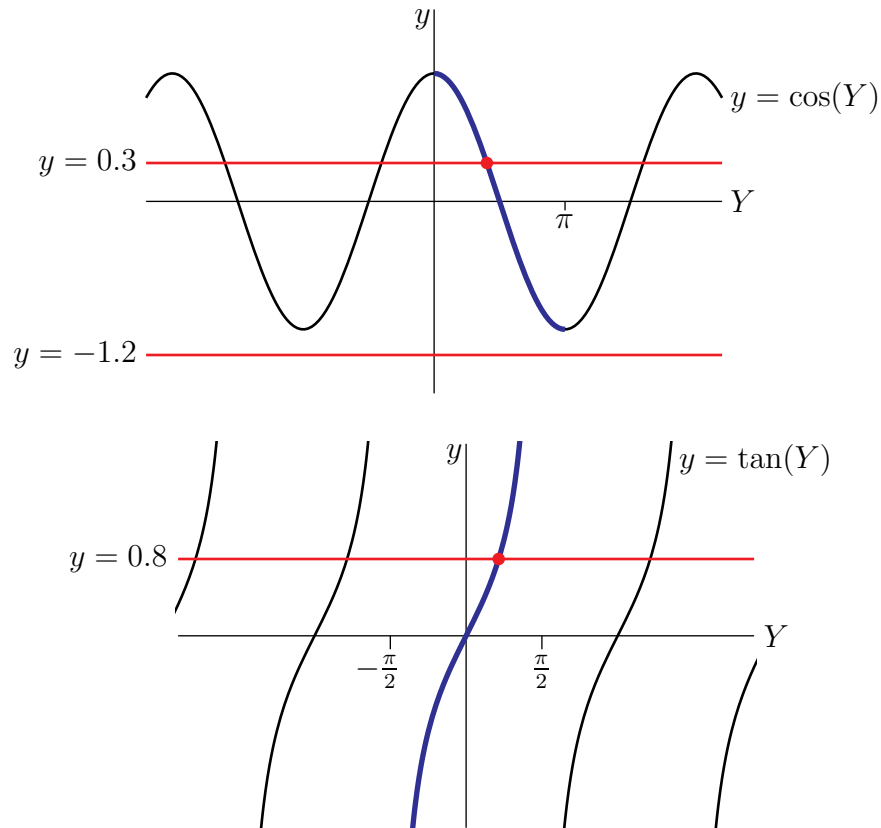
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Since $\sin(\theta) = x$ (see (3)), we have made the side opposite the angle θ of length x and the hypoteneuse of length 1. Then, by Pythagorous, the side adjacent to θ has length $\sqrt{1 - x^2}$ and so

$$\cos\big(\arcsin(x)\big) = \cos(\theta) = \sqrt{1 - x^2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1 - x^2}} \tag{4}$$

The definitions for arccos and arctan are developed in the same way. Here are the graphs that are used.

The definitions for the remaining three inverse trigonometric functions may also be developed in the same way. But it's a little easier to use

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

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> **Definition 6.**
>
> $\arcsin x$ is defined for $|x| \leq 1$. It is the unique number obeying
>
> $$\sin\big(\arcsin(x)\big) = x \qquad \text{and} \qquad -\frac{\pi}{2} \leq \arcsin(x) \leq \frac{\pi}{2}$$
>
> $\arccos x$ is defined for $|x| \leq 1$. It is the unique number obeying
>
> $$\cos\big(\arccos(x)\big) = x \qquad \text{and} \qquad 0 \leq \arccos(x) \leq \pi$$
>
> $\arctan x$ is defined for all $x \in \mathbb{R}$. It is the unique number obeying
>
> $$\tan\big(\arctan(x)\big) = x \qquad \text{and} \qquad -\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2}$$
>
> $\operatorname{arccsc} x = \arcsin\frac{1}{x}$ is defined for $|x| \geq 1$. It is the unique number obeying
>
> $$\csc\big(\operatorname{arccsc}(x)\big) = x \qquad \text{and} \qquad -\frac{\pi}{2} \leq \operatorname{arccsc}(x) \leq \frac{\pi}{2}$$
>
> $\operatorname{arcsec} x = \arccos\frac{1}{x}$ is defined for $|x| \geq 1$. It is the unique number obeying
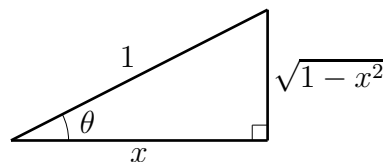>
> $$\sec\big(\operatorname{arcsec}(x)\big) = x \qquad \text{and} \qquad 0 \leq \operatorname{arccsc}(x) \leq \pi$$
>
> $\operatorname{arccot} x = \arctan\frac{1}{x}$ is defined for all $0 \neq x \in \mathbb{R}$. It is the unique number obeying
>
> $$\cot\big(\operatorname{arccot}(x)\big) = x \qquad \text{and} \qquad -\frac{\pi}{2} < \operatorname{arccot}(x) < \frac{\pi}{2}$$
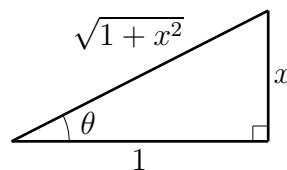
To find the derivative of arccos we apply (2) with $f(x) = \cos x$ and $f'(x) = -\sin x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) = -\frac{1}{\sin\big(\arccos(x)\big)} = -\frac{1}{\sqrt{1-x^2}}$$

(In the triangle, $\theta = \arccos(x)$.) To find the derivative of arctan we apply (2) with $f(x) = \tan x$ and $f'(x) = \sec^2 x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{\sec^2\big(\arctan(x)\big)} = \frac{1}{1+x^2}$$

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(In the triangle, $\theta = \arctan(x)$.) To find the derivatives of the remaining three inverse trig functions, we just use their definitions, derivatives we already know and the chain rule.

$$\frac{d}{dx}\operatorname{arccsc}(x) = \frac{d}{dx}\arcsin\left(\frac{1}{x}\right) = \frac{1}{\sqrt{1 - 1/x^2}} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}\operatorname{arcsec}(x) = \frac{d}{dx}\arccos\left(\frac{1}{x}\right) = -\frac{1}{\sqrt{1 - 1/x^2}} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}\operatorname{arccot}(x) = \frac{d}{dx}\arctan\left(\frac{1}{x}\right) = \frac{1}{1 + 1/x^2} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{1 + x^2}$$

By way of summary, we have

Theorem 7.

The derivatives of the inverse trgonometric functions are

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1 - x^2}} \qquad\qquad \frac{d}{dx}\operatorname{arccsc}(x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1 - x^2}} \qquad\qquad \frac{d}{dx}\operatorname{arcsec}(x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1 + x^2} \qquad\qquad \frac{d}{dx}\operatorname{arccot}(x) = -\frac{1}{1 + x^2}$$

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