Mixed Partial Derivatives

In these notes we prove that the mixed partial derivatives $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are equal at points where both are continuous. This result goes under several different names including "equality of mixed partials" and "Clairaut's theorem". Following the proof there is an example which shows that, when $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are not continuous, they can be different.

Theorem 1.

If the partial derivatives $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ exist and are continuous at (a, b), then

$$\frac{\partial^2 f}{\partial y \partial x}(a, b) = \frac{\partial^2 f}{\partial x \partial y}(a, b)$$

Proof. Here is an outline of the proof. The details are given as "footnotes" at the end of the outline. Fix a and b and define$^{(1)}$

$$F(h, k) = \frac{1}{hk}\big[f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(a, b)\big]$$

Then, by the mean value theorem,

$$\begin{aligned} F(h, k) &\overset{2}{=} \frac{1}{h}\Big[\frac{\partial f}{\partial y}(a+h, b+\theta_1 k) - \frac{\partial f}{\partial y}(a, b+\theta_1 k)\Big] \\ &\overset{3}{=} \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(a+\theta_2 h, b+\theta_1 k) \\ F(h, k) &\overset{4}{=} \frac{1}{k}\Big[\frac{\partial f}{\partial x}(a+\theta_3 h, b+k) - \frac{\partial f}{\partial x}(a+\theta_3 h, b)\Big] \\ &\overset{5}{=} \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(a+\theta_3 h, b+\theta_4 k) \end{aligned}$$

for some $0 < \theta_1, \theta_2, \theta_3, \theta_4 < 1$. Here "$\overset{2}{=}$" means that the equality is shown in "footnote" number 2. All of $\theta_1, \theta_2, \theta_3, \theta_4$ depend on a, b, h, k. Hence

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(a+\theta_2 h, b+\theta_1 k) = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(a+\theta_3 h, b+\theta_4 k)$$

for all h and k. Taking the limit $(h, k) \to (0, 0)$ and using the assumed continuity of both partial derivatives at (a, b) gives

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(a, b) = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(a, b)$$

The Details

1. We define $F(h,k)$ in this way because both partial derivatives $\frac{\partial^2 f}{\partial x \partial y}(a,b)$ and $\frac{\partial^2 f}{\partial y \partial x}(a,b)$ are defined as limits of $F(h,k)$ as $h,k \to 0$. For example,

$$\begin{aligned} \lim_{h\to 0} F(h,k) &= \frac{1}{k}\Big[\frac{\partial f}{\partial x}(a,b+k) - \frac{\partial f}{\partial x}(a,b)\Big] \\ \implies \qquad \lim_{k\to 0}\lim_{h\to 0} F(h,k) &= \lim_{k\to 0}\frac{1}{k}\Big[\frac{\partial f}{\partial x}(a,b+k) - \frac{\partial f}{\partial x}(a,b)\Big] \\ &= \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(a,b) \end{aligned}$$

Similarly,

$$\begin{aligned} \lim_{k\to 0} F(h,k) &= \frac{1}{h}\Big[\frac{\partial f}{\partial y}(a+h,b) - \frac{\partial f}{\partial y}(a,b)\Big] \\ \implies \qquad \lim_{h\to 0}\lim_{k\to 0} F(h,k) &= \lim_{h\to 0}\frac{1}{h}\Big[\frac{\partial f}{\partial y}(a+h,b) - \frac{\partial f}{\partial y}(a,b)\Big] \\ &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(a,b) \end{aligned}$$

2. Define $G(y) = f(a+h,y) - f(a,y)$. By the mean value theorem

$$\begin{aligned} F(h,k) &= \frac{1}{h}\Big[\frac{G(b+k)-G(b)}{k}\Big] \\ &= \frac{1}{h}\frac{dG}{dy}(b+\theta_1 k) \qquad \text{for some } 0<\theta_1<1 \\ &= \frac{1}{h}\Big[\frac{\partial f}{\partial y}(a+h,b+\theta_1 k) - \frac{\partial f}{\partial y}(a,b+\theta_1 k)\Big] \end{aligned}$$

3. Define $H(x) = \frac{\partial f}{\partial y}(x,b+\theta_1 k)$. By the mean value theorem

$$\begin{aligned} F(h,k) &= \frac{1}{h}\Big[H(a+h)-H(a)\Big] \\ &= \frac{dH}{dx}(a+\theta_2 h) \qquad \text{for some } 0<\theta_2<1 \\ &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(a+\theta_2 h,b+\theta_1 k) \end{aligned}$$

4. Define $A(x) = f(x,b+k) - f(x,b)$. By the mean value theorem

$$\begin{aligned} F(h,k) &= \frac{1}{k}\Big[\frac{A(a+h)-A(a)}{h}\Big] \\ &= \frac{1}{k}\frac{dA}{dx}(a+\theta_3 h) \qquad \text{for some } 0<\theta_3<1 \\ &= \frac{1}{k}\Big[\frac{\partial f}{\partial x}(a+\theta_3 h,b+k) - \frac{\partial f}{\partial x}(a+\theta_3 h,b)\Big] \end{aligned}$$

5. Define $B(y) = \frac{\partial f}{\partial x}(a + \theta_3 h, y)$. By the mean value theorem

$$\begin{aligned} F(h,k) &= \frac{1}{k}\Big[B(b+k) - B(b)\Big] \\ &= \frac{dB}{dy}(b + \theta_4 k) \qquad \text{for some } 0 < \theta_4 < 1 \\ &= \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(a + \theta_3 h, b + \theta_4 k) \end{aligned}$$

$\square$

Example 2

Here is an example which shows that it is **not always true** that $\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$. Define

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

This function is continuous everywhere. We now compute the first order partial derivatives. For $(x,y) \neq (0,0)$

$$\begin{aligned} \frac{\partial f}{\partial x}(x,y) &= y\frac{x^2-y^2}{x^2+y^2} + xy\frac{2x}{x^2+y^2} - xy\frac{2x(x^2-y^2)}{(x^2+y^2)^2} = y\frac{x^2-y^2}{x^2+y^2} + xy\frac{4xy^2}{(x^2+y^2)^2} \\ \frac{\partial f}{\partial y}(x,y) &= x\frac{x^2-y^2}{x^2+y^2} - xy\frac{2y}{x^2+y^2} - xy\frac{2y(x^2-y^2)}{(x^2+y^2)^2} = x\frac{x^2-y^2}{x^2+y^2} - xy\frac{4yx^2}{(x^2+y^2)^2} \end{aligned}$$

For $(x,y) = (0,0)$

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \Big[\frac{d}{dx}f(x,0)\Big]_{x=0} = \Big[\frac{d}{dx}0\Big]_{x=0} = 0 \\ \frac{\partial f}{\partial y}(0,0) &= \Big[\frac{d}{dy}f(0,y)\Big]_{y=0} = \Big[\frac{d}{dy}0\Big]_{y=0} = 0 \end{aligned}$$

By way of summary, the two first order partial derivatives are

$$f_x(x,y) = \begin{cases} y\frac{x^2-y^2}{x^2+y^2} + \frac{4x^2y^3}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$f_y(x,y) = \begin{cases} x\frac{x^2-y^2}{x^2+y^2} - \frac{4x^3y^2}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Both $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous. Finally, we compute

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y}(0,0) &= \big[\frac{d}{dx}f_y(x,0)\big]_{x=0} = \lim_{h\to 0}\frac{1}{h}\big[f_y(h,0) - f_y(0,0)\big] = \lim_{h\to 0}\frac{1}{h}\big[h\frac{h^2-0^2}{h^2+0^2} - 0\big] = 1 \\ \frac{\partial^2 f}{\partial y \partial x}(0,0) &= \big[\frac{d}{dy}f_x(0,y)\big]_{y=0} = \lim_{k\to 0}\frac{1}{k}\big[f_x(0,k) - f_x(0,0)\big] = \lim_{k\to 0}\frac{1}{k}\big[k\frac{0^2-k^2}{0^2+k^2} - 0\big] = -1 \end{aligned}$$

Example 2