

Sketching Surfaces in 3d

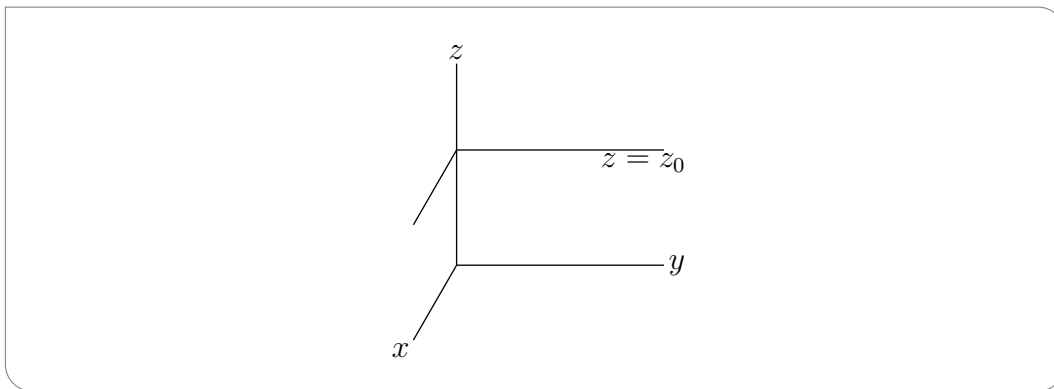
In practice students taking multivariable calculus regularly have great difficulty visualising surfaces in three dimensions, despite the fact that we all live in three dimensions. In these notes we'll develop some technique to help us sketch surfaces in three dimensions.

We all have a fair bit of experience drawing curves in two dimensions. Typically the intersection of a surface (in three dimensions) with a plane is a curve lying in the (two dimensional) plane. We'll call such an intersection a cross-section. In the special case that the plane is one of the coordinate planes, the intersection is sometimes called a trace. One can often get a pretty good idea of what a surface looks like by sketching a bunch of cross-sections. Here are some examples.

Example 1 ($4x^2 + y^2 - z^2 = 1$)

Sketch $4x^2 + y^2 - z^2 = 1$.

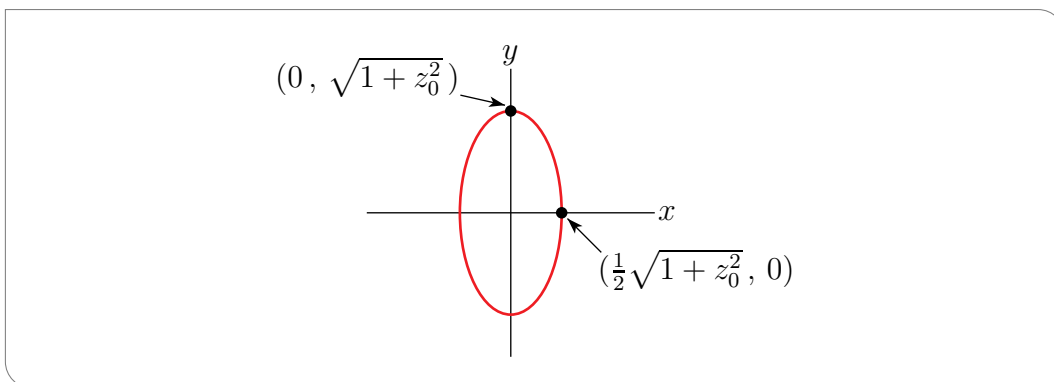
Solution. We'll start by fixing any number z_0 and sketching the part of the surface that lies in the horizontal plane $z = z_0$.



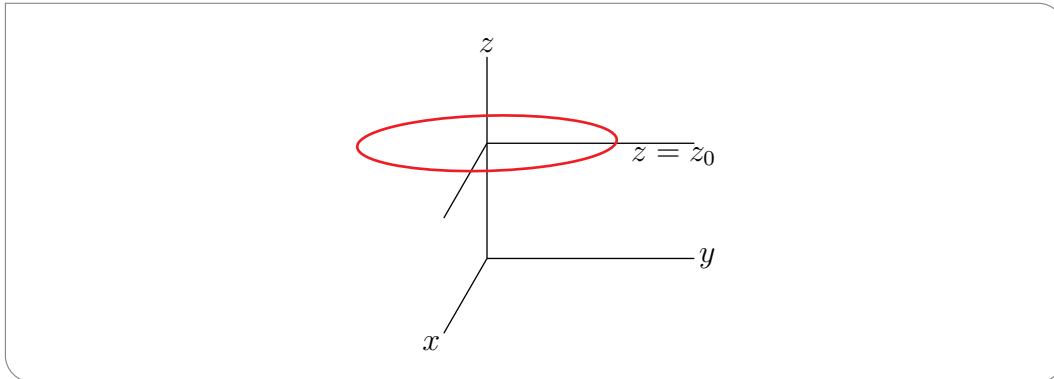
The intersection of our surface with that horizontal plane is a horizontal cross-section. Any point (x, y, z) lying on that horizontal cross-section has both

$$z = z_0 \text{ and } 4x^2 + y^2 - z^2 = 1 \iff z = z_0 \text{ and } 4x^2 + y^2 = 1 + z_0^2$$

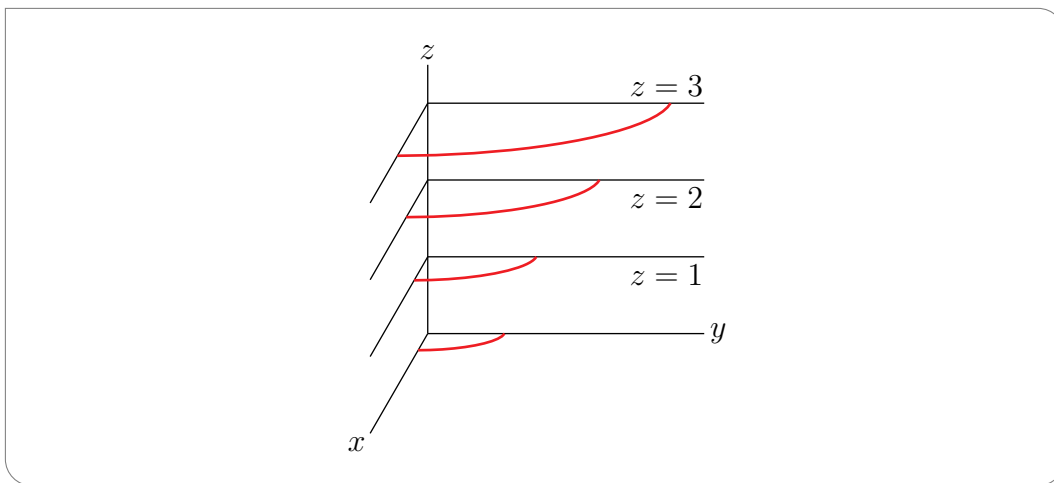
Think of z_0 as a constant. Then $4x^2 + y^2 = 1 + z_0^2$ is a curve in the xy -plane. When $y = 0$, we have $x = \pm \frac{1}{2}\sqrt{1 + z_0^2}$ and when $x = 0$, $y = \pm\sqrt{1 + z_0^2}$. The curve is just an ellipse with x semi-axis $\frac{1}{2}\sqrt{1 + z_0^2}$ and y semi-axis $\sqrt{1 + z_0^2}$. It's easy to sketch.



Remember that this ellipse is the part of our surface that lies in the plane $z = z_0$. Imagine that the sketch of the ellipse is on a single sheet of paper. Lift the sheet of paper up, rotate it around so that the x - and y -axes point in the directions of the three dimensional x - and y -axes and place the sheet of paper into the three dimensional sketch at height z_0 . This gives a single horizontal ellipse in 3d, as in the figure below.



The full surface consists of many of these horizontal ellipses — one for each possible height z_0 . Our surface is a stack of horizontal ellipses. So draw a few of them as in the figure below. To reduce the amount of clutter in the sketch, only the first octant (i.e. the part of three dimensions that has $x \geq 0$, $y \geq 0$ and $z \geq 0$) has been drawn.



Here is why it is OK, in this case, to just sketch the first octant. Replacing x by $-x$ in the equation $4x^2 + y^2 - z^2 = 1$ does not change the equation. That means that a point (x, y, z) is on the surface if and only if the point $(-x, y, z)$ is on the surface. So the surface is invariant under reflection in the yz -plane. Similarly, the equation $4x^2 + y^2 - z^2 = 1$ does not change when y is replaced by $-y$ or z is replaced by $-z$. Our surface is also invariant reflection in the xz - and yz -planes. Once we have the part in the first octant, the remaining octants can be gotten simply by reflecting about the coordinate planes.

We can get a more visually meaningful sketch by adding in some vertical cross-sections. The $x = 0$ and $y = 0$ cross-sections (also called traces — they are the parts of our surface that are in the yz - and xz -planes, respectively) are

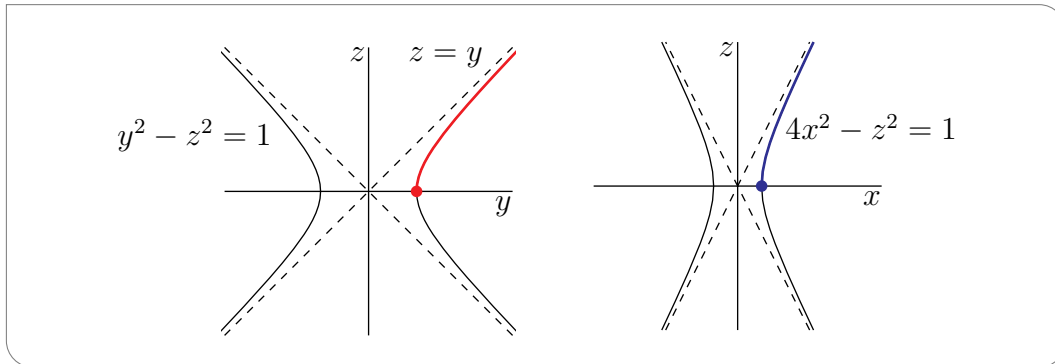
$$x = 0, y^2 - z^2 = 1 \quad \text{and} \quad y = 0, 4x^2 - z^2 = 1$$

They are both hyperbolae. We'll first sketch them in 2d. Since

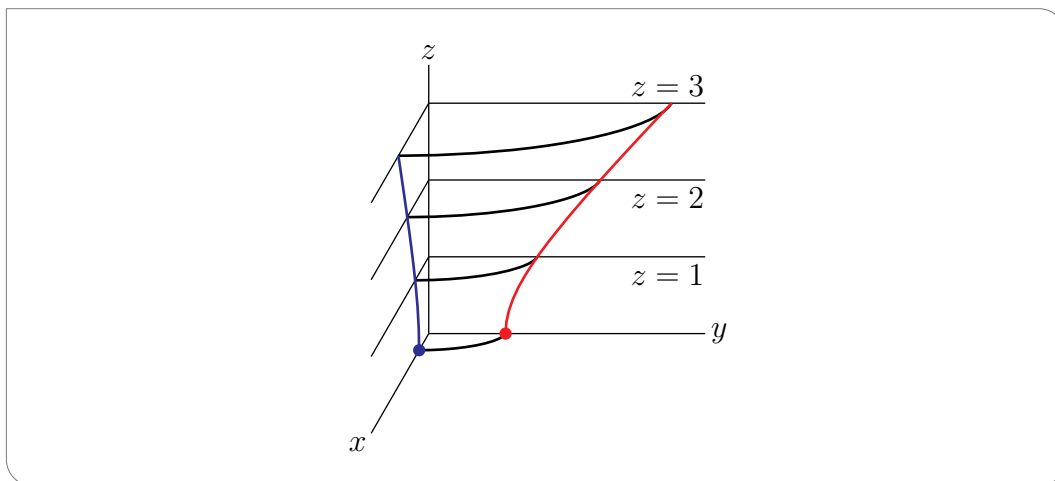
$$y^2 = 1 + z^2 \implies |y| \geq 1 \quad \text{and} \quad y = \pm 1 \text{ when } z = 0 \quad \text{and} \quad \text{for large } z, y \approx \pm z$$

$$4x^2 = 1 + z^2 \implies |x| \geq \frac{1}{2} \quad \text{and} \quad x = \pm \frac{1}{2} \text{ when } z = 0 \quad \text{and} \quad \text{for large } z, x \approx \pm \frac{1}{2}z$$

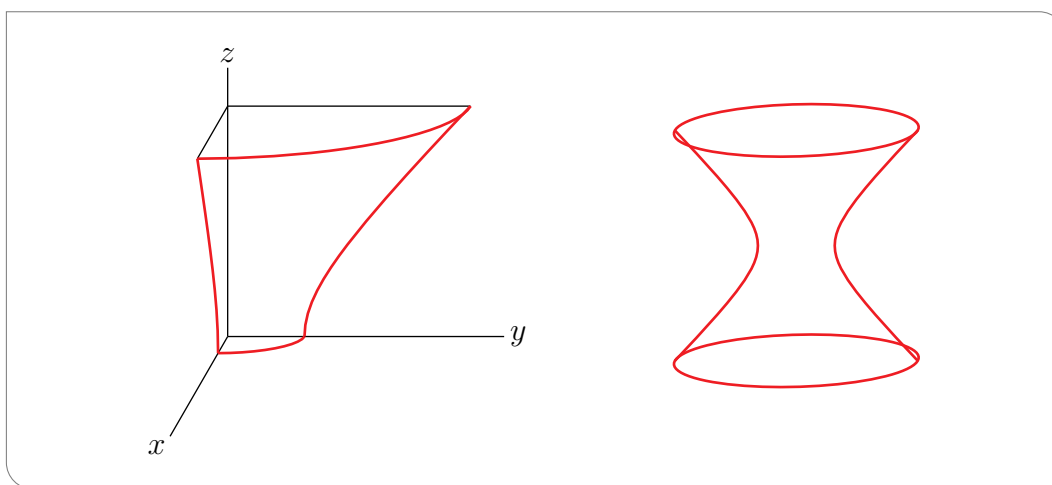
the sketches are



Now we'll incorporate them into the 3d sketch. Once again imagine that each is a single sheet of paper. Pick each up and move it into the 3d sketch, carefully matching up the axes. The red (blue) parts of the hyperbolas above become the red (blue) parts of the 3d sketch below (assuming of course that you are looking at this on a colour screen).



Now that we have a pretty good idea of what the surface looks like we can clean up and simplify the sketch. Here are a couple of possibilities.



This type of surface is called a hyperboloid of one sheet.

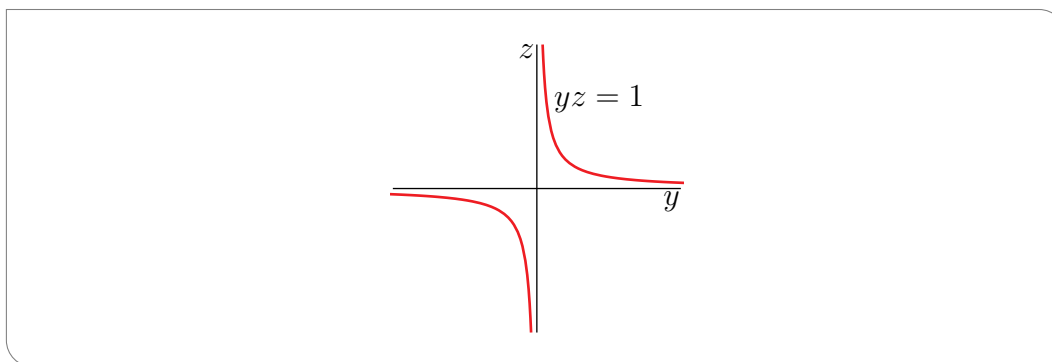
Example 1

Example 2 ($yz = 1$)

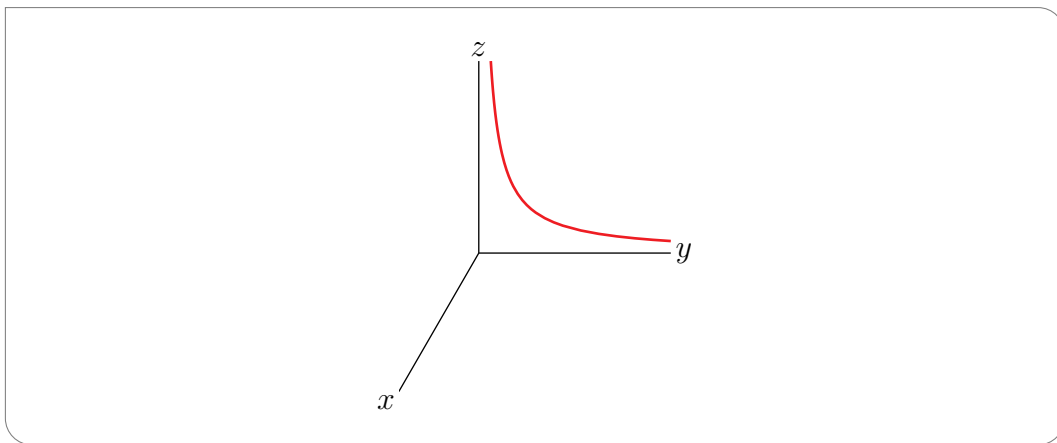
Sketch the surface $yz = 1$.

Solution. This surface has a special property that makes it relatively easy to sketch. There are no x 's in the equation $yz = 1$. That means that if some y_0 and z_0 obey $y_0z_0 = 1$, then the point (x, y_0, z_0) lies on the surface $yz = 1$ for all values of x . As x runs from $-\infty$ to ∞ , the point (x, y_0, z_0) sweeps out a straight line parallel to the x -axis. So the surface $yz = 1$ is a union of lines parallel to the x -axis. It is invariant under translations parallel to the x -axis. To sketch $yz = 1$, we just need to sketch its intersection with the yz -plane and then translate the resulting curve parallel to the x -axis.

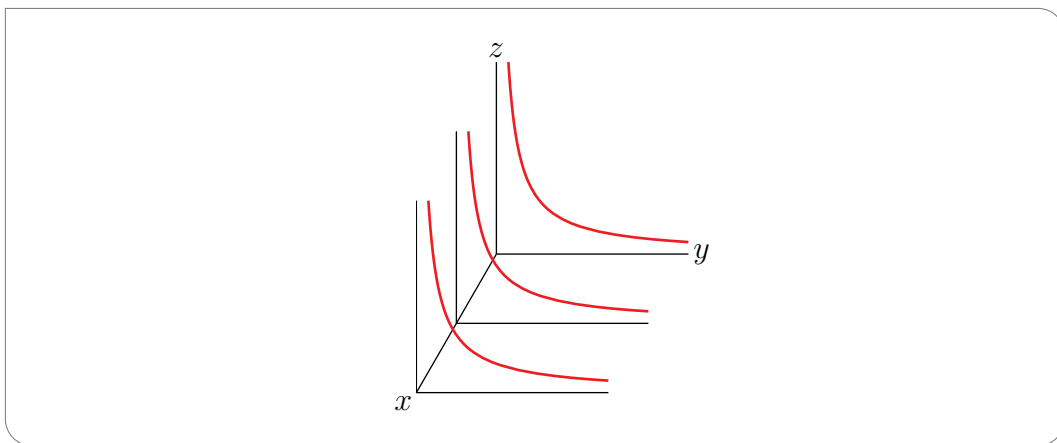
We'll start with a sketch of the hyperbola $yz = 1$ in two dimensions.



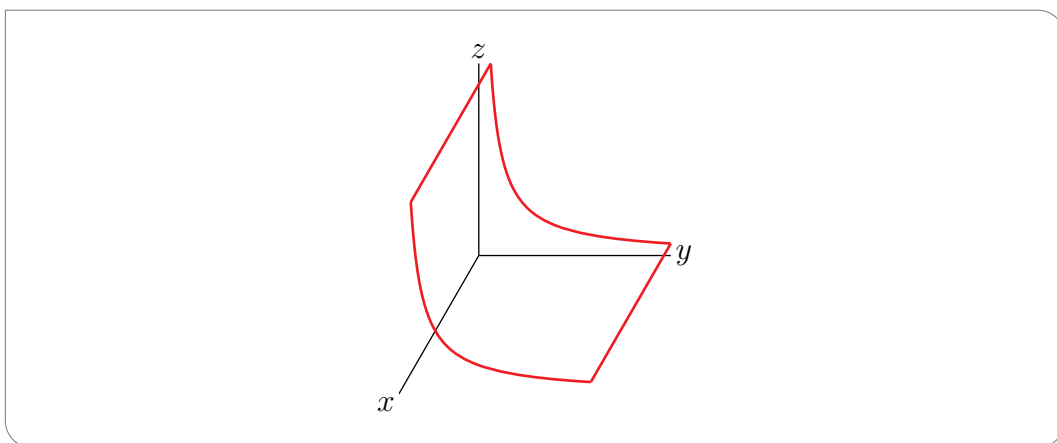
Next we'll move this 2d sketch into the yz -plane, i.e. the plane $x = 0$, in 3d, except that we'll only draw in the part in the first octant.



The we'll draw in $x = x_0$ cross-sections for a couple of more values of x_0



and clean up the sketch a bit



Example 2

Often the reason you are interested in a surface in 3d is that it is the graph $z = f(x, y)$ of a function of two variables $f(x, y)$. Another good way to visualize the behaviour of a function $f(x, y)$ is to sketch what are called its level curves. By definition, a level curve of $f(x, y)$ is a curve whose equation is $f(x, y) = C$, for some constant C . It is the set of points in the

xy -plane where f takes the value C . Because it is a curve in 2d, it is usually easier to sketch than the graph of f . Here are a couple of examples.

Example 3 ($f(x, y) = x^2 + 4y^2 - 2x + 2$)

Sketch the level curves of $f(x, y) = x^2 + 4y^2 - 2x + 2$.

Solution. Fix any real number C . Then, for the specified function f , the level curve $f(x, y) = C$ is the set of points (x, y) that obey

$$\begin{aligned} x^2 + 4y^2 - 2x + 2 = C &\iff x^2 - 2x + 1 + 4y^2 + 1 = C \\ &\iff (x - 1)^2 + 4y^2 = C - 1 \end{aligned}$$

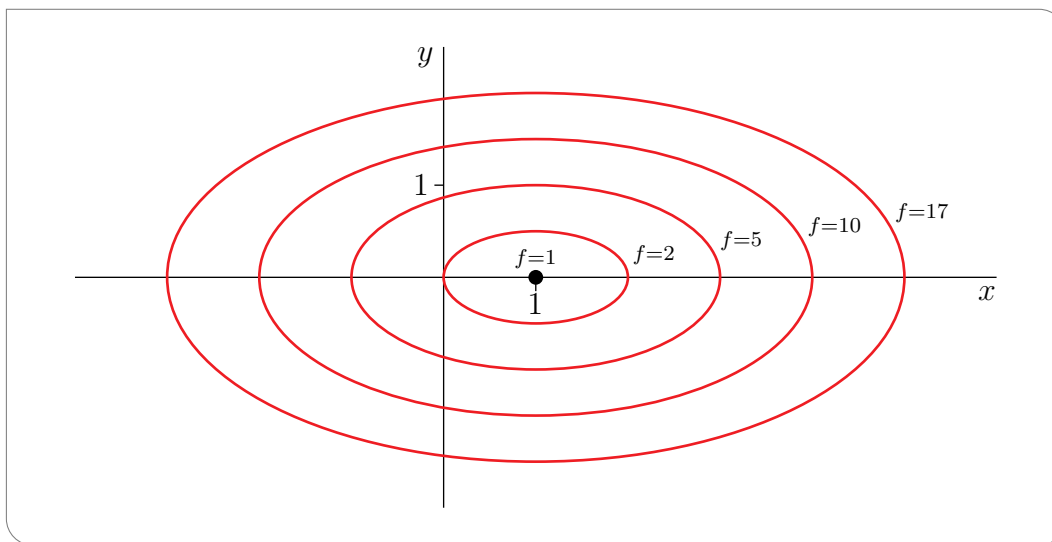
Of course $(x - 1)^2 + 4y^2$ is always at least zero. So if $C - 1 < 0$, i.e. if $C < 1$, there is no curve $f(x, y) = C$. If $C - 1 = 0$, i.e. if $C = 1$, then $f(x, y) = C = 1$ if and only if both $(x - 1)^2 = 0$ and $4y^2 = 0$ and so consists of the single point $(1, 0)$. If $C > 1$, then $f(x, y) = C$ is an ellipse. It intersects the x -axis when $y = 0$ and

$$(x - 1)^2 = C - 1 \iff x - 1 = \pm\sqrt{C - 1} \iff x = 1 \pm \sqrt{C - 1}$$

and it intersects the y -axis when $x = 0$ and

$$4y^2 = C - 1 \iff 2y = \pm\sqrt{C - 1} \iff y = \pm\frac{1}{2}\sqrt{C - 1}$$

So, when $C > 1$, $f(x, y) = C$ is the ellipse centred on $(1, 0)$ with x semi-axis $\sqrt{C - 1}$ and y semi-axis $\frac{1}{2}\sqrt{C - 1}$. Here is a sketch of some representative level curves of $f(x, y) = x^2 + 4y^2 - 2x + 2$.



Example 3

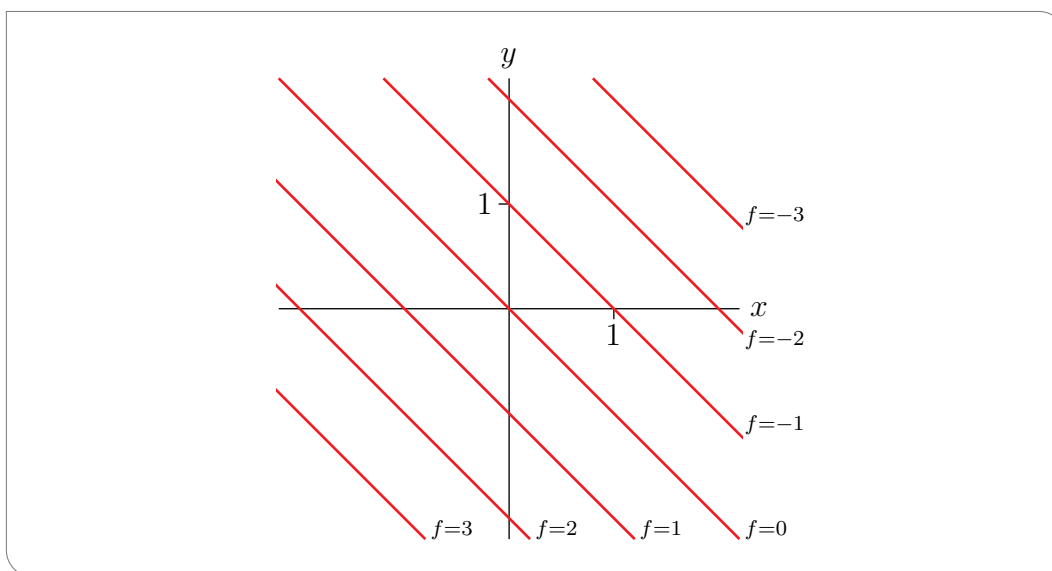
Example 4 ($e^{x+y+z} = 1$)

The function $f(x, y)$ is given implicitly by the equation $e^{x+y+z} = 1$. Sketch the level curves of f .

Solution. Fix any real number C . That “ $f(x, y)$ is given implicitly by the equation $e^{x+y+z} = 1$ ” means that, for each x, y , the solution z of $e^{x+y+z} = 1$ is $f(x, y)$. So, for the specified function f , the level curve $f(x, y) = C$ is the set of points (x, y) that obey

$$\begin{aligned} e^{x+y+C} = 1 &\iff x + y + C = 0 \\ &\iff x + y = -C \end{aligned}$$

This is of course a straight line. It intersects the x -axis when $y = 0$ and $x = -C$ and it intersects the y -axis when $x = 0$ and $y = -C$. Here is a sketch of some representative level curves.



Example 4