Partial Derivatives

The derivative of a function, $f(x)$, of one variable tells you how quickly $f(x)$ changes as you increase the value of the variable x. For a function $f(x, y)$ of two variables, there are two corresponding derivatives.

- One is called the partial derivative with respect to x. It is denoted $\frac{\partial f}{\partial x}(x, y)$ and tells you how quickly $f(x, y)$ changes as you increase the value of the variable x while holding the value of the other variable, y, fixed. The funny looking symbol "∂" is a stylized "d" and is read "partial". The symbols $\frac{\partial f}{\partial x}$ is read " partial dee f by dee x".
- The other is called the partial derivative with respect to y. It is denoted $\frac{\partial f}{\partial y}(x, y)$ and tells you how quickly $f(x, y)$ changes as you increase the value of the variable y while holding the value of the variable x fixed.

For example, if $f(x, y)$ is the output of a factory when its labor force has size x and the amount of capital that has been invested is y, then

- $\frac{\partial f}{\partial x}(x, y)$ tells you what impact increasing the labour force has on output, when no additional capital is invested and
- $\frac{\partial f}{\partial y}(x, y)$ tells you what impact additional capital investment would have on output, when no more employees are hired.

Here are the precise definitions.

Definition 1.

Let the function $f(x, y)$ be defined for all (x, y) in some disk of nonzero radius centred on (a, b). The partial derivative of f with respect to x, evaluated at (a, b), is

$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

and the partial derivative of f with respect to y, evaluated at (a, b), is

$$\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

when the limits exist.

Notation 2.

There are quite a few commonly used notations for partial derivatives. Here are some of them.

$$\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial x}\Big|_{(a,b)} = f_x(a, b) \qquad \frac{\partial f}{\partial y}(a, b) = \frac{\partial f}{\partial y}\Big|_{(a,b)} = f_y(a, b)$$

In practice, you compute the partial derivative $\frac{\partial f}{\partial x}(x,y)$ by pretending that y is a constant, like π, rather than a variable. When y is a constant, $f(x,y)$ is a function of the one variable x and you differentiate it in the normal way. Similarly you compute the partial derivative $\frac{\partial f}{\partial y}(x,y)$ by pretending that x is a constant rather than a variable. When x is a constant, $f(x,y)$ is a function of the one variable y and you differentiate it in the normal way.

Here are some examples. In these examples,

- if y is to be thought of as a variable, we'll write it "y", and if y is to be thought of as a constant, we'll write if "Y".
- Similarly if x is to be thought of as a variable, we'll write it "x", and if x is to be thought of as a constant, we'll write if "X".

Once we have had a little more experience computing partial derivates, we'll stop using X and Y to highlight when x and y, respectively, are to be thought of as constants.

Example 3 ($f(x,y) = 2x^2 + y^2$)

Let $f(x,y) = 2x^2 + y^2$. Evaluate $\frac{\partial f}{\partial x}(1,3)$ and $\frac{\partial f}{\partial y}(1,3)$.

Solution. We'll first compute $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ and then evaluate the results at $x=1$, $y=3$. When x is to be thought of as a constant, we'll write it X and when y is to be thought of as a constant, we'll write it Y. Here we go

$$\frac{\partial f}{\partial x}(x,\mathrm{Y}) = \frac{\partial}{\partial x}\big(2x^2 + \mathrm{Y}^2\big) = 4x + 0 = 4x$$

$$\frac{\partial f}{\partial y}(\mathrm{X},y) = \frac{\partial}{\partial y}\big(2\mathrm{X}^2 + y^2\big) = 0 + 2y = 2y$$

and

$$\frac{\partial f}{\partial x}(1,2) = 4x\Big|_{\substack{x=1\\y=3}} = 4$$

$$\frac{\partial f}{\partial y}(1,2) = 2y\Big|_{\substack{x=1\\y=3}} = 6$$

Example 3

Example 4 ($f(x,y) = 2x^2 + y + 3xy^2 - x^3y^4$)

Let $f(x,y) = 2x^2 + y + 3xy^2 - x^3y^4$. Find $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$.

Solution. We'll again write X when x is to be thought of as a constant, and write Y when y is to be thought of as a constant.

$$\frac{\partial f}{\partial x}(x,\mathrm{Y}) = \frac{\partial}{\partial x}\big(2x^2 + \mathrm{Y} + 3x\mathrm{Y}^2 - x^3\mathrm{Y}^4\big) = 4x + 0 + 3\mathrm{Y}^2 - 3x^2\mathrm{Y}^4 \qquad = 4x + 3\mathrm{Y}^2 - 3x^2\mathrm{Y}^4$$

$$\frac{\partial f}{\partial y}(\mathrm{X},y) = \frac{\partial}{\partial y}\big(2\mathrm{X}^2 + y + 3\mathrm{X}y^2 - \mathrm{X}^3y^4\big) = 0 + 1 + 3\mathrm{X}(2y) - \mathrm{X}^3(4y^3) = 1 + 6\mathrm{X}y - 4\mathrm{X}^3y^3$$

Restoring the original notation,

$$\frac{\partial f}{\partial x}(x,y) = 4x + 3y^2 - 3x^2y^4 \qquad \frac{\partial f}{\partial y}(x,y) = 1 + 6xy - 4x^3y^3$$

Example 4

Example 5 ($f(x,y) = x\sin y^2 + ye^{xy}$)

Find the partial derivatives of $f(x,y) = x\sin y + ye^{xy}$.

Solution. We'll again, probably for the last time, write X when x is to be thought of as a constant, and write Y when y is to be thought of as a constant.

$$\frac{\partial f}{\partial x}(x,\mathrm{Y}) = \frac{\partial}{\partial x}\big(x\sin \mathrm{Y}^2 + \mathrm{Y}e^{\mathrm{Y}x}\big) = \sin \mathrm{Y}^2 + \mathrm{Y}^2e^{x\mathrm{Y}}$$
$$\frac{\partial f}{\partial y}(\mathrm{X},y) = \frac{\partial}{\partial y}\big(\mathrm{X}\sin y^2 + ye^{\mathrm{X}y}\big) = \mathrm{X}(\cos y^2)(2y) + e^{\mathrm{X}y} + y\mathrm{X}e^{\mathrm{X}y} = 2\mathrm{X}y\cos y^2 + e^{\mathrm{X}y} + \mathrm{X}ye^{\mathrm{X}y}$$

Restoring the original notation,

$$\frac{\partial f}{\partial x}(x,y) = \sin y^2 + y^2e^{xy} \qquad \frac{\partial f}{\partial y}(x,y) = 2xy\cos y^2 + e^{xy} + xye^{xy}$$

Example 5

Higher Order Partial Derivatives

For a function of one variable $f(x)$, the second order derivative $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ (with the name "second order" indicating that two derivatives are being applied) is found by differentiating $f(x)$ once to get $\frac{\mathrm{d}f}{\mathrm{d}x}$ and then differentiating the result to get $\frac{\mathrm{d}}{\mathrm{d}x}\big(\frac{\mathrm{d}f}{\mathrm{d}x}\big) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$. We can do the same thing for functions of two variables, except that then we have two different types of derivatives that we can apply — namely $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. A second order derivative of a function $f(x,y)$ has two derivatives applied to it, and each of those two derivatives can be either $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$. So a function of two variables has four second order derivatives. Here are the definitions of, and two notations used for, those four second order derivatives.

$$\frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial x}\Big) = \frac{\partial^2 f}{\partial x^2} \qquad \big(f_x\big)_x = f_{xx}$$
$$\frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}\Big) = \frac{\partial^2 f}{\partial y\partial x} \qquad \big(f_x\big)_y = f_{xy}$$
$$\frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big) = \frac{\partial^2 f}{\partial x\partial y} \qquad \big(f_y\big)_x = f_{yx}$$
$$\frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial y}\Big) = \frac{\partial^2 f}{\partial y^2} \qquad \big(f_y\big)_y = f_{yy}$$

In the notation $\frac{\partial^2 f}{\partial y \partial x}$, the derivative closest to f, in this case the x derivative, is applied first. Similarly, in the notation f_{xy}, the derivative closest to f, in this case the x derivative, is applied first.

Example 6 ($f(x,y) = x\sin y^2 + ye^{xy}$)

Find all second order partial derivatives of $f(x,y) = e^{xy}\sin y$.

Solution. We first find the first order partial derivatives.

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= \frac{\partial}{\partial x}\big(e^{xy}\sin y\big) = ye^{xy}\sin y \\
\frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y}\big(e^{xy}\sin y\big) = xe^{xy}\sin y + e^{xy}\cos y
\end{aligned}$$

Then we differentiate a second time, in all possible ways.

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}(x,y) &= \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial x}\Big) = \frac{\partial}{\partial x}\big(ye^{xy}\sin y\big) \\
&= y^2e^{xy}\sin y \\
\frac{\partial^2 f}{\partial y\partial x}(x,y) &= \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}\Big) = \frac{\partial}{\partial y}\big(ye^{xy}\sin y\big) \\
&= e^{xy}\sin y + xye^{xy}\sin y + ye^{xy}\cos y \\
\frac{\partial^2 f}{\partial x\partial y}(x,y) &= \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big) = \frac{\partial}{\partial x}\big(xe^{xy}\sin y + e^{xy}\cos y\big) \\
&= e^{xy}\sin y + xye^{xy}\sin y + ye^{xy}\cos y \\
\frac{\partial^2 f}{\partial y^2}(x,y) &= \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial y}\Big) = \frac{\partial}{\partial y}\big(xe^{xy}\sin y + e^{xy}\cos y\big) \\
&= x^2e^{xy}\sin y + xe^{xy}\cos y + xe^{y}\cos y - e^{xy}\sin y \\
&= x^2e^{xy}\sin y + 2xe^{xy}\cos y - e^{xy}\sin y
\end{aligned}$$

Example 6

Notice that, in Example 6, $\frac{\partial^2 f}{\partial y\partial x}(x,y)$ and $\frac{\partial^2 f}{\partial x\partial y}(x,y)$ turned out to be the same. This was not an accident. It is true for all "reasonable" functions. That's the content of the following theorem, which we state without proof. This theorem has a number of different names including "equality of mixed partials" and "Clairaut's theorem".

Theorem 7.

If the partial derivatives $\frac{\partial^2 f}{\partial y\partial x}$ and $\frac{\partial^2 f}{\partial x\partial y}$ exist and are continuous at (a,b), then

$$\frac{\partial^2 f}{\partial y\partial x}(a,b) = \frac{\partial^2 f}{\partial x\partial y}(a,b)$$

Example 8

Find, if possible a function $f(x,y)$ obeying

$$\frac{\partial f}{\partial x}(x,y)=\sin y \qquad \frac{\partial f}{\partial y}(x,y)=\cos x$$

Solution. This is a trick question. No such function $f(x,y)$ exists. If it were to exist, it would have to obey

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x}(x,y) &= \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}\Big) = \frac{\partial}{\partial y}\big(\sin y\big) = \cos y\\ \frac{\partial^2 f}{\partial x \partial y}(x,y) &= \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big) = \frac{\partial}{\partial x}\big(\cos x\big) = -\sin x\end{aligned}$$

So $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ would be different and that would violate Clairaut's theorem.

Example 8