Sequences and Series

You have probably learned about Taylor polynomials and, in particular, that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + E_n(x)$$

where $E_n(x)$ is the error introduced when you approximate e^x by its Taylor polynomial of degree n. You may have even seen a formula for $E_n(x)$. We are now going to ask what happens as n goes to infinity? Does the error go zero, giving an exact formula for e^x? We shall later see that it does and that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

But we shall also see other functions for which the corresponding error obeys $\lim_{n\to\infty} E_n(x) = 0$ for some values of x and not for other values of x. Before we can deal with such questions, we have to build some foundations.

Sequences

Definition 1.

A sequence is a list of infinitely many numbers with a specified order. It is denoted

$$\big\{a_1,\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots\big\} \quad \text{or} \quad \big\{a_n\big\} \quad \text{or} \quad \big\{a_n\big\}_{n=1}^{\infty}$$

Example 2

Here are three sequences.

$$\Big\{1,\ \frac{1}{2},\ \frac{1}{3},\ \cdots,\ \frac{1}{n},\ \cdots\Big\} \qquad \text{or} \qquad \Big\{a_n = \frac{1}{n}\Big\}_{n=1}^{\infty}$$

$$\Big\{1,\ 2,\ 3,\ \cdots,\ n,\ \cdots\Big\} \qquad \text{or} \qquad \Big\{a_n = n\Big\}_{n=1}^{\infty}$$

$$\Big\{1,\ -1,\ 1,\ -1,\ \cdots,\ (-1)^{n-1},\ \cdots\Big\} \qquad \text{or} \qquad \Big\{a_n = (-1)^{n-1}\Big\}_{n=1}^{\infty}$$

It is not necessary that there be a simple explicit formula for the n^{th} term of a sequence. For example the decimal digits of π is a perfectly good sequence.

$$\big\{3,\ 1,\ 4,\ 1,\ 5,\ 9,\ 2,\ 6,\ 5,\ 3,\ 5,\ 8,\ 9,\ 7,\ 9,\ 3,\ 2,\ 3,\ 8,\ 4,\ 6,\ 2,\ 6,\ 4,\ 3,\ 3,\ 8,\ \cdots\ \big\}$$

Example 2

Our primary concern with sequences wil be the behaviour of a_n as n tends to infinity and, in particular, whether or not a_n "settles down" to some value as n tends to infinity.

Definition 3.

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge to the limit A if a_n approaches A as n tends to infinity. If so, we write

$$\lim_{n\to\infty} a_n = A \qquad \text{or} \qquad a_n \to A \text{ as } n \to \infty$$

A sequence is said to converge if it converges to some limit. Otherwise it is said to diverge.

Example 4

Three of the four sequences in Example 2 diverge:

- The sequence $\{a_n = n\}_{n=1}^{\infty}$ diverges because a_n grows without bound, rather than approaching some finite value, as n tends to infinity.
- The sequence $\{a_n = (-1)^{n-1}\}_{n=1}^{\infty}$ diverges because a_n oscillates between $+1$ and -1 rather than approaching a singe value as n tends to infinity.
- The sequence of the decimal digits of π also diverges, though the proof that this is the case is a bit beyond us right now.

The other sequence in Example 2 has $a_n = \frac{1}{n}$. As n tends to infinity, $\frac{1}{n}$ tends to zero. So

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Example 4

Example 5 ($\lim_{n\to\infty} \frac{n}{2n+1}$)

Here is a little less trivial example. To study the behaviour of $\frac{n}{2n+1}$ as $n \to \infty$, it is a good idea to write it as

$$\frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

As $n \to \infty$, the $\frac{1}{n}$ in the denominator tends to zero, so that the denominator $2+\frac{1}{n}$ tends to 2 and $\frac{1}{2+\frac{1}{n}}$ tends to $\frac{1}{2}$. So

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}$$

Example 5

You already have already had a fair bit of experience dealing with limits like $\lim_{x\to\infty} f(x)$. This experience can be easily transferred to dealing with $\lim_{n\to\infty} a_n$ limits by using the following result.

Theorem 6.

If
$$\lim_{x\to\infty} f(x) = L$$
and if $f(n) = a_n$ for all positive integers n, then
$$\lim_{n\to\infty} a_n = L$$

Example 7 ($\lim_{n\to\infty} e^{-n}$)

Set $f(x) = e^{-x}$. Then $e^{-n} = f(n)$ and
$$\lim_{x\to\infty} e^{-x} = 0 \implies \lim_{n\to\infty} e^{-n} = 0$$

Example 7

The bulk of the rules that you have used to work with limits like $\lim_{x\to\infty} f(x)$ also apply to limits like $\lim_{n\to\infty} a_n$.

Theorem 8 (Arithmetic of limits).

Let A, B and C be real numbers and let the two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge to A and B respectively. That is, assume that
$$\lim_{n\to\infty} a_n = A \qquad\qquad \lim_{n\to\infty} b_n = B$$
Then the following limits hold.

(a) $\lim_{n\to\infty} \big[a_n + b_n\big] = A + B$
(The limit of the sum is the sum of the limits.)

(b) $\lim_{n\to\infty} \big[a_n - b_n\big] = A - B$
(The limit of the difference is the difference of the limits.)

(c) $\lim_{n\to\infty} Ca_n = CA$.

(d) $\lim_{n\to\infty} a_n\, b_n = A\, B$
(The limit of the product is the product of the limits.)

(e) If $B \neq 0$ then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$
(The limit of the quotient is the quotient of the limits *provided* the limit of the denominator is not zero.)

Theorem 9 (Squeeze theorem)**.**

If $a_n \le c_b \le b_n$ for all natural numbers n, and if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

then

$$\lim_{n\to\infty} c_n = L$$

Theorem 10 (Continuous functions of limits)**.**

If $\lim\limits_{n\to\infty} a_n = L$ and if the function $g(x)$ is continuous at L, then

$$\lim_{n\to\infty} g(a_n) = g(L)$$

Example 11 ($\lim\limits_{n\to\infty} \sin \frac{\pi n}{2n+1}$)

Write $\sin \frac{\pi n}{2n+1} = g\big(\frac{n}{2n+1}\big)$ with $g(x) = \sin(\pi x)$. We saw, in Example 5 that

$$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

Since $g(x) = \sin(\pi x)$ is continuous at $x = \frac{1}{2}$, which is the limit of $\frac{n}{2n+1}$, we have

$$\lim_{n\to\infty} \sin \frac{\pi n}{2n+1} = \lim_{n\to\infty} g\Big(\frac{n}{2n+1}\Big) = g\Big(\frac{1}{2}\Big) = \sin \frac{\pi}{2} = 1$$

Example 11

Series

A series is a sum

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

of infinitely many terms. In summation notation, it is written

$$\sum_{n=1}^{\infty} a_n$$

An example is the decimal expansion of $\frac{1}{3}$ which you will recall is $0.3333\cdots$. You will also recall that $0.3333\cdots$ means

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots = \sum_{n=1}^{\infty} \frac{3}{10^n}$$

The summation index n is of course a dummy index. You can use any symbol you like (within reason) for the summation index.

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \sum_{i=1}^{\infty} \frac{3}{10^i} = \sum_{j=1}^{\infty} \frac{3}{10^j} = \sum_{\ell=1}^{\infty} \frac{3}{10^\ell}$$

A series can be expressed using summation in notation in many different ways. For example

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \overbrace{\frac{3}{10}}^{n=1} + \overbrace{\frac{3}{100}}^{n=2} + \overbrace{\frac{3}{1000}}^{n=3} + \cdots$$

$$\sum_{j=2}^{\infty} \frac{3}{10^{j-1}} = \overbrace{\frac{3}{10}}^{j=2} + \overbrace{\frac{3}{100}}^{j=3} + \overbrace{\frac{3}{1000}}^{j=4} + \cdots$$

$$\frac{3}{10} + \sum_{n=2}^{\infty} \frac{3}{10^n} = \frac{3}{10} + \overbrace{\frac{3}{100}}^{n=2} + \overbrace{\frac{3}{1000}}^{n=3} + \cdots$$

all represent exactly the same series. To get from the first line to the second, substitute $n = j-1$ everywhere, including in the limits of summation (so that $n = 1$ becomes $j-1 = 1$ which is rewritten as $j = 2$). Whenever you are in doubt as to what series a summation notation expression represents, write out the first few terms, as above.

Of course a sum of infinitely many terms may or may not add up to a finite number. To decide whether or not it does, we approximate it by a finite sum, say of N terms, and take the limit as N tends to infinity. Here are the associated definitions.

Definition 12.

The N^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is

$$S_N = \sum_{n=1}^{N} a_n$$

If the sequence $\big\{S_N\big\}_{N=1}^{\infty}$ converges as $N \to \infty$, say to S, then we say that the series $\sum_{n=1}^{\infty} a_n$ converges and we write

$$\sum_{n=1}^{\infty} a_n = S$$

If the sequence of partial sums diverges, we say that the series diverges.

Example 13 (Geometric Series)

Let a and r be any two fixed real numbers with $a \neq 0$. The series

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n$$

is called the geometric series with first term a and ratio r. Note that we have chosen to start the summation index at $n = 0$. That's fine. The first term is the $n = 0$ term, which is $ar^0 = a$. The second term is the $n = 1$ term, which is $ar^1 = ar$. And so on. We could have also written the series $\sum_{n=1}^{\infty} ar^{n-1}$. That's exactly the same series — the first term is $ar^{n-1}\big|_{n=1} = ar^{1-1} = a$, the second term is $ar^{n-1}\big|_{n=2} = ar^{2-1} = ar$, and so on. Regardless of how a geometric series is written, a is the first term and r is the ratio between successive terms.

The partial sums of any geometric series can be computed exactly. Define

$$S_N = \sum_{n=0}^{N} ar^n = a + ar + ar^2 + \cdots + ar^N$$

The secret to evaluating this sum is to see what happens when we multiply it r:

$$\begin{aligned} rS_N &= r\big(a + ar + ar^2 + \cdots + ar^N\big) \\ &= ar + ar^2 + ar^3 + \cdots + ar^{N+1} \end{aligned}$$

This is almost the same as S_N. The only differences are that the first term, a, is missing and one additional term, ar^{N+1}, has been tacked on the end. So

$$rS_N = S_N - a + ar^{N+1} \iff (1-r)S_N = a\big(1 - r^{N+1}\big)$$

If $r \neq 1$, we can now solve for S_N just by dividing the $1 - r$ across. If $r = 1$, S_N is exactly $N + 1$ copies of a added together. So

$$S_N = \begin{cases} a\frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

If $|r| < 1$, then r^{N+1} tends to zero as $N \to \infty$, so that S_N converges to $\frac{1}{1-r}$ as $N \to \infty$ and

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \qquad \text{if } |r| < 1$$

On the other hand if $|r| \geq 1$, S_N diverges because

- if $r > 1$, then r^N grows to ∞ as $N \to \infty$.
- If $r < -1$, then the magnitude of r^N grows to ∞, and the sign of r^N oscillates between $+$ and $-$, as $N \to \infty$.
- If $r = +1$, then $N + 1$ grows to ∞ as $N \to \infty$.
- If $r = -1$, then r^N just oscillates between $+1$ and -1 as $N \to \infty$.

So if $|r| \geq 1$ the geometric series $\sum_{n=0}^{\infty} ar^n$ diverges.

Example 13

Example 14 (Decimal Expansions)

The decimal expansion

$$0.3333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots = \sum_{n=1}^{\infty} \frac{3}{10^n}$$

is a geometric series with the first term $a = \frac{3}{10}$ and the ratio $r = \frac{1}{10}$. So, by Example 13,

$$0.3333\cdots = \sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3/10}{1 - 1/10} = \frac{3/10}{9/10} = \frac{1}{3}$$

just as we would have expected. Similarly,

$$0.16161616\cdots = \frac{16}{100} + \frac{16}{10000} + \frac{16}{1000000} + \cdots$$

is a geometric series with the first term $a = \frac{16}{100}$ and the ratio $r = \frac{1}{100}$. So, by Example 13,

$$0.16161616\cdots = \sum_{n=1}^{\infty} \frac{16}{100^n} = \frac{16/100}{1 - 1/100} = \frac{16/100}{99/100} = \frac{1}{6}$$

again, as expected. In this way any periodic decimal expansion converges to a ratio of two integers — that is, to a rational number.

Example 14

Example 15 (Telescoping Series)

In this example we are going to study the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. This is a rather artificial series that has been rigged to illustrate a phenomenon call "telescoping". Because

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

we can compute the partial sums for this series exactly.

$$\begin{aligned} S_N &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{N\cdot(N+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \end{aligned}$$

The second term of each bracket exactly cancels the first term of the following bracket. So the sum "telescopes" leaving just

$$S_N = 1 - \frac{1}{N+1}$$

and we can now easily compute

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1$$

Example 15

The usual addition and multiplication by constants rules also apply to series.

Theorem 16 (Arithmetic of series).

Let A, B and C be real numbers and let the two series $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ converge to A and B respectively. That is, assume that

$$\sum_{n=1}^\infty a_n = A \qquad\qquad \sum_{n=1}^\infty b_n = B$$

Then the following hold.

(a) $\displaystyle\sum_{n=1}^\infty \big[a_n + b_n\big] = A + B$ and $\displaystyle\sum_{n=1}^\infty \big[a_n - b_n\big] = A - B$

(b) $\displaystyle\sum_{n=1}^\infty C a_n = CA$.

Convergence Tests

It is very common to encounter series for which it is difficult, or even virtually impossible, to determine the sum exactly. Often you try to evaluate the sum approximately by truncating it, i.e. having the index run only up to some finite N, rather than infinity. But there is no point in doing so if the series diverges. So you like to at least know if the series converges or diverges. Furthermore you would also like to know what error is introduced when you approximate $\sum_{n=1}^\infty a_n$ by the "truncated series" $\sum_{n=1}^N a_n$. That's called the truncation error. There are a number of "convergence tests" to help you with this.

The Divergence Test

Our first test is very easy to apply, but it is also rarely useful. It just allows us to quickly reject some "trivially divergent" series. It is based on the observation that

- by definition, a series $\sum_{n=1}^\infty a_n$ converges to S when the partial sums $S_N = \sum_{n=1}^N a_n$ converge to S.
- Then, as $N \to \infty$, we have $S_N \to S$ and, because $N-1 \to \infty$ too, we also have $S_{N-1} \to S$.
- So $a_N = S_N - S_{N-1} \to S - S = 0$.

Theorem 17 (Divergence Test).

If the sequence $\big\{a_n\big\}_{n=1}^\infty$ fails to converge to zero as $n \to \infty$, then the series $\sum_{n=1}^\infty a_n$ diverges.

Example 18

Let $a_n = \frac{n}{n+1}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+{}^1\!/_n} = 1 \neq 0$$

So the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

Example 18

Warning 19

The divergence test is a "one way test". It tells us that if $\lim_{n\to\infty} a_n$ is nonzero, or fails to exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges. But it tells us *absolutely nothing* when $\lim_{n\to\infty} a_n = 0$. In particular, it is perfectly possible for a series $\sum_{n=1}^{\infty} a_n$ to *diverge* even though $\lim_{n\to\infty} a_n = 0$. An example is $\sum_{n=1}^{\infty} \frac{1}{n}$. We'll show in Example 21, below, that it diverges.

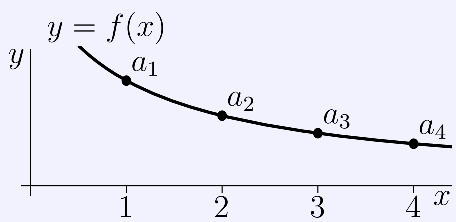
Warning 19

The Integral Test

Theorem 20 (The Integral Test)**.**

Let c be any real number. If $f(x)$ is a function which is defined and continuous for all $x \geq c$ and which obeys

(i) $f(x) \geq 0$ for all $x \geq c$ and
(ii) $f(x)$ decreases as x increases and
(iii) $f(n) = a_n$ for all $n \geq c$.



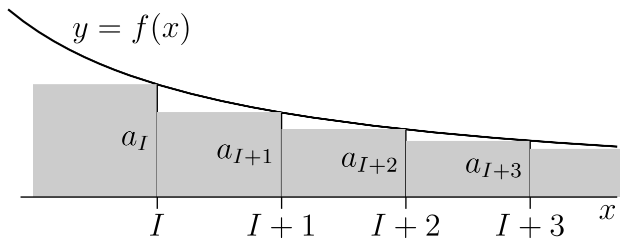
Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_c^{\infty} f(x)\, dx \text{ converges}$$

Furthermore, when the series converges, the truncation error

$$\left| \sum_{n=1}^{N} a_n - \sum_{n=1}^{\infty} a_n \right| \leq \int_N^{\infty} f(x)\, dx$$

Proof. Let I be any fixed integer bigger than $c+1$. Then $\sum_{n=1}^\infty a_n$ converges if and only if $\sum_{n=I}^\infty a_n$ converges — removing a fixed finite number of terms from a series cannot impact whether or not it converges. Since $a_n \ge 0$ for all $n \ge I > c$, the sequence of partial sums $s_\ell = \sum_{n=I}^{\ell} a_n$ increases as ℓ increases. It must either converge to some finite number or increase to infinity. That is, either $\sum_{n=I}^\infty a_n$ converges to a finite number or it is $+\infty$.



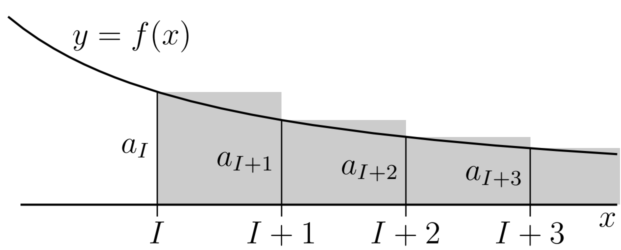
Look at the figure above. The shaded area in the figure is $\sum_{n=I}^\infty a_n$ because

- the first shaded rectangle has height a_I and width 1, and hence area a_I and
- the second shaded rectangle has height a_{I+1} and width 1, and hence area a_{I+1}, and so on

This shaded area is smaller than the area under the curve $y = f(x)$ for $I-1 \le x < \infty$. So

$$\sum_{n=I}^{\infty} a_n \le \int_{I-1}^{\infty} f(x)\ dx \tag{1}$$

and, if the integral is finite, the sum $\sum_{n=I}^\infty a_n$ is finite too.



For the "divergence case" look at the figure above. The (new) shaded area in the figure is again $\sum_{n=I}^\infty a_n$ because

- the first shaded rectangle has height a_I and width 1, and hence area a_I and
- the second shaded rectangle has height a_{I+1} and width 1, and hence area a_{I+1}, and so on

This time the shaded area is larger than the area under the curve $y = f(x)$ for $I \le x < \infty$. So

$$\sum_{n=I}^{\infty} a_n \ge \int_{I}^{\infty} f(x)\ dx$$

and, if the integral is infinite, the sum $\sum_{n=I}^{\infty} a_n$ is infinite too.

Finally, the bound on the trunction error is just the special case of (1) with $I = N + 1$:

$$\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=N+1}^{\infty} a_n \le \int_N^{\infty} f(x)\, dx$$

$\square$

Example 21 ($\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$)

Let $p > 0$. We'll now use the integral test to determine whether or not the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. To do so, we need a function $f(x)$ that obeys $f(n) = a_n = \frac{1}{n^p}$ for all n bigger than some c. Certainly $f(x) = \frac{1}{x^p}$ obeys $f(n) = \frac{1}{n^p}$ for all $n \ge 1$. So let's pick this f and try $c = 1$. (We can always increase c later if we need to.) This function also obeys the other two conditions of Theorem 20:

(i) $f(x) > 0$ for all $x \ge c = 1$ and

(ii) $f(x)$ decreases as x increases because $f'(x) = -p\frac{1}{x^{p+1}} < 0$ for all $x \ge c = 1$.

So the integral test tells us that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if the integral $\int_1^{\infty} \frac{dx}{x^p}$ converges. We have already seen, in Example 4 of the notes "Improper Integrals", that the integral $\int_1^{\infty} \frac{dx}{x^p}$ converges if and only if $p > 1$. So we conclude that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. In particular the series $\sum_{n=1}^{\infty} \frac{1}{n}$, which is called the harmonic series, diverges.

On the other hand the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. If we approximate $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by the truncated series $\sum_{n=1}^{N} \frac{1}{n^2}$, we make an error of at most

$$\int_N^{\infty} \frac{dx}{x^2} = \lim_{R\to\infty} \int_N^R \frac{dx}{x^2} = \lim_{R\to\infty} \Big[-\frac{1}{R} + \frac{1}{N} \Big] = \frac{1}{N}$$

Example 21

Example 22 ($\sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^p}$)

Let $p > 0$. We'll now use the integral test to determine whether or not the series $\sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges. As in the last example, we start by choosing a function that obeys $f(n) = a_n = \frac{1}{n(\log n)^p}$ for all n bigger than some c. Certainly $f(x) = \frac{1}{x(\log x)^p}$ obeys $f(n) = \frac{1}{n(\log n)^p}$ for all $n \ge 2$. So let's use that f and try $c = 2$. Now let's check the other two conditions of Theorem 20:

(i) Both x and $\log x$ are positive for all $x > 1$, so $f(x) > 0$ for all $x \ge c = 2$.

(ii) As x increases both x and $\log x$ increase and so $x(\log x)^p$ increases and $f(x)$ decreases.

So the integral test tells us that the series $\sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if the integral $\int_2^\infty \frac{dx}{x(\log x)^p}$ converges. To test the convergence of the integral, we make the substitution $u = \log x$, $du = \frac{dx}{x}$.

$$\int_2^R \frac{dx}{x(\log x)^p} = \int_{\log 2}^{\log R} \frac{du}{u^p}$$

We already know that the integral the integral $\int_1^\infty \frac{du}{u^p}$, and hence the integral $\int_2^R \frac{dx}{x(\log x)^p}$, converges if and only if $p > 1$. So we conclude that $\sum\limits_{n=1}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.

Example 22

The Comparison Test

Our next convergence test is called the comparison test.

Theorem 23 (The Comparison Test)**.**

Let N_0 be a natural number and let $K > 0$.

(a) If $|a_n| \le K c_n$ for all $n \ge N_0$ and $\sum\limits_{n=0}^{\infty} c_n$ converges, then $\sum\limits_{n=0}^{\infty} a_n$ converges.

(b) If $a_n \ge K d_n \ge 0$ for all $n \ge N_0$ and $\sum\limits_{n=0}^{\infty} d_n$ diverges, then $\sum\limits_{n=0}^{\infty} a_n$ diverges.

"Proof". We will not prove this theorem. We'll just observe that it is very reasonable. That's why there are quotation marks around "Proof".

(a) If $\sum\limits_{n=0}^{\infty} c_n$ converges to a finite number and if the terms in $\sum\limits_{n=0}^{\infty} a_n$ are smaller than the terms in $\sum\limits_{n=0}^{\infty} c_n$, then it is no surprise that $\sum\limits_{n=0}^{\infty} a_n$ converges too.

(b) If $\sum\limits_{n=0}^{\infty} d_n$ diverges (i.e. adds up to ∞) and if the terms in $\sum\limits_{n=0}^{\infty} a_n$ are larger than the terms in $\sum\limits_{n=0}^{\infty} d_n$, then of course $\sum\limits_{n=0}^{\infty} a_n$ adds up to ∞, and so diverges, too.

$\square$

Example 24 ($\sum_{n=1}^\infty \frac{1}{n^2+2n+3}$)

We could determine whether or not the series $\sum_{n=1}^\infty \frac{1}{n^2+2n+3}$ converges by applying the integral test. But it is not worth the effort. Whether or not any series converges is determined by the behaviour of the summand for very large n. So the first step in tackling such a problem is to develop some intuition about the behaviour of a_n when n is very large.

- *Step 1: Develop intuition.* In this case, when n is very large $n^2 \gg 2n \gg 3$ so that $\frac{1}{n^2+2n+3} \approx \frac{1}{n^2}$. We already know from Example 21, with $p = 2$, that $\sum_{n=1}^\infty \frac{1}{n^2}$ converges, so we would expect that $\sum_{n=1}^\infty \frac{1}{n^2+2n+3}$ converges too.
- *Step 2: Verify intuition.* We can use the comparison test to confirm that this is indeed the case. For any $n \geq 1$, $n^2 + 2n + 3 > n^2$, so that $\frac{1}{n^2+2n+3} \leq \frac{1}{n^2}$. So the comparison test, Theorem 23, with $a_n = \frac{1}{n^2+2n+3}$ and $c_n = \frac{1}{n^2}$, tells us that $\sum_{n=1}^\infty \frac{1}{n^2+2x+3}$ converges.

Example 24

Of course the previous example was "rigged" to give an easy application of the comparison test. It is often relatively easy, using arguments like those used in Example 24, to find a "simple" comparison series $\sum_{n=1}^\infty b_n$. However it is pretty rare that $a_n \leq b_n$. It is much more common that $a_n \leq K b_n$ for some constant K. This is enough to allow application of the comparison test. However finding the constant K can be really tedious. Here is a variant of the comparison test that eliminates the need to find K.

Theorem 25 (Limiting Comparison Theorem)**.**

Let $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ be two series with $b_n > 0$ for all n. Assume that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

exists.

(a) If $\sum_{n=1}^\infty b_n$ converges, then $\sum_{n=1}^\infty a_n$ converges too.

(b) If $L \neq 0$ and $\sum_{n=1}^\infty b_n$ diverges, then $\sum_{n=1}^\infty a_n$ diverges too.

"Proof". We will not prove this theorem, but we will explain the idea behind the proof.

Let's start with part (a). Because we are told that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, we know that, when n is large, $\frac{a_n}{b_n}$ is very close to L, so that $\left|\frac{a_n}{b_n}\right|$ is very close to $|L|$. In particular, there is some natural number N so that $\left|\frac{a_n}{b_n}\right| \leq |L| + 1$, and hence $|a_n| \leq K b_n$ with $K = |L| + 1$, for all $n \geq N$. The comparison Theorem 23 now implies that $\sum_{n=1}^\infty a_n$ converges.

Now for part (b). Let's suppose that $L > 0$. (If $L < 0$, just replace a_n with $-a_n$.) Because we are told that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, we know that, when n is large, $\frac{a_n}{b_n}$ is very close to L. In particular, there is some natural number N so that $\frac{a_n}{b_n} \geq \frac{L}{2}$, and hence $a_n \geq K b_n$ with $K = \frac{L}{2} > 0$, for all $n \geq N$. The comparison Theorem 23 now implies that $\sum_{n=1}^{\infty} a_n$ diverges. $\square$

Example 26 ($\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2-2n+3}$)

Set $a_n = \frac{\sqrt{n+1}}{n^2-2n+3}$. We first try to develop some intuition about the behaviour of a_n for large n and then we confirm that our intuition was correct.

- *Step 1: Develop intuition.* When $n \gg 1$, the numerator $\sqrt{n+1} \approx \sqrt{n}$, and the denominator $n^2-2n+3 \approx n^2$ so that $a_n \approx \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ and it looks like our series should converge by Example 21 with $p = \frac{3}{2}$.
- *Step 2: Verify intuition.* To confirm our intuition we set $b_n = \frac{1}{n^{3/2}}$ and compute

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{n^2-2n+3}}{\frac{1}{n^{3/2}}} &&= \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n+1}}{n^2-2n+3} \\
&= \lim_{n\to\infty} \frac{n^2\sqrt{1+{}^1\!/\!_n}}{n^2-2n+3} &&= \lim_{n\to\infty} \frac{n^2\sqrt{1+{}^1\!/\!_n}}{n^2-2n+3}\ \frac{{}^1\!/\!_{n^2}}{{}^1\!/\!_{n^2}} \\
&= \lim_{n\to\infty} \frac{\sqrt{1+{}^1\!/\!_n}}{1-{}^2\!/\!_n+{}^3\!/\!_{n^2}} &&= 1
\end{aligned}$$

We already know that the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by Example 21 with $p = \frac{3}{2}$. So our series converges by the limiting comparison test, Theorem 25.

Example 26

The Ratio Test

Theorem 27 (Ratio Test)**.**

Let N be any natural number and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

(b) If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = +\infty$, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Proof. (a) Pick any number R obeying $L < R < 1$. We are assuming that $\left|\frac{a_{n+1}}{a_n}\right|$ approaches L as $n \to \infty$. In particular there must be some natural number M so that $\left|\frac{a_{n+1}}{a_n}\right| \leq R$ for all

$n \geq M$. So $|a_{n+1}| \leq R|a_n|$ for all $n \geq M$. In particular

$$\begin{aligned}
|a_{M+1}| &\leq R\,|a_M| \\
|a_{M+2}| &\leq R\,|a_{M+1}| \leq R^2\,|a_M| \\
|a_{M+3}| &\leq R\,|a_{M+2}| \leq R^3\,|a_M| \\
&\vdots \\
|a_{M+\ell}| &\leq R^\ell\,|a_M|
\end{aligned}$$

for all $\ell \geq 0$. The series $\sum_{\ell=0}^\infty R^\ell\,|a_M|$ is a geometric series with ratio R smaller than one in magnitude and so converges. Consequently, by the comparison test with a_n replaced by $A_\ell = a_{n+\ell}$ and c_n replaced by $C_\ell = R^\ell\,|a_M|$, the series $\sum\limits_{\ell=1}^\infty a_{M+\ell} = \sum\limits_{n=M+1}^\infty a_n$ converges. So the series $\sum\limits_{n=1}^\infty a_n$ converges too.

(b) We are assuming that $\left|\frac{a_{n+1}}{a_n}\right|$ approaches $L > 1$ as $n \to \infty$. In particular there must be some natural number $M > N$ so that $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq M$. So $|a_{n+1}| \geq |a_n|$ for all $n \geq M$. That is, $|a_n|$ increases as n increases as long as $n \geq M$. So $|a_n| \geq |a_M|$ for all $n \geq M$ and a_n cannot converge to zero as $n \to \infty$. So the series diverges by the divergence test.

$\square$

Warning 28

Beware that the ratio test provides absolutely no conclusion about the convergence or divergence of the series $\sum\limits_{n=1}^\infty a_n$ if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$. See Example 30, below.

Warning 28

Example 29 ($\sum_{n=0}^\infty anx^{n-1}$)

Fix any two nonzero real numbers a and x. We have already seen in Example 13 — we have just renamed r to x — that the geometric series $\sum_{n=0}^\infty ax^n$ converges when $|x| < 1$ and diverges when $|x| \geq 1$. We are now going to consider a new series, constructed by differentiating each term in the geometric series $\sum_{n=0}^\infty ax^n$. This new series is

$$\sum_{n=0}^\infty a_n \qquad \text{with } a_n = a\,n\,x^{n-1}$$

Let's apply the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a\,(n+1)\,x^n}{a\,n\,x^{n-1}}\right| = \frac{n+1}{n}|x| = \Big(1+\frac{1}{n}\Big)|x| \to L = |x| \quad \text{as } n \to \infty$$

The ratio test now tells us that the series $\sum_{n=0}^\infty a\,n\,x^{n-1}$ converges if $|x| < 1$ and diverges if $|x| > 1$. It says nothing about the cases $x = \pm 1$. But in both of those cases $a_n = a\,n\,(\pm 1)^n$ does not converge to zero as $n \to \infty$ and the series diverges by the divergence test.

Example 29

Example 30 ($L = 1$)

In this example, we are going to see two different series that have $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$. One is going to converge and the other is going to diverge.

The first series is the harmonic series

$$\sum_{n=1}^{\infty} a_n \qquad \text{with } a_n = \frac{1}{n}$$

We have already seen, in Example 21, that this series diverges. It has

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to L = 1 \quad \text{as } n \to \infty$$

The second series is

$$\sum_{n=1}^{\infty} a_n \qquad \text{with } a_n = \frac{1}{n^2}$$

We have already seen, also in Example 21, that this series converges. But it also has

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = \frac{n^2}{(n+1)^2} = \frac{1}{(1+{}^1\!/\!{}_n)^2} \to L = 1 \quad \text{as } n \to \infty$$

Example 30

Power Series

Remember that we set as our goal, in studying sequences and series, the development of machinery which would allow us to answer questions like, "Is $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$?". We are now ready to start working on series like $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$. We'll start with the definition of a power series.

Definition 31.

A series of the form

$$A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots = \sum_{n=0}^{\infty} A_n(x-c)^n$$

is called a *power series in* $(x - c)$ or a *power series centered about* c. The numbers A_n are called the coefficients of the power series. Often $c = 0$ and then the series reduces to

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots = \sum_{n=0}^{\infty} A_nx^n$$

The x in a power series is to be thought of as a variable. So each power series is really a whole family of series — a different series for each value of x. Notice what happens if we apply the ratio test to try and determine which series in this family converge. The $n^{\rm th}$ term in the series $\sum_{n=0}^{\infty} A_n(x-c)^n$ is $a_n = A_n(x-c)^n$. So the ratio test tells us to compute

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{A_{n+1}(x-c)^{n+1}}{A_n(x-c)^n}\right| = \left|\frac{A_{n+1}}{A_n}\right| |x-c|$$

Now we are to try and take the limit $n\to\infty$. There are several possibilities.

- If the limit $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|$ exists and equals some nonzero value, say A, then the ratio test says that the series $\sum_{n=0}^{\infty} A_n(x-c)^n$ converges when $A|x-c| < 1$, i.e. when $|x-c| < \frac{1}{A}$, and diverges when $A|x-c| > 1$, i.e. when $|x-c| > \frac{1}{A}$. This $R = \frac{1}{A}$ is called the *radius of convergence* of the series.

- If the limit $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|$ exists and equals zero, then $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right||x-c| = 0$ for every x and the ratio test tells us that the series $\sum_{n=0}^{\infty} A_n(x-c)^n$ converges for every number x. In this case we say that the series has an infinite radius of convergence.

- If the $\left|\frac{A_{n+1}}{A_n}\right|$ tends to $+\infty$ as $n\to 0$, then $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right||x-c| = +\infty$ for every $x\neq c$ and the ratio test tells us that the series $\sum_{n=0}^{\infty} A_n(x-c)^n$ diverges for every number $x\neq c$. When $x=c$, the series reduces to $A_0+0+0+0+0+\cdots$, which of course converges. In this case we say that the series has radius of convergence zero.

- If $\left|\frac{A_{n+1}}{A_n}\right|$ does not approach a limit as $n\to\infty$, then we learn nothing from the ratio test.

All of these possibilities do happen. Here is an example of each.

Example 32

If $a\neq 0$, the geometric series $\sum_{n=0}^{\infty} ax^n$ has $A_n = a$. So

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty} 1 = 1$$

and $\sum\limits_{n=0}^{\infty} ax^n$ has radius of convergence 1. Of course, we already knew that.

Example 32

Example 33

Recall that $n! = 1\times 2\times 3\times\cdots\times n$ is called "n factorial". The series $\sum\limits_{n=0}^{\infty}\frac{x^n}{n!}$ has $A_n = \frac{1}{n!}$. So

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| &= \lim_{n\to\infty}\frac{1/(n+1)!}{1/n!} = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1\times 2\times 3\times\cdots\times n}{1\times 2\times 3\times\cdots\times n\times(n+1)} = \lim_{n\to\infty}\frac{1}{n+1}\\ &= 0\end{aligned}$$

and $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence ∞. It converges for every x.

Example 33

Example 34

The series $\sum_{n=0}^{\infty} n!x^n$ has $A_n = n!$. So

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}\frac{1\times 2\times 3\times 4\times\cdots\times n\times(n+1)}{1\times 2\times 3\times 4\times\cdots\times n} = \lim_{n\to\infty}(n+1) = +\infty$$

and $\sum_{n=0}^{\infty} n!x^n$ has radius of convergence zero. It converges only for $x = 0$.

Example 34

Example 35

Let $A_0 = 4$ and for each natural number n, let A_n be one plus the n^{th} decimal digit of π. So every A_n is an integer between 1 and 10 and the series

$$\sum_{n=0}^{\infty} A_n x^n = 4 + 2x + 5x^2 + 2x^3 + 6x^4 + 10x^5 + \cdots$$

Because π is an irrational number $\frac{A_{n+1}}{A_n}$ cannot have a limit as $n\to\infty$. (If you don't know why this is the case, don't worry about it.) So the ratio test tells us nothing about the convergence of this series. But we can still figure out for which x's it converges.

- Because every coefficient A_n is no bigger (in magnitude) than 10, the n^{th} term in our series obeys
$$\big|A_n x^n\big| \le 10|x|^n$$
and so is smaller than the n^{th} term in the geometric series $\sum_{n=0}^{\infty} 10|x|^n$. This geometric series converges if $|x| < 1$. So, by the comparison test, our series converges for $|x| < 1$ too.

- Since every A_n is at least one, the n^{th} term in our series obeys
$$\big|A_n x^n\big| \ge |x|^n$$
If $|x| \ge 1$, this $a_n = A_n x^n$ cannot converge to zero as $n\to\infty$, and our series diverges by the divergence test.

In conclusion, our series converges if and only if $|x| < 1$. We say that it has radius of convergence 1.

Example 35

Though we won't prove it, it is true that every power series has a radius of convergence, whether or not the limit of $\left|\frac{A_{n+1}}{A_n}\right|$ exists.

Theorem 36.

Let $\sum_{n=0}^{\infty} A_n(x-c)^n$ be a power series. Then one of the following alternatives must hold.

(i) The power converges for every number x. In this case we say that the radius of convergence is ∞.

(ii) There is a number $0 < R < \infty$ such that the series converges for $|x-c| < R$ and diverges for $|x-c| > R$. Then R is called the radius of convergence.

(iii) The series converges for $x = 0$ and diverges for all $x \neq 0$. In this case, we say that the radius of convergence is 0.

Working With Power Series

Here is a theorem that can be used help build power series representations for complicated functions from power series representations of simple functions.

Theorem 37 (Operations on Power Series)**.**

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n$$

for all x obeying $|x-c| < R$. In particular, we are assuming that both power series have radius of convergence at least R. Also let A be a constant. Then

$$\begin{aligned}
f(x) + g(x) &= \sum_{n=0}^{\infty} [A_n + B_n]\,(x-c)^n \\
A f(x) &= \sum_{n=0}^{\infty} A\, A_n\,(x-c)^n \\
(x-c)^N f(x) &= \sum_{n=0}^{\infty} A_n\,(x-c)^{n+N} \quad \text{for any natural number } N \\
&= \sum_{k=N}^{\infty} A_{k-N}\,(x-c)^k \quad \text{where } k = n+N \\
f'(x) &= \sum_{n=0}^{\infty} A_n\, n\,(x-c)^{n-1} \\
\int_c^x f(t)\;dt &= \sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1} \\
\int f(x)\;dx &= \left[\sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1}\right] + C
\end{aligned}$$

for all x obeying $|x-c| < R$. In particular the radius of convergence of each of the five power series on the right hand side is at least R. The C in the last formula is of course an arbitrary constant.

We'll now use this theorem to build power series representations for a bunch of functions out of the one simple power series representation that we know — the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for all } |x| < 1$$

Example 38 ($\frac{x}{2+x^2}$)

Find a power series representation for $\frac{x}{2+x^2}$.

Solution. The secret to finding power series representations for a good many functions is to manipulate them into a form in which $\frac{1}{1-y}$ appears and use the geometric series representation

$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$. We have deliberately renamed the variable to y here — it does not have to be x. We can do that for the given function.

$$\begin{aligned}
\frac{x}{2+x^2} = \frac{x}{2}\ \frac{1}{1+x^2/2} \qquad &= \frac{x}{2}\ \frac{1}{1-\big(-x^2/2\big)} \\
= \frac{x}{2}\ \frac{1}{1-y}\bigg|_{y=-x^2/2} \quad &= \frac{x}{2}\left[\sum_{n=0}^{\infty} y^n\right]_{y=-x^2/2} \\
= \frac{x}{2}\ \sum_{n=0}^{\infty}\Big(-\frac{x^2}{2}\Big)^n \ &= \ \frac{x}{2}\ \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^{2n} \\
= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{2n+1}
\end{aligned}$$

This is a perfectly good power series. There is nothing wrong with the power of x being $2n+1$. In fact, you should try to always write power series in forms that are as easy to understand as possible. The geometric series that we used in the second line converges for

$$|y| < 1 \iff \big|-x^2/2\big| < 1 \iff |x|^2 < 2 \iff |x| < \sqrt{2}$$

So our power series has radius of convergence $\sqrt{2}$. It converges for $-\sqrt{2} < x < \sqrt{2}$.

Example 38

Example 39 ($\frac{1}{(1-x)^2}$)

Find a power series representation for $\frac{1}{(1-x)^2}$.

Solution. Once again the trick is to express $\frac{1}{(1-x)^2}$ in terms of $\frac{1}{1-x}$.

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n \\
&= \sum_{n=1}^{\infty} nx^{n-1}
\end{aligned}$$

Note that the $n=0$ term has disappeared because, for $n=0$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = \frac{\mathrm{d}}{\mathrm{d}x}1 = 0$$

Our power series has radius of convergence 1. It converges if and only if $-1 < x < 1$, since the series diverges for $|x| \geq 1$, by the divergence test.

Example 39

Example 40 ($\ln(1+x)$)

Find a power series representation for $\ln(1+x)$.

Solution. Recall that $\frac{\mathrm{d}}{\mathrm{d}x}\ln(1+x) = \frac{1}{1+x}$ so that $\ln(1+t)$ is an antiderivative of $\frac{1}{1+t}$ and

$$\begin{aligned}
\ln(1+x) &= \int_0^x \frac{dt}{1+t} = \int_0^x \Big[\sum_{n=0}^\infty (-t)^n\Big]\,dt = \sum_{n=0}^\infty \int_0^x (-t)^n\,dt \\
&= \sum_{n=0}^\infty (-1)^n \frac{x^{n+1}}{n+1} \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots
\end{aligned}$$

Theorem 37 guarantees that the radius of convergence is at least one (the radius of convergence of the geometric series $\sum_{n=0}^\infty (-t)^n$). When $x=-1$ our series reduces to minus $\sum_{n=0}^\infty \frac{1}{n+1}$, which is the harmonic series and so diverges. That's no surprise — $\ln(1+(-1)) = -\infty$. So the radius of convergence is exactly 1. It is possible to prove, though we won't do so here, that when $x=1$, the series converges to $\ln 2$. So the series converges when $-1 < x \le 1$.

Example 40

Example 41 ($\arctan x$)

Find a power series representation for $\arctan x$.

Solution. Recall that $\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$ so that $\arctan t$ is an antiderivative of $\frac{1}{1+t^2}$ and

$$\begin{aligned}
\arctan x &= \int_0^x \frac{dt}{1+t^2} = \int_0^x \Big[\sum_{n=0}^\infty \left(-t^2\right)^n\Big]\,dt = \sum_{n=0}^\infty \int_0^x (-1)^n t^{2n}\,dt \\
&= \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1}
\end{aligned}$$

Theorem 37 guarantees that the radius of convergence is at least one (the radius of convergence of the geometric series $\sum_{n=0}^\infty \left(-t^2\right)^n$). When $|x|>1$, the series diverges by the ratio test or the divergence test. It is possible to prove, though once again we won't do so here, that when $x=\pm 1$, the series converges to $\arctan(\pm 1) = \pm\frac{\pi}{2}$. So the series converges when $-1 \le x \le 1$.

Example 41

Taylor Series

Recall that Taylor polynomials provide a hierarchy of approximations to a given function $f(x)$ near a given point a.

- The crudest approximation is the constant approximation $f(x) \approx f(a)$.
- Then comes the linear, or tangent line, approximation $f(x) \approx f(a) + f'(a)\,(x-a)$.
- Then comes the quadratic approximation $f(x) \approx f(a)+f'(a)\,(x-a)+\frac{1}{2}f''(a)\,(x-a)^2$.
- In general, the Taylor polynomial of degree n, for the function $f(x)$ about the expansion point a, is the polynomial, $T_n(x)$, determined by the requirements that $f^{(m)}(a) = T_n^{(m)}(a)$ for all $0 \le m \le n$. That is, f and T_n have the same derivatives at a, up to order n. Explicitly,

$$\begin{aligned} f(x) \approx T_n(x) &= f(a) + f'(a)\,(x-a) + \frac{1}{2}f''(a)\,(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)\,(x-a)^n \\ &= \sum_{m=0}^{n} \frac{1}{m!}f^{(m)}(a)\,(x-a)^n \end{aligned}$$

These are of course approximations — often very good approximations near $x = a$ — but still just approximations. Can we get exact representations by taking the limit as $n \to \infty$? That's the question we'll consider now.

Fix a real number a and suppose that all derivatives of the function $f(x)$ exist. Then, for any natural number n,

$$f(x) = T_n(x) + E_n(x) \tag{2}$$

where

$$T_n(x) = f(a) + f'(a)\,(x-a) + \cdots + \tfrac{1}{n!}f^{(n)}(a)\,(x-a)^n \tag{2a}$$

is the Taylor polynomial of degree n for the function $f(x)$ and expansion point a, and $E_n(x) = f(x) - T_n(x)$ is the error introduced when we approximate $f(x)$ by the polynomial $T_n(x)$. It is true, though we won't prove it, that

$$E_n(x) = \tfrac{1}{(n+1)!}f^{(n+1)}(c)\,(x-a)^{n+1} \tag{2b}$$

for some (usually unknown) c strictly between a and x.

If it happens that $E_n(x)$ tends to zero as $n \to \infty$, then we have the exact formula

$$f(x) = \lim_{n\to\infty} P_n(x)$$

for $f(x)$. This is usually written

$$f(x) = \sum_{n=0}^{\infty} \tfrac{1}{n!}f^{(n)}(a)\,(x-a)^n \tag{3}$$

and is called the Taylor series of $f(x)$ with expansion point a. (When $a = 0$ it is also called the Maclaurin series of $f(x)$.) It is a power series representation for $f(x)$.

Example 42 (Exponential Series)

This happens with the exponential function $f(x) = e^x$. We'll first find $f^{(m)}(0)$ for all integers $m \ge 0$.

$$\begin{array}{lllllll} f(x) = e^x & \Rightarrow & f'(x) = e^x & \Rightarrow & f''(x) = e^x & \cdots \\ f(0) = e^0 = 1 & \Rightarrow & f'(0) = e^0 = 1 & \Rightarrow & f''(0) = e^0 = 1 & \cdots \end{array}$$

Applying (2) with $f(x) = e^x$ and $a = 0$, and using that $f^{(m)}(a) = e^{x_0} = e^0 = 1$ for all m,

$$e^x = f(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{1}{(n+1)!} e^c x^{n+1} \tag{4}$$

for some c between 0 and x. Now consider any fixed real number x. As c runs from 0 to x, e^c runs from $e^0 = 1$ to e^x. In particular, e^c is always between 1 and e^x and so is smaller than $1 + e^x$. Thus the error term

$$|E_n(x)| = \left| \frac{e^c}{(n+1)!} x^{n+1} \right| \le [e^x + 1] \frac{|x|^{n+1}}{(n+1)!}$$

Let's call $e_n(x) = \frac{|x|^{n+1}}{(n+1)!}$. We claim that as n increases towards infinity, $e_n(x)$ decreases (quickly) towards zero. To see this, let's compare $e_n(x)$ and $e_{n+1}(x)$.

$$\frac{e_{n+1}(x)}{e_n(x)} = \frac{\frac{|x|^{n+2}}{(n+2)!}}{\frac{|x|^{n+1}}{(n+1)!}} = \frac{|x|}{n+2}$$

So, when n is bigger than, for example $2|x|$, we have $\frac{e_{n+1}(x)}{e_n(x)} < \frac{1}{2}$. That is, increasing the index on $e_n(x)$ by one decreases the size of $e_n(x)$ by a factor of at least two. As a result $e_n(x)$ must tend to zero as $n \to \infty$. Consequently $\lim\limits_{n\to\infty} E_n(x) = 0$ and

$$e^x = \lim_{n\to\infty} \left[1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \cdots + \frac{1}{n!} x^n \right] = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{5}$$

Example 42

Example 43 (Sine and Cosine Series)

The trigonometric functions $\sin x$ and $\cos x$ also have widely used Taylor series expansions about $a = 0$. To find them, we first, compute all derivatives at general x.

$$\begin{array}{llllll} f(x) = \sin x & f'(x) = \cos x & f''(x) = -\sin x & f^{(3)}(x) = -\cos x & f^{(4)}(x) = \sin x & \cdots \\ g(x) = \cos x & g'(x) = -\sin x & g''(x) = -\cos x & g^{(3)}(x) = \sin x & g^{(4)}(x) = \cos x & \cdots \end{array} \tag{6}$$

The pattern starts over again with the fourth derivative being the same as the original function. Now set $x = a = 0$.

$$\begin{array}{lllllll} f(x) = \sin x & f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f^{(3)}(0) = -1 & f^{(4)}(0) = 0 & \cdots \\ g(x) = \cos x & g(0) = 1 & g'(0) = 0 & g''(0) = -1 & g^{(3)}(0) = 0 & g^{(4)}(0) = 1 & \cdots \end{array} \tag{7}$$

For $\sin x$, all even numbered derivatives are zero. The odd numbered derivatives alternate between 1 and -1. For $\cos x$, all odd numbered derivatives are zero. The even numbered derivatives alternate between 1 and -1. So, the Taylor polynomials that best approximate $\sin x$ and $\cos x$ near $x = a = 0$ are

$$\begin{aligned} \sin x &\approx x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots \\ \cos x &\approx 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots \end{aligned}$$

Reviewing (6) we see that every derivative of $\sin x$ and $\cos x$ is one of $\pm\sin x$ and $\pm\cos x$. Consequently, when we apply (2b) we always have $\left|f^{(n+1)}(c)\right| \le 1$ and hence $|E_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$. We have already seen in Example 42, that $\frac{|x|^{n+1}}{(n+1)!}$ (which we called $e_n(x)$ in Example 42) converges to zero as $n \to \infty$. Consequently, for both $f(x) = \sin x$ and $f(x) = \cos x$, we have $\lim\limits_{n\to\infty} E_n(x) = 0$ and

$$f(x) = \lim_{n\to\infty}\left[f(0) + f'(0)\,x + \cdots + \tfrac{1}{n!}f^{(n)}(0)\,x^n\right]$$

Reviewing (7), we conclude that

$$\begin{aligned} \sin x &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots = \sum_{n=0}^{\infty}(-1)^n\tfrac{1}{(2n+1)!}x^{2n+1} \\ \cos x &= 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots = \sum_{n=0}^{\infty}(-1)^n\tfrac{1}{(2n)!}x^{2n} \end{aligned} \tag{8}$$

Example 43