Taylor Polynomials — Approximating Functions Near a Specified Point

Suppose that you are interested in the values of some function $f(x)$ for x near some fixed point x_0. The function is too complicated to work with directly. So you wish to work instead with some other function $F(x)$ that is both simple and a good approximation to $f(x)$ for x near x_0. We'll consider a couple of examples of this scenario later. First, we develop several different approximations.

1. Zeroth Approximation — the Constant Approximation

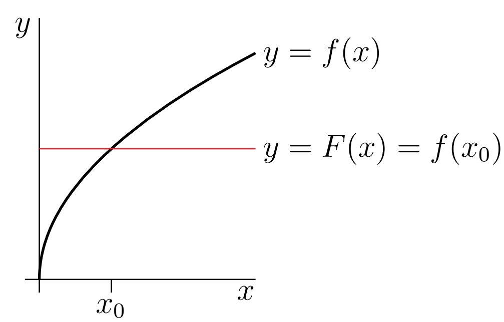
The simplest functions are those that are constants. The first approximation will be by a constant function. That is, the approximating function will have the form $F(x) = A$, for some constant A. To ensure that $F(x)$ is a good approximation for x close to x_0, we choose A so that $f(x)$ and $F(x)$ take exactly the same value when $x = x_0$.

$$F(x) = A \qquad \text{so} \qquad F(x_0) = A = f(x_0) \implies A = f(x_0)$$

Our first, and crudest, approximation rule is

$$f(x) \approx f(x_0) \tag{1}$$

Here is a figure showing the graphs of a typical $f(x)$ and approximating function $F(x)$. At



$x = x_0$, $f(x)$ and $F(x)$ take the same value. For x very near x_0, the values of $f(x)$ and $F(x)$ remain close together. But the quality of the approximation deteriorates fairly quickly as x moves away from x_0.

2. First Approximation — the Tangent Line, or Linear, Approximation

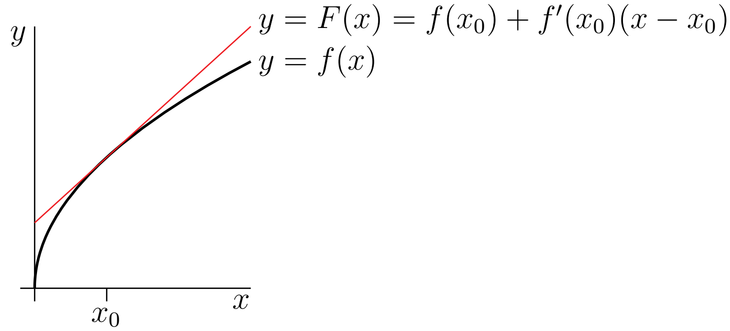
We now develop a better approximation by allowing the approximating function to be a linear function of x and not just a constant function. That is, we allow $F(x)$ to be of the form $A + Bx$, for some constants A and B. To ensure that $F(x)$ is a good approximation for x close to x_0, we choose A and B so that $f(x_0) = F(x_0)$ and $f'(x_0) = F'(x_0)$. Then $f(x)$ and $F(x)$ will have both the same value and the same slope at $x = x_0$.

$$\begin{aligned} F(x) &= A + Bx & \implies \quad F(x_0) &= A + Bx_0 = f(x_0) \\ F'(x) &= B & \implies \quad F'(x_0) &= B = f'(x_0) \end{aligned}$$

Substituting $B = f'(x_0)$ into $A + Bx_0 = f(x_0)$ gives $A = f(x_0) - x_0 f'(x_0)$ and consequently $F(x) = A + Bx = f(x_0) - x_0 f'(x_0) + x f'(x_0) = f(x_0) + f'(x_0)(x - x_0)$. So, our second approximation is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \tag{2}$$

You may recall that $y = f(x_0) + f'(x_0)(x - x_0)$ is exactly the equation of the tangent line to the curve $y = f(x)$ at x_0. Here is a figure showing the graphs of a typical $f(x)$ and approximating function $F(x)$. Observe that the graph of $f(x_0) + f'(x_0)(x - x_0)$ remains



close to the graph of $f(x)$ for a much larger range of x than did the graph of $f(x_0)$.

3. Second Approximation — the Quadratic Approximation

We next develop a still better approximation by allowing the approximating function be to a quadratic function of x. That is, we allow $F(x)$ to be of the form $A + Bx + Cx^2$, for some constants A, B and C. To ensure that $F(x)$ is a good approximation for x close to x_0, we choose A, B and C so that $f(x_0) = F(x_0)$ and $f'(x_0) = F'(x_0)$ and $f''(x_0) = F''(x_0)$.

$$\begin{aligned} F(x) &= A + Bx + Cx^2 & \implies \quad F(x_0) &= A + Bx_0 + Cx_0^2 = f(x_0) \\ F'(x) &= B + 2Cx & \implies \quad F'(x_0) &= B + 2Cx_0 = f'(x_0) \\ F''(x) &= 2C & \implies \quad F''(x_0) &= 2C = f''(x_0) \end{aligned}$$

Solve for C first, then B and finally A.

$$\begin{aligned} C = \tfrac{1}{2} f''(x_0) \implies & B = f'(x_0) - 2Cx_0 = f'(x_0) - x_0 f''(x_0) \\ \implies & A = f(x_0) - Bx_0 - Cx_0^2 = f(x_0) - x_0[f'(x_0) - x_0 f''(x_0)] - \tfrac{1}{2} f''(x_0) x_0^2 \end{aligned}$$

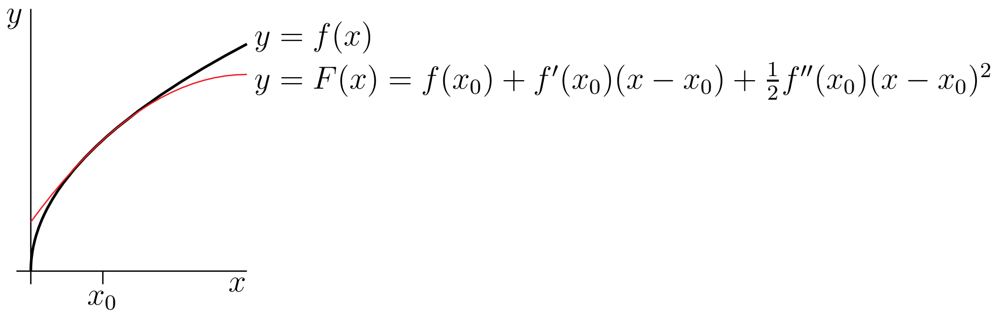
Then build up $F(x)$.

$$\begin{aligned} F(x) &= f(x_0) - f'(x_0)x_0 + \tfrac{1}{2} f''(x_0)x_0^2 && \text{(this line is } A\text{)} \\ &\quad + f'(x_0)\,x - f''(x_0)x_0 x && \text{(this line is } Bx\text{)} \\ &\quad + \tfrac{1}{2} f''(x_0)x^2 && \text{(this line is } Cx^2\text{)} \\ &= f(x_0) + f'(x_0)(x - x_0) + \tfrac{1}{2} f''(x_0)(x - x_0)^2 \end{aligned}$$

Our third approximation is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \tfrac{1}{2}f''(x_0)(x - x_0)^2 \tag{3}$$

It is called the quadratic approximation. Here is a figure showing the graphs of a typical $f(x)$ and approximating function $F(x)$. This third approximation looks better than both



the first and second.

4. Still Better Approximations – Taylor Polynomials

We can use the same strategy to generate still better approximations by polynomials of any degree we like. Let's approximate by a polynomial of degree n. The algebra will be simpler if we make the approximating polynomial $F(x)$ of the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$$

Because x_0 is itself a constant, this is really just a rewriting of $A_0 + A_1x + A_2x^2 + \cdots + A_nx^n$. For example,

$$\begin{aligned} a_0 + a_1(x - x_0) + a_2(x - x_0)^2 &= a_0 + a_1x - a_1x_0 + a_2x^2 - 2a_2xx_0 + a_2x_0^2 \\ &= (a_0 - a_1x_0 + a_2x_0^2) + (a_1 - 2a_2x_0)x + a_2x^2 \\ &= A_0 + A_1x + A_2x^2 \end{aligned}$$

with $A_0 = a_0 - a_1x_0 + a_2x_0^2$, $A_1 = a_1 - 2a_2x_0$ and $A_2 = a_2$. The advantage of the form $a_0 + a_1(x - x_0) + \cdots$ is that $x - x_0$ is zero when $x = x_0$, so lots of terms in the computation drop out. We determine the coefficients a_i by the requirements that $f(x)$ and its approximator $F(x)$ have the same value and the same first n derivatives at $x = x_0$.

$$\begin{aligned} F(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n \\ &\qquad\qquad \implies F(x_0) = a_0 = f(x_0) \\ F'(x) &= a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1} \\ &\qquad\qquad \implies F'(x_0) = a_1 = f'(x_0) \\ F''(x) &= 2a_2 + 3 \times 2a_3(x - x_0) + \cdots + n(n-1)a_n(x - x_0)^{n-2} \\ &\qquad\qquad \implies F''(x_0) = 2a_2 = f''(x_0) \end{aligned}$$

$$F^{(3)}(x) = 3 \times 2a_3 + \cdots + n(n-1)(n-2)a_n(x-x_0)^{n-3}$$
$$\implies F^{(3)}(x_0) = 3 \times 2a_3 = f^{(3)}(x_0)$$
$$\vdots$$
$$F^{(n)}(x) = n!a_n \qquad \implies F^{(n)}(x_0) = n!a_n = f^{(n)}(x_0)$$

Here $n! = n(n-1)(n-2)\cdots 1$ is called n factorial. Hence

$$a_0 = f(x_0) \quad a_1 = f'(x_0) \quad a_2 = \tfrac{1}{2!}f''(x_0) \quad a_3 = \tfrac{1}{3!}f^{(3)}(x_0) \quad \cdots \quad a_n = \tfrac{1}{n!}f^{(n)}(x_0)$$

and the approximator, which is called the Taylor polynomial of degree n for $f(x)$ at $x = x_0$, is

$$f(x) \approx f(x_0) + f'(x_0)\,(x-x_0) + \tfrac{1}{2!}f''(x_0)\,(x-x_0)^2 + \tfrac{1}{3!}f^{(3)}(x_0)\,(x-x_0)^3 + \cdots + \tfrac{1}{n!}f^{(n)}(x_0)\,(x-x_0)^n$$

or, in summation notation,

$$f(x) \approx \sum_{\ell=0}^{n} \tfrac{1}{\ell!}f^{(\ell)}(x_0)\,(x-x_0)^\ell \tag{4}$$

where we are using the standard convention that $0! = 1$.

5. The Δx, Δy Notation

Suppose that we have two variables x and y that are related by $y = f(x)$, for some function f. For example, x might be the number of cars manufactured per week in some factory and y the cost of manufacturing those x cars. Let x_0 be some fixed value of x and let $y_0 = f(x_0)$ be the corresponding value of y. Now suppose that x changes by an amount Δx, from x_0 to $x_0 + \Delta x$. As x undergoes this change, y changes from $y_0 = f(x_0)$ to $f(x_0 + \Delta x)$. The change in y that results from the change Δx in x is

$$\Delta y = f(x_0 + \Delta x) - f(x_0)$$

Substituting $x = x_0 + \Delta x$ into the linear approximation (2) yields the approximation

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)(x_0 + \Delta x - x_0) = f(x_0) + f'(x_0)\Delta x$$

for $f(x_0 + \Delta x)$ and consequently the approximation

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \approx f(x_0) + f'(x_0)\Delta x - f(x_0) \implies \Delta y \approx f'(x_0)\Delta x \tag{5}$$

for Δy. In the automobile manufacturing example, when the production level is x_0 cars per week, increasing the production level by Δx will cost approximately $f'(x_0)\Delta x$. The additional cost per additional car, $f'(x_0)$, is called the "marginal cost" of a car.

If we use the quadratic approximation (3) in place of the linear approximation (2)

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x + \tfrac{1}{2}f''(x_0)\Delta x^2$$

we arrive at the quadratic approximation

$$\begin{aligned} \Delta y &= f(x_0+\Delta x) - f(x_0) \\ &\approx f(x_0) + f'(x_0)\Delta x + \tfrac{1}{2} f''(x_0)\Delta x^2 - f(x_0) \\ \implies \Delta y &\approx f'(x_0)\Delta x + \tfrac{1}{2} f''(x_0)\Delta x^2 \end{aligned} \tag{6}$$

for Δy.

6. Examples

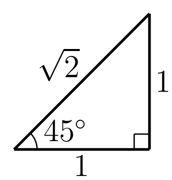
Example 1

As an initial example, we compute, approximately, $\tan 46°$, using the constant approximation (1), the linear approximation (2) and the quadratic approximation (3). To do so, we choose $f(x) = \tan x$, $x = 46\frac{\pi}{180}$ radians and $x_0 = 45\frac{\pi}{180} = \frac{\pi}{4}$ radians. This is a good choice for x_0 because

- $x_0 = 45°$ is close to $x = 46°$. Generally, the closer x is to x_0, the better the quality of our various approximations.
- We know the values of all trig functions at $45°$.

The first step in applying our approximations is to compute f and its first two derivatives at $x = x_0$.

$$\begin{aligned} f(x) &= \tan x & &\implies & f(x_0) &= \tan\tfrac{\pi}{4} = 1 \\ f'(x) &= (\cos x)^{-2} & &\implies & f'(x_0) &= \tfrac{1}{\cos^2(\pi/4)} = \tfrac{1}{(1/\sqrt{2})^2} = 2 \\ f''(x) &= -2\tfrac{-\sin x}{\cos^3 x} & &\implies & f''(x_0) &= 2\tfrac{\sin(\pi/4)}{\cos^3(\pi/4)} = 2\tfrac{1/\sqrt{2}}{(1/\sqrt{2})^3} = 2\tfrac{1}{1/2} = 4 \end{aligned}$$



As $x - x_0 = 46\frac{\pi}{180} - 45\frac{\pi}{180} = \frac{\pi}{180}$ radians, the three approximations are

$$\begin{aligned} f(x) &\approx f(x_0) & & &&= 1 \\ f(x) &\approx f(x_0) + f'(x_0)(x-x_0) & &= 1 + 2\tfrac{\pi}{180} &&= 1.034907 \\ f(x) &\approx f(x_0) + f'(x_0)(x-x_0) + \tfrac{1}{2} f''(x_0)(x-x_0)^2 & &= 1 + 2\tfrac{\pi}{180} + \tfrac{1}{2}4\big(\tfrac{\pi}{180}\big)^2 &&= 1.035516 \end{aligned}$$

For comparison purposes, $\tan 46°$ really is 1.035530 to 6 decimal places.

Example 1

Warning 2.

All of our derivative formulae for trig functions were developed under the assumption that angles are measured in radians. Those derivatives appeared in the approximation formulae that we used in Example 1, so we were obliged to express $x - x_0$ in radians.

Example 3

Let's find all Taylor polyomials for $\sin x$ and $\cos x$ at $x_0 = 0$. To do so we merely need compute all derivatives of $\sin x$ and $\cos x$ at $x_0 = 0$. First, compute all derivatives at general x.

$$\begin{array}{llllll} f(x) = \sin x & f'(x) = \cos x & f''(x) = -\sin x & f^{(3)}(x) = -\cos x & f^{(4)}(x) = \sin x & \cdots \\ g(x) = \cos x & g'(x) = -\sin x & g''(x) = -\cos x & g^{(3)}(x) = \sin x & g^{(4)}(x) = \cos x & \cdots \end{array} \tag{7}$$

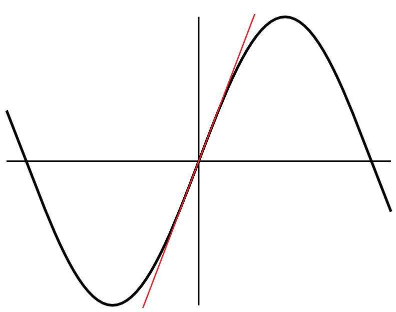
The pattern starts over again with the fourth derivative being the same as the original function. Now set $x = x_0 = 0$.

$$\begin{array}{llllllll} f(x) = \sin x & f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f^{(3)}(0) = -1 & f^{(4)}(0) = 0 & \cdots \\ g(x) = \cos x & g(0) = 1 & g'(0) = 0 & g''(0) = -1 & g^{(3)}(0) = 0 & g^{(4)}(0) = 1 & \cdots \end{array} \tag{8}$$

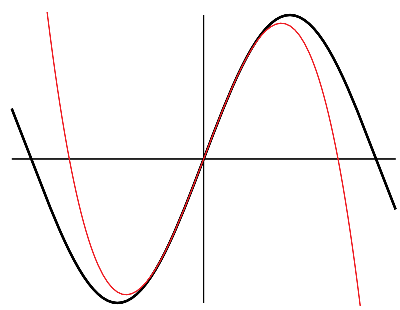
For $\sin x$, all even numbered derivatives are zero. The odd numbered derivatives alternate between 1 and -1. For $\cos x$, all odd numbered derivatives are zero. The even numbered derivatives alternate between 1 and -1. So, the Taylor polynomials that best approximate $\sin x$ and $\cos x$ near $x = x_0 = 0$ are

$$\begin{aligned} \sin x &\approx x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots \\ \cos x &\approx 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots \end{aligned}$$

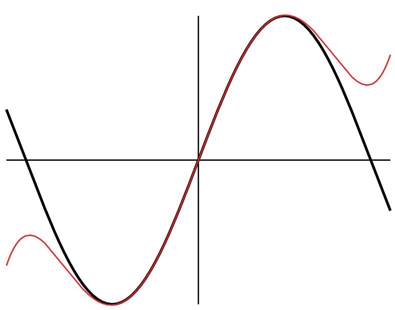
Here are graphs of $\sin x$ and its Taylor poynomials (about $x_0 = 0$) up to degree seven.



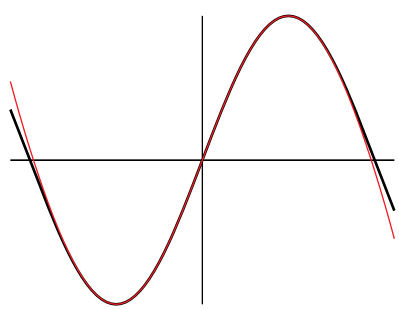
$\sin x \approx x$



$\sin x \approx x - \frac{1}{3!}x^3$



$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$



$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$

To get an idea of how good these Taylor polynomials are at approximating sin and cos, let's concentrate on $\sin x$ and consider x's whose magnitude $|x| \le 1$. (If you're writing software to evaluate $\sin x$, you can always use the trig identity $\sin(x) = \sin(x - 2n\pi)$, to easily restrict to $|x| \le \pi$, and then use the trig identity $\sin(x) = -\sin(x \pm \pi)$ to reduce to $|x| \le \frac{\pi}{2}$ and then use the trig identity $\sin(x) = \mp\cos(\frac{\pi}{2} \pm x))$ to reduce to $|x| \le \frac{\pi}{4}$.) If $|x| \le 1$ radians (recall that the derivative formulae that we used to derive the Taylor polynomials are valid only when x is in radians), or equivalently if $|x|$ is no larger than $\frac{180}{\pi} \approx 57°$, then the magnitudes of the successive terms in the Taylor polynomials for $\sin x$ are bounded by

$$\begin{array}{lll} |x| \le 1 & \frac{1}{3!}|x|^3 \le \frac{1}{6} & \frac{1}{5!}|x|^3 \le \frac{1}{120} \approx 0.0083 \\ \frac{1}{7!}|x|^7 \le \frac{1}{7!} \approx 0.0002 & \frac{1}{9!}|x|^9 \le \frac{1}{9!} \approx 0.000003 & \frac{1}{11!}|x|^{11} \le \frac{1}{11!} \approx 0.000000025 \end{array}$$

From these inequalities, and the graphs on the previous page, it certainly looks like, for x not too large, even relatively low degree Taylor polynomials give very good approximations. We'll see later how to get rigorous error bounds on our Taylor polynomial approximations.

Example 3

Example 4

Suppose that you are ten meters from a vertical pole. You were contracted to measure the height of the pole. You can't take it down or climb it. So you measure the angle subtended by the top of the pole. You measure $\theta = 30°$, which gives

$$h = 10\tan 30° = \tfrac{10}{\sqrt{3}} \approx 5.77\text{m}$$

But there's a catch. Angles are hard to measure accurately. Your contract specifies that the height must be measured to within an accuracy of 10 cm. How accurate did your measurement of θ have to be?

Solution. For simplicity, we are going to assume that the pole is perfectly straight and perfectly vertical and that your distance from the pole was exactly 10 m. Write $h = h_0 + \Delta h$, where h is the exact height and $h_0 = \frac{10}{\sqrt{3}}$ is the computed height. Their difference, Δh, is the error. Similarly, write $\theta = \theta_0 + \Delta\theta$ where θ is the exact angle, θ_0 is the measured angle and $\Delta\theta$ is the error. Then

$$h_0 = 10\tan\theta_0 \qquad h_0 + \Delta h = 10\tan(\theta_0 + \Delta\theta)$$

We apply $\Delta y \approx f'(x_0)\Delta x$, with y replaced by h and x replaced by θ. That is, we apply $\Delta h \approx f'(\theta_0)\Delta\theta$. Choosing $f(\theta) = 10\tan\theta$ and $\theta_0 = 30°$ and substituting in

$$f'(\theta_0) = 10\sec^2\theta_0 = 10\sec^2 30° = 10\big(\tfrac{2}{\sqrt{3}}\big)^2 = \tfrac{40}{3}$$

we see that the error in the computed value of h and the error in the measured value of θ are related by

$$\Delta h \approx \tfrac{40}{3}\Delta\theta \qquad \text{or} \qquad \Delta\theta \approx \tfrac{3}{40}\Delta h$$

To achieve $|\Delta h| \le 0.1$m, we better have $|\Delta\theta|$ smaller than $0.1\frac{3}{40}$ radians or $0.1\frac{3}{40}\frac{180}{\pi} = 0.43°$.

Example 4

Example 5

Suppose that the radius of a sphere has been measured with a percentage error of at most ε%. Find the corresponding approximate percentage error in the surface area and volume of the sphere.

Solution. Suppose that the exact radius is r_0 and that the measured radius is $r_0 + \Delta r$. Then the absolute error in the measurement is $|\Delta r|$ and, by definition, the percentage error is $100\frac{|\Delta r|}{r_0}$. We are told that $100\frac{|\Delta r|}{r_0} \le \varepsilon$. The surface area of a sphere of radius r is $A(r) = 4\pi r^2$. The error in the surface area computed with the measured radius is

$$\Delta A = A(r_0 + \Delta r) - A(r_0) \approx A'(r_0)\Delta r$$

The corresponding percentage error is

$$100\tfrac{|\Delta A|}{A(r_0)} \approx 100\tfrac{|A'(r_0)\Delta r|}{A(r_0)} = 100\tfrac{8\pi r_0|\Delta r|}{4\pi r_0^2} = 2 \times 100\tfrac{|\Delta r|}{r_0} \le 2\varepsilon$$

The volume of a sphere of radius r is $V(r) = \frac{4}{3}\pi r^3$. The error in the volume computed with the measured radius is

$$\Delta V = V(r_0 + \Delta r) - V(r_0) \approx V'(r_0)\Delta r$$

The corresponding percentage error is

$$100\tfrac{|\Delta V|}{V(r_0)} \approx 100\tfrac{|V'(r_0)\Delta r|}{V(r_0)} = 100\tfrac{4\pi r_0^2|\Delta r|}{4\pi r_0^3/3} = 3 \times 100\tfrac{|\Delta r|}{r_0} \le 3\varepsilon$$

We have just computed an approximation to ΔV. In this problem, we can compute the exact error

$$V(r_0 + \Delta r) - V(r_0) = \tfrac{4}{3}\pi(r_0 + \Delta r)^3 - \tfrac{4}{3}\pi r_0^3$$

Applying $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ with $a = r_0$ and $b = \Delta r$, gives

$$\begin{aligned} V(r_0 + \Delta r) - V(r_0) &= \tfrac{4}{3}\pi[r_0^3 + 3r_0^2\Delta r + 3r_0\,\Delta r^2 + \Delta r^3 - r_0^3] \\ &= \tfrac{4}{3}\pi[3r_0^2\Delta r + 3r_0\,\Delta r^2 + \Delta r^3] \end{aligned}$$

The linear approximation, $\Delta V \approx 4\pi r_0^2 \times \Delta r$, is recovered by retaining only the first of the three terms in the square brackets. Thus the difference between the exact error and the linear approximation to the error is obtained by retaining only the last two terms in the square brackets. This has magnitude

$$\tfrac{4}{3}\pi\big|3r_0\,\Delta r^2 + \Delta r^3\big| = \tfrac{4}{3}\pi\big|3r_0 + \Delta r\big|\Delta r^2$$

or in percentage terms

$$100\frac{1}{\frac{4}{3}\pi r_0^3}\tfrac{4}{3}\pi\big|3r_0\,\Delta r^2 + \Delta r^3\big| = 100\big|3\tfrac{\Delta r^2}{r_0^2} + \tfrac{\Delta r^3}{r_0^3}\big| = \big(100\tfrac{3\Delta r}{r_0}\big)\big(\tfrac{\Delta r}{r_0}\big)\big|1 + \tfrac{\Delta r}{3r_0}\big| \le 3\varepsilon\big(\tfrac{\varepsilon}{100}\big)\big(1 + \tfrac{\varepsilon}{300}\big)$$

Thus the difference between the exact error and the linear approximation is roughly a factor of $\frac{\varepsilon}{100}$ smaller than the linear approximation 3ε.

Example 5

Example 6

When an aircraft crosses the Atlantic ocean at a speed of u mph, the flight costs the company

$$C(u) = 100 + \tfrac{u}{3} + \tfrac{240{,}000}{u}$$

dollars per passenger. When there is no wind, the aircraft flies at an airspeed of 550mph. Find the approximate savings, per passenger, when there is a 35 mph tail wind and estimate the cost when there is a 50 mph head wind.

Solution. Let $u_0 = 550$. When the aircraft flies at speed u_0, the cost per passenger is $C(u_0)$. By (5), a change of Δu in the airspeed results in an change of

$$\Delta C \approx C'(u_0)\Delta u = \left[\tfrac{1}{3} - \tfrac{240{,}000}{u_0^2}\right]\Delta u = \left[\tfrac{1}{3} - \tfrac{240{,}000}{550^2}\right]\Delta u \approx -.460\Delta u$$

in the cost per passenger. With the tail wind $\Delta u = 35$ and the resulting

$$\Delta C \approx -.460 \times 35 = -16.10$$

so there is a savings of \$16.10. With the head wind $\Delta u = -50$ and the resulting

$$\Delta C \approx -.4601 \times (-50) = 23.01$$

so there is an additional cost of about \$23.00.

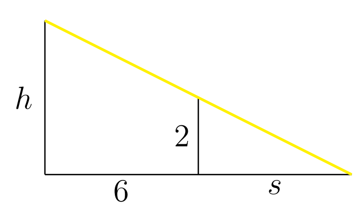
Example 6

Example 7

To compute the height h of a lamp post, the length s of the shadow of a two meter pole is measured. The pole is 6 m from the lamp post. If the length of the shadow was measured to be 4 m, with an error of at most one cm, find the height of the lamp post and estimate the percentage error in the height.

Solution. By similar triangles,

$$\frac{s}{2} = \frac{6+s}{h} \implies h = (6+s)\frac{2}{s} = \frac{12}{s} + 2$$



The length of the shadow was measured to be $s_0 = 4$ m. The corresponding height of the lamp post is $h_0 = \frac{12}{s_0} + 2 = \frac{12}{4} + 2 = 5$ m. If the error in the measurement of the length of the shadow was Δs, then the exact shadow length was $s = s_0 + \Delta s$ and the exact lamp post height is $h = f(s_0 + \Delta s)$, where $f(s) = \frac{12}{s} + 2$. The error in the computed lamp post height is $\Delta h = h - h_0 = f(s_0 + \Delta s) - f(s_0)$. By (5),

$$\Delta h \approx f'(s_0)\Delta s = -\tfrac{12}{s_0^2}\Delta s = -\tfrac{12}{4^2}\Delta s$$

We are told that $|\Delta s| \le \frac{1}{10}$ m. Consequently $|\Delta h| \le \frac{12}{4^2}\frac{1}{10} = \frac{3}{40}$ (approximately). The percentage error is then approximately

$$100\frac{|\Delta h|}{h_0} \le 100\frac{3}{40\times 5} = 1.5\%$$

Example 7

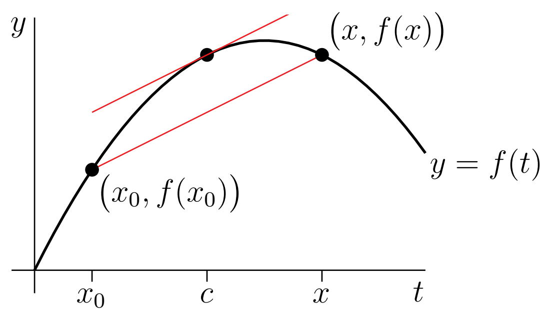
7. The Error in the Taylor Polynomial Approximations

Any time you make an approximation, it is desirable to have some idea of the size of the error you introduced. We will now develop a formula for the error introduced by the approximation $f(x) \approx f(x_0)$. This formula can be used to get an upper bound on the size of the error, even when you cannot determine $f(x)$ exactly.

By simple algebra

$$f(x) = f(x_0) + \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \tag{9}$$

The coefficient $\frac{f(x)-f(x_0)}{x-x_0}$ of $(x - x_0)$ is the average slope of $f(t)$ as t moves from $t = x_0$ to $t = x$. In the figure below, it is the slope of the secant joining the points $(x_0, f(x_0))$ and



$(x, f(x))$. As t moves x_0 to x, the instantaneous slope $f'(t)$ keeps changing. Sometimes it is larger than the average slope $\frac{f(x)-f(x_0)}{x-x_0}$ and sometimes it is smaller than the average slope. But there is a theorem, called the Mean–Value Theorem, which says that there must be some number c, strictly between x_0 and x, for which $f'(c) = \frac{f(x)-f(x_0)}{x-x_0}$. Substituting this into formula (9) gives

$$f(x) = f(x_0) + f'(c)(x - x_0) \qquad \text{for some } c \text{ strictly between } x_0 \text{ and } x \tag{10}$$

Thus the error in the approximation $f(x) \approx f(x_0)$ is exactly $f'(c)(x - x_0)$ for some c strictly between x_0 and x. There are formulae similar to (10), that can be used to bound the error in our other approximations. One is

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\tfrac{1}{2}f''(c)(x-x_0)^2 \qquad \text{for some } c \text{ strictly between } x_0 \text{ and } x \tag{11}$$

It implies that the error in the approximation $f(x) \approx f(x_0) + f'(x_0)\,(x - x_0)$ is exactly $\frac{1}{2} f''(c)\,(x-x_0)^2$ for some c strictly between x_0 and x. In general

$$\begin{aligned} f(x) =& f(x_0) + f'(x_0)\,(x-x_0) + \cdots + \tfrac{1}{n!} f^{(n)}(x_0)\,(x-x_0)^n \\ &+ \tfrac{1}{(n+1)!} f^{(n+1)}(c)\,(x-x_0)^{n+1} \qquad \text{for some } c \text{ strictly between } x_0 \text{ and } x \end{aligned} \tag{12}$$

That is, the error introduced when $f(x)$ is approximated by its Taylor polynomial of degree n, is precisely the last term of the Taylor polynomial of degree $n+1$, but with the derivative evaluated at some point between x_0 and x, rather than exactly at x_0. These error formulae are proven in the next (optional) section.

Example 8

Suppose we wish to approximate $\sin 46°$ using Taylor polynomials about $x_0 = 45°$. Then, we would define

$$f(x) = \sin x \qquad x_0 = 45° = 45\tfrac{\pi}{180}\text{radians} \quad x = 46° = 46\tfrac{\pi}{180}\text{radians} \quad x - x_0 = \tfrac{\pi}{180}\text{radians}$$

The first few derivatives of f at x_0 are

$$\begin{aligned} f(x) &= \sin x & f(x_0) &= \tfrac{1}{\sqrt{2}} \\ f'(x) &= \cos x & f'(x_0) &= \tfrac{1}{\sqrt{2}} \\ f''(x) &= -\sin x & f''(x_0) &= -\tfrac{1}{\sqrt{2}} \\ f^{(3)}(x) &= -\cos x \end{aligned}$$

The constant, linear and quadratic approximations for $\sin 46°$ are

$$\begin{aligned} \sin 46° &\approx f(x_0) &&= \tfrac{1}{\sqrt{2}} &&= 0.70710678 \\ \sin 46° &\approx f(x_0) + f'(x_0)(x-x_0) &&= \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{180}\right) &&= 0.71944812 \\ \sin 46° &\approx f(x_0) + f'(x_0)(x-x_0) + \tfrac{1}{2} f''(x_0)(x-x_0)^2 &&= \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{180}\right) - \tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{180}\right)^2 &&= 0.71934042 \end{aligned}$$

The errors in those approximations are

$$\begin{aligned} \text{error in } 0.70710678 &= f'(c)(x-x_0) &&= (\cos c)\ \left(\tfrac{\pi}{180}\right) \\ \text{error in } 0.71944812 &= \tfrac{1}{2} f''(c)(x-x_0)^2 &&= -\tfrac{1}{2}(\sin c)\ \left(\tfrac{\pi}{180}\right)^2 \\ \text{error in } 0.71923272 &= \tfrac{1}{3!} f^{(3)}(c)(x-x_0)^3 &&= -\tfrac{1}{3!}(\cos c)\ \left(\tfrac{\pi}{180}\right)^3 \end{aligned}$$

In each of these three cases c must lie somewhere between $45°$ and $46°$. No matter what c is, we know that $|\sin c| \le 1$ and $\cos c| \le 1$. Hence

$$\begin{aligned} \big|\text{error in } 0.70710678\big| &\le \left(\tfrac{\pi}{180}\right) && < 0.018 \\ \big|\text{error in } 0.71944812\big| &\le \tfrac{1}{2}\left(\tfrac{\pi}{180}\right)^2 && < 0.00015 \\ \big|\text{error in } 0.71934042\big| &\le \tfrac{1}{3!}\left(\tfrac{\pi}{180}\right)^3 && < 0.0000009 \end{aligned}$$

Example 8

Example 9 (e^x and e)

Let $f(x) = e^x$. Then

$$\begin{array}{lllll} f(x) = e^x & \Rightarrow & f'(x) = e^x & \Rightarrow & f''(x) = e^x & \cdots \\ f(0) = e^0 = 1 & \Rightarrow & f'(0) = e^0 = 1 & \Rightarrow & f''(0) = e^0 = 1 & \cdots \end{array}$$

Applying (12) with $f(x) = e^x$ and $x_0 = 0$, and using that $f^{(m)}(x_0) = e^{x_0} = e^0 = 1$ for all m,

$$e^x = f(x) = 1 + x + \tfrac{x^2}{2!} + \cdots + \tfrac{x^n}{n!} + \tfrac{1}{(n+1)!} e^c x^{n+1} \tag{13}$$

for some c between 0 and x. We can use this to find approximate values for the number e, with any desired degree of accuracy. Just setting $x = 1$ in (13) gives

$$e = 1 + 1 + \tfrac{1}{2!} + \cdots + \tfrac{1}{n!} + \tfrac{1}{(n+1)!} e^c \tag{14}$$

for some c between 0 and 1. Since e^c increases as c increases, this says that $1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}$ is an approximate value for e with error at most $\frac{e}{(n+1)!}$. The only problem with this error bound is that it contains the number e, which we do not know. Fortunately, we can again use (14) to get a simple upper bound on how big e can be. Just setting $n = 2$ in (14), and again using that $e^c \le e$, gives

$$e \le 1 + 1 + \tfrac{1}{2!} + \tfrac{e}{3!} \implies \big(1 - \tfrac{1}{6}\big)e \le 1 + 1 + \tfrac{1}{2!} = \tfrac{5}{2} \implies e \le \tfrac{5}{2} \times \tfrac{6}{5} = 3$$

So we now know that $1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$ is an approximate value for e with error at most $\frac{3}{(n+1)!}$. For example, when $n = 9$, $\frac{3}{(n+1)!} = \frac{3}{10!} < 10^{-6}$ so that

$$1 + 1 + \tfrac{1}{2!} + \cdots + \tfrac{1}{9!} \le e \le 1 + 1 + \tfrac{1}{2!} + \cdots + \tfrac{1}{9!} + 10^{-6}$$

with

$$\begin{aligned} &1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \tfrac{1}{4!} + \tfrac{1}{5!} + \tfrac{1}{6!} + \tfrac{1}{7!} + \tfrac{1}{8!} + \tfrac{1}{9!} \\ =&1 + 1 + 0.5 + 0.1\dot{6} + 0.041\dot{6} + 0.008\dot{3} + 0.0013\dot{8} + 0.0001984 + 0.0000248 + 0.0000028 \\ =&2.718282 \end{aligned}$$

to six decimal places.

Example 9

Example 10 (Example 4 Revisited)

In Example 4 (measuring the height of the pole), we used the linear approximation

$$f(\theta_0 + \Delta\theta) \approx f(\theta_0) + f'(\theta_0)\Delta\theta \tag{15}$$

with $f(\theta) = 10 \tan\theta$ and $\theta_0 = 30\frac{\pi}{180}$ to get

$$\Delta h = f(\theta_0 + \Delta\theta) - f(\theta_0) \approx f'(\theta_0)\Delta\theta \implies \Delta\theta \approx \frac{\Delta h}{f'(\theta_0)}$$

While this procedure is fairly reliable, it did involve an approximation. So that you could not 100% guarantee to your client's lawyer that an accuracy of 10 cm was achieved. If we use the exact formula (10), with the replacements $x \to \theta_0 + \Delta\theta$ and $x_0 \to \theta_0$

$$f(\theta_0 + \Delta\theta) = f(\theta_0) + f'(c)\Delta\theta \text{ for some } c \text{ between } \theta_0 \text{ and } \theta_0 + \Delta\theta$$

in place of the approximate formula (2), this legality is taken care of.

$$\Delta h = f(\theta_0 + \Delta\theta) - f(\theta_0) = f'(c)\Delta\theta \implies \Delta\theta = \frac{\Delta h}{f'(c)} \text{ for some } c \text{ between } \theta_0 \text{ and } \theta_0 + \Delta\theta$$

Of course we do not know exactly what c is. But suppose that we know that the angle was somewhere between 25° and 35°. In other words suppose that, even though we don't know precisely what our measurement error was, it was certainly no more than 5°. Since $\sec(c)$ increases with c (for c between 0 and 90°), $f'(c) = 10\sec^2(c)$ must certainly be smaller than $10\sec^2 35° < 14.91$, which means that $\frac{\Delta h}{f'(c)}$ must be at least $\frac{.1}{14.91}$ radians or $\frac{.1}{14.91}\frac{180}{\pi} = .38°$. A measurement error of 0.38° or less is certainly acceptable.

Example 10

8. Derivation of the Error Formulae (Optional)

Fix any real number x_0 and natural number n. Define

$$E_n(x) = f(x) - f(x_0) - f'(x_0)(x - x_0) - \cdots - \tfrac{1}{n!}f^{(n)}(x_0)(x - x_0)^n$$

This is the error introduced when one approximates $f(x)$ by its Taylor polynomial of degree n (about x_0). We shall now prove that

$$E_n(x) = \tfrac{1}{(n+1)!}f^{(n+1)}(c)\,(x - x_0)^{n+1} \tag{16_n}$$

for some c strictly between x_0 and x. In fact, we have already used the Mean–Value Theorem to prove that $E_0(x) = f'(c)\,(x - x_0)$, for some c strictly between x_0 and x. This was the content of (10). To deal with $n \geq 1$, we need the following generalization of the Mean–Value Theorem. (Choosing $G(x) = x$ reduces Theorem 11 to the Mean–Value Theorem.)

Theorem 11 (Generalized Mean–Value Theorem)**.**

Let the functions $F(x)$ and $G(x)$ both be defined and continuous on $a \leq x \leq b$ and both be differentiable on $a < x < b$. Furthermore, suppose that $G'(x) \neq 0$ for all $a < x < b$. Then, there is a number c obeying $a < c < b$ such that

$$\tfrac{F(b)-F(a)}{G(b)-G(a)} = \tfrac{F'(c)}{G'(c)}$$

Proof. Define

$$h(x) = \big[F(b) - F(a)\big]\big[G(x) - G(a)\big] - \big[F(x) - F(a)\big]\big[G(b) - G(a)\big]$$

Observe that $h(a) = h(b) = 0$. So, by the Mean–Value Theorem, there is a number c obeying $a < c < b$ such that

$$0 = \tfrac{h(b)-h(a)}{b-a} = h'(c) = \big[F(b) - F(a)\big]G'(c) - F'(c)\big[G(b) - G(a)\big]$$

As $G(a) \neq G(b)$ (otherwise the Mean–Value Theorem would imply the existence of an $a < x < b$ obeying $G'(x) = 0$), we may divide by $G'(c)\big[G(b) - G(a)\big]$ which gives the desired result. ☐

Proof of (16_n). To prove (16_1), that is (16_n) for $n = 1$, simply apply the Generalized Mean–Value Theorem with $F(x) = E_1(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$, $G(x) = (x - x_0)^2$, $a = x_0$ and $b = x$. Then $F(a) = G(a) = 0$, so that

$$\tfrac{F(b)}{G(b)} = \tfrac{F'(\tilde c)}{G'(\tilde c)} \implies \tfrac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2} = \tfrac{f'(\tilde c)-f'(x_0)}{2(\tilde c-x_0)}$$

for some $\tilde c$ strictly between x_0 and x. By the Mean–Value Theorem (the standard one, but with $f(x)$ replaced by $f'(x)$), $\frac{f'(\tilde c)-f'(x_0)}{\tilde c-x_0} = f''(c)$, for some c strictly between x_0 and $\tilde c$ (which forces c to also be strictly between x_0 and x). Hence

$$\tfrac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2} = \tfrac{1}{2}f''(c)$$

which is exactly (16_1).

At this stage, we know that (16_n) applies to all (sufficiently differentiable) functions for $n = 0$ and $n = 1$. To prove it for general n, we proceed by induction. That is, we assume that we already know that (16_n) applies to $n = k - 1$ for some k (as is the case for $k = 1, 2$) and we wish to prove that it also applies to $n = k$. We apply the Generalized Mean–Value Theorem with $F(x) = E_k(x)$, $G(x) = (x - x_0)^{k+1}$, $a = x_0$ and $b = x$. Then $F(a) = G(a) = 0$, so that

$$\frac{F(b)}{G(b)} = \frac{F'(\tilde c)}{G'(\tilde c)} \implies \frac{E_k(x)}{(x - x_0)^{k+1}} = \frac{E_k'(\tilde c)}{(k+1)(\tilde c - x_0)^k} \tag{17}$$

for some $\tilde c$ between x_0 and x. But

$$\begin{aligned} E_k'(\tilde c) &= \frac{\mathrm{d}}{\mathrm{d}x}\Big[f(x) - f(x_0) - f'(x_0)\,(x - x_0) - \cdots - \tfrac{1}{k!}f^{(k)}(x_0)\,(x - x_0)^k\Big]_{x=\tilde c} \\ &= \Big[f'(x) - f'(x_0) - \cdots - \tfrac{1}{(k-1)!}f^{(k)}(x_0)(x - x_0)^{k-1}\Big]_{x=\tilde c} \\ &= f'(\tilde c) - f'(x_0) - \cdots - \tfrac{1}{(k-1)!}f^{(k)}(x_0)(\tilde c - x_0)^{k-1} \end{aligned} \tag{18}$$

The last expression is exactly the definition of $E_{k-1}(\tilde c)$, but for the function $f'(x)$, instead of the function $f(x)$. But we already know that (16_{k-1}) is true. So, substituting $n \to k - 1$, $f \to f'$ and $x \to \tilde c$ into (16_n), we already know that (18), i.e. $E_k'(\tilde c)$, equals

$$\tfrac{1}{(k-1+1)!}\big(f'\big)^{(k-1+1)}(c)(\tilde c - x_0)^{k-1+1} = \tfrac{1}{k!}f^{(k+1)}(c)\,(\tilde c - x_0)^k$$

for some c strictly between x_0 and $\tilde{c}$, and hence also strictly between x_0 and x. Substituting this into (17) gives

$$\frac{E_k(x)}{(x-x_0)^{k+1}} = \frac{E_k'(\tilde{c})}{(k+1)(\tilde{c}-x_0)^k} = \frac{f^{(k+1)}(c)\,(\tilde{c}-x_0)^k}{(k+1)\,k!\,(\tilde{c}-x_0)^k} = \frac{1}{(k+1)!}f^{(k+1)}(c)$$

which is exactly (16_k).

So we now know that

- if, for some k, (16_{k-1}) is true for all k times differentiable functions,
- then (16_k) is true for all $k+1$ times differentiable functions.

Repeatedly applying this for $k = 2, 3, 4, \cdots$ (and recalling that (16_1) is true) gives (16_k) for all k. □

9. Taylor Series

Fix a real number x_0 and suppose that all derivatives of the function $f(x)$ exist. We have seen in (12) that, for any natural number n,

$$f(x) = P_n(x) + E_n(x) \tag{19}$$

where

$$P_n(x) = f(x_0) + f'(x_0)\,(x-x_0) + \cdots + \tfrac{1}{n!}f^{(n)}(x_0)\,(x-x_0)^n \tag{19a}$$

is the Taylor polynomial of degree n for the function $f(x)$ and expansion point x_0 and

$$E_n(x) = f(x) - P_n(x) = \tfrac{1}{(n+1)!}f^{(n+1)}(c)\,(x-x_0)^{n+1} \tag{19b}$$

is the error introduced when we approximate $f(x)$ by the polynomial $P_n(x)$. If it happens that $E_n(x)$ tends to zero as $n \to \infty$, then we have the exact formula

$$f(x) = \lim_{n\to\infty} P_n(x)$$

for $f(x)$. This is usually written

$$f(x) = \sum_{n=0}^{\infty} \tfrac{1}{n!}f^{(n)}(x_0)\,(x-x_0)^n \tag{20}$$

and is called the Taylor series of $f(x)$ with expansion point x_0.

Example 12 (Exponential Series)

This happens with the exponential function $f(x) = e^x$. Recall from (13) that, for all natural numbers n and all real numbers x,

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \cdots + \tfrac{1}{n!}x^n + \tfrac{e^c}{(n+1)!}x^{n+1}$$

for some c strictly between 0 and x. Now consider any fixed real number x. As c runs from 0 to x, e^c runs from $e^0 = 1$ to e^x. In particular, e^c is always between 1 and e^x and so is smaller than $1 + e^x$. Thus the error term

$$|E_n(x)| = \left|\frac{e^c}{(n+1)!}x^{n+1}\right| \le [e^x + 1]\frac{|x|^{n+1}}{(n+1)!}$$

Let's call $e_n(x) = \frac{|x|^{n+1}}{(n+1)!}$. We claim that as n increases towards infinity, $e_n(x)$ decreases (quickly) towards zero. To see this, let's compare $e_n(x)$ and $e_{n+1}(x)$.

$$\frac{e_{n+1}(x)}{e_n(x)} = \frac{\frac{|x|^{n+2}}{(n+2)!}}{\frac{|x|^{n+1}}{(n+1)!}} = \frac{|x|}{n+2}$$

So, when n is bigger than, for example $2|x|$, we have $\frac{e_{n+1}(x)}{e_n(x)} < \frac{1}{2}$. That is, increasing the index on $e_n(x)$ by one decreases the size of $e_n(x)$ by a factor of at least two. As a result $e_n(x)$ must tend to zero as $n \to \infty$. Consequently $\lim\limits_{n\to\infty} E_n(x) = 0$ and

$$e^x = \lim_{n\to\infty}\left[1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \cdots + \tfrac{1}{n!}x^n\right] = \sum_{n=0}^{\infty} \tfrac{1}{n!}x^n \tag{21}$$

Example 12

Example 13 (Sine and Cosine Series)

The trigonometric functions $\sin x$ and $\cos x$ also have widely used Taylor series expansions about $x_0 = 0$. Reviewing (7) we see that every derivative of $\sin x$ and $\cos x$ is one of $\pm\sin x$ and $\pm\cos x$. Consequently, when we apply (19b) we always have $\left|f^{(n+1)}(c)\right| \le 1$ and hence $|E_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$. We have already seen in Example 12, that $\frac{|x|^{n+1}}{(n+1)!}$ (which we called $e_n(x)$ in Example 12) converges to zero as $n \to \infty$. Consequently, for both $f(x) = \sin x$ and $f(x) = \cos x$, we have $\lim\limits_{n\to\infty} E_n(x) = 0$ and

$$f(x) = \lim_{n\to\infty}\left[f(0) + f'(0)\,x + \cdots + \tfrac{1}{n!}f^{(n)}(0)\,x^n\right]$$

Reviewing (8), we conclude that

$$\begin{aligned} \sin x &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots = \sum_{n=0}^{\infty}(-1)^n \tfrac{1}{(2n+1)!}x^{2n+1} \\ \cos x &= 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots = \sum_{n=0}^{\infty}(-1)^n \tfrac{1}{(2n)!}x^{2n} \end{aligned} \tag{22}$$

Example 13

10. Evaluating Limits Using Taylor Expansions

Taylor polynomials provide a good way to understand the behaviour of a function near a specified point and so are useful for evaluating complicated limits. We'll see examples of this shortly.

We'll just start by recalling, from (12), that if, for some natural number n, the function $f(x)$ has $n+1$ derivatives near the point x_0, then

$$f(x) = P_n(x) + E_n(x)$$

where

$$P_n(x) = f(x_0) + f'(x_0)\,(x-x_0) + \cdots + \tfrac{1}{n!} f^{(n)}(x_0)\,(x-x_0)^n$$

is the Taylor polynomial of degree n for the function $f(x)$ and expansion point x_0 and

$$E_n(x) = f(x) - P_n(x) = \tfrac{1}{(n+1)!} f^{(n+1)}(c)\,(x-x_0)^{n+1}$$

is the error introduced when we approximate $f(x)$ by the polynomial $P_n(x)$. Here c is some unknown number between x_0 and x. As c is not known, we do not know exactly what the error $E_n(x)$ is. But that is usually not a problem. In taking the limit $x \to x_0$, we are only interested in x's that are very close to x_0, and when x is very close x_0, c must also be very close to x_0. As long as $f^{(n+1)}(x)$ is continuous at x_0, $f^{(n+1)}(c)$ must approach $f^{(n)}(x_0)$ as $x \to x_0$. In particular there must be constants $M, D > 0$ such that $\big|f^{(n+1)}(c)\big| \le M$ for all c's within a distance D of x_0. If so, there is another constant C (namely $\frac{M}{(n+1)!}$) such that

$$\big|E_n(x)\big| \le C|x-x_0|^{n+1} \qquad \text{whenever } |x-x_0| \le D$$

There is some notation for this behaviour.

11. The Big O Notation

Definition 14 (Big O).

Let x_0 and m be real numbers. We say "$F(x)$ is of order $|x-x_0|^m$ near x_0" and we write $F(x) = O\big(|x-x_0|^m\big)$ if there exist constants $C, D > 0$ such that

$$\big|F(x)\big| \le C|x-x_0|^m \qquad \text{whenever } |x-x_0| \le D \tag{23}$$

Whenever $O\big(|x-x_0|^m\big)$ appears in an algebraic expression, it just stands for some (unknown) function $F(x)$ that obeys (23). This is called "big O" notation.

Example 15

Let $f(x) = \sin x$ and $x_0 = 0$. Then

$$\begin{array}{llllll} f(x) = \sin x & f'(x) = \cos x & f''(x) = -\sin x & f^{(3)}(x) = -\cos x & f^{(4)}(x) = \sin x & \cdots \\ f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f^{(3)}(0) = -1 & f^{(4)}(0) = 0 & \cdots \end{array}$$

and the pattern repeats. Thus $\left|f^{(n+1)}(c)\right| \le 1$ for all real numbers c and all natural numbers n. So the Taylor polynomial of, for example, degree 3 and its error term are

$$\begin{aligned}\sin x &= x - \tfrac{1}{3!}x^3 + \tfrac{\cos c}{5!}x^5 \\ &= x - \tfrac{1}{3!}x^3 + O(|x|^5)\end{aligned}$$

under Definition 14, with $C = \frac{1}{5!}$ and any $D > 0$. Similarly, for any natural number n,

$$\begin{aligned}\sin x &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots + (-1)^n \tfrac{1}{(2n+1)!}x^{2n+1} + O\big(|x|^{2n+3}\big) \\ \cos x &= 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots + (-1)^n \tfrac{1}{(2n)!}x^{2n} + O\big(|x|^{2n+2}\big)\end{aligned}$$

Example 15

Example 16

Let n be any natural number. Since $\frac{d^m}{dx^m}e^x = e^x$ for every integer $m \ge 0$,

$$e^x = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots + \tfrac{x^n}{n!} + \tfrac{e^c}{(n+1)!}x^{n+1}$$

for some c between 0 and x. If, for example, $|x| \le 1$, then $|e^c| \le e$, so that the error term

$$\left|\tfrac{e^c}{(n+1)!}x^{n+1}\right| \le C|x|^{n+1} \qquad \text{with } C = \tfrac{e}{(n+1)!} \qquad \text{whenever } |x| \le 1$$

So, under Definition 14, with $C = \frac{e}{(n+1)!}$ and $D = 1$,

$$e^x = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots + \tfrac{x^n}{n!} + O\big(|x|^{n+1}\big)$$

Example 16

Example 17

Let $f(x) = \ln(1+x)$ and $x_0 = 0$. Then

$$\begin{array}{lllll} f'(x) = \frac{1}{1+x} & f''(x) = -\frac{1}{(1+x)^2} & f^{(3)}(x) = \frac{2}{(1+x)^3} & f^{(4)}(x) = -\frac{2\times 3}{(1+x)^4} & f^{(5)}(x) = \frac{2\times 3\times 4}{(1+x)^5} \\ f'(0) = 1 & f''(0) = -1 & f^{(3)}(0) = 2 & f^{(4)}(0) = -3! & f^{(5)}(0) = 4! \end{array}$$

We can see a pattern for $f^{(n)}(x)$ forming here — $f^{(n)}(x)$ is a sign times a ratio with

- the sign being $+$ when n is odd and being $-$ when n is even. So the sign is $(-1)^{n-1}$.
- The denominator is a power of $(1+x)$. The power is just n.
- The numerator is a product $2 \times 3 \times 4 \times \cdots$. The last integer in the power is $n-1$, at least for $n \ge 2$. So the product, for $n \ge 2$, is $2 \times 3 \times 4 \times \cdots \times (n-1)$. The notation $n!$, read "n factorial", means $1 \times 2 \times 3 \times \cdots \times n$, so the numerator is $(n-1)!$, at least for $n \ge 2$. By convention, $0! = 1$, so the numerator is $(n-1)!$ for $n = 1$ too.

Thus, for any natural number n,

$$f^{(n)}(x) = (-1)^{n-1}\tfrac{(n-1)!}{(1+x)^n} \qquad \tfrac{1}{n!}f^{(n)}(0)\,x^n = (-1)^{n-1}\tfrac{(n-1)!}{n!}x^n = (-1)^{n-1}\tfrac{x^n}{n}$$

so

$$\ln(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \cdots + (-1)^{n-1}\tfrac{x^n}{n} + E_n(x)$$

with

$$E_n(x) = \tfrac{1}{(n+1)!}f^{(n+1)}(c)\,(x-x_0)^{n+1} = (-1)^n\tfrac{1}{(n+1)(1+c)^{n+1}}x^{n+1}$$

If we choose, for example $D = \frac{1}{2}$, then for any x obeying $|x| \le \frac{1}{2}$, we have $|c| \le \frac{1}{2}$ and $|1+c| \ge \frac{1}{2}$ so that

$$|E_n(x)| \le \tfrac{1}{(n+1)(1/2)^{n+1}}|x|^{n+1} = O\big(|x|^{n+1}\big)$$

under Definition 14, with $C = \frac{2^{n+1}}{n+1}$ and $D = 1$. Thus we may write

$$\ln(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \cdots + (-1)^{n-1}\tfrac{x^n}{n} + O\big(|x|^{n+1}\big) \tag{24}$$

Example 17

Remark 18.

The big O notation has a few properties that are useful in computations and taking limits. All follow immediately from Definition 14.

1. If $p > 0$, then $\lim\limits_{x\to 0} O(|x|^p) = 0$.
2. For any real numbers p and q, $O(|x|^p)\;O(|x|^q) = O(|x|^{p+q})$.
   (This is just because $C|x|^p \times C'|x|^q = (CC')|x|^{p+q}$.)
   In particular, $ax^m\,O(|x|^p) = O(|x|^{p+m})$, for any constant a and any integer m.
3. For any real numbers p and q, $O(|x|^p) + O(|x|^q) = O(|x|^{\min\{p,q\}})$.
   (For example, if $p = 2$ and $q = 5$, then $C|x|^2 + C'|x|^5 = \big(C + C'|x|^3\big)|x|^2 \le (C + C')|x|^2$ whenever $|x| \le 1$.)
4. For any real numbers p and q with $p > q$, any function which is $O(|x|^p)$ is also $O(|x|^q)$ because $C|x|^p = C|x|^{p-q}|x|^q \le C|x|^q$ whenever $|x| \le 1$.

12. Evaluating Limits Using Taylor Expansion — Examples

Example 19

In this example we'll use the Taylor polynomial of Example 17 to evaluate $\lim_{x\to 0} \frac{\ln(1+x)}{x}$ and $\lim_{x\to 0}(1+x)^{a/x}$. The Taylor expansion (24) with $n = 1$ tells us that

$$\ln(1+x) = x + O(|x|^2)$$

That is, for small x, $\ln(1+x)$ is the same as x, up to an error that is bounded by some constant times x^2. So, dividing by x, $\frac{1}{x}\ln(1+x)$ is the same as 1, up to an error that is bounded by some constant times $|x|$. That is

$$\tfrac{1}{x}\ln(1+x) = 1 + O(|x|)$$

But any function that is bounded by some constant times $|x|$, for all x smaller than some constant $D > 0$, necessarily tends to 0 as $x \to 0$. Thus

$$\lim_{x\to 0} \tfrac{\ln(1+x)}{x} = \lim_{x\to 0} \tfrac{x+O(|x|^2)}{x} = \lim_{x\to 0} \big[1 + O(|x|)\big] = 1$$

and

$$\lim_{x\to 0}(1+x)^{a/x} = \lim_{x\to 0} e^{a/x\ \ln(1+x)} = \lim_{x\to 0} e^{a/x\,[x+O(|x|^2)]} = \lim_{x\to 0} e^{a+O(|x|)} = e^a$$

Here we have used that if $F(x) = O(|x|^2)$, that is if $|F(x)| \le C|x|^2$ for some constant C, then $\big|\frac{a}{x}F(x)\big| \le C'|x|$ for the new constant $C' = |a|C$, so that $F(x) = O(|x|)$. We have also used that the exponential is continuous — as x tends to zero, the exponent of $e^{a+O(|x|)}$ tends to a so that $e^{a+O(|x|)}$ tends to e^a.

Example 19

Example 20

In this example we'll evaluate the harder limit

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x\sin x}{[\ln(1+x)]^4}$$

The first thing to notice about this limit is that, as x tends to zero, the numerator, which is $\cos x - 1 + \frac{1}{2}x\sin x$, tends to $\cos 0 - 1 + \frac{1}{2}\cdot 0\cdot\sin 0 = 0$ and the denominator $[\ln(1+x)]^4$ tends to $[\ln(1+0)]^4 = 0$ too. So both the numerator and denominator tend to zero and we may not simply evaluate the limit of the ratio by taking the limits of the numerator and denominator and dividing. To find the limit, or show that it does not exist, we are going to have to exhibit a cancellation between the numerator and the denominator. To develop a strategy for evaluating this limit, let's do a "little scratch work", starting by taking a closer look at the denominator. By Example 17,

$$\ln(1+x) = x + O(x^2)$$

This tells us that $\ln(1+x)$ looks a lot like x for very small x. So the denominator $[x+O(x^2)]^4$ looks a lot like x^4 for very small x. Now, what about the numerator?

- If the numerator looks like some constant times x^p with $p > 4$, for very small x, then the ratio will look like the constant times $\frac{x^p}{x^4} = x^{p-4}$ and will tend to 0 as x tends to zero.
- If the numerator looks like some constant times x^p with $p < 4$, for very small x, then the ratio will look like the constant times $\frac{x^p}{x^4} = x^{p-4}$ and will tend to plus or minus ∞ (depending on the sign of the constant) as x tends to zero.
- If the numerator looks like Cx^4, for very small x, then the ratio will look like $\frac{Cx^4}{x^4} = C$ and will tend to C as x tends to zero.

The moral of the above "scratch work" is that we need to know the behaviour of the numerator, for small x, up to order x^4. Any contributions of order x^p with $p > 4$ may be put into error terms $O(|x|^p)$. Now we are ready to evaluate the limit. Using Examples 15 and 17,

$$
\begin{aligned}
\lim_{x\to 0}\frac{\cos x - 1 + \frac{1}{2}x\sin x}{[\ln(1+x)]^4} &= \lim_{x\to 0}\frac{\big[1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + O(x^6)\big] - 1 + \frac{1}{2}x\big[x - \frac{1}{3!}x^3 + O(|x|^5)\big]}{[x + O(x^2)]^4} \\
&= \lim_{x\to 0}\frac{(\frac{1}{4!} - \frac{1}{2\times 3!})x^4 + O(x^6) + \frac{x}{2}\,O(|x|^5)}{[x + O(x^2)]^4} \\
&= \lim_{x\to 0}\frac{(\frac{1}{4!} - \frac{1}{2\times 3!})x^4 + O(x^6) + O(x^6)}{[x + O(x^2)]^4} && \text{by Remark 18, part 2.} \\
&= \lim_{x\to 0}\frac{(\frac{1}{4!} - \frac{1}{2\times 3!})x^4 + O(x^6)}{[x + x\,O(|x|)]^4} && \text{by Remark 18, parts 2, 3.} \\
&= \lim_{x\to 0}\frac{(\frac{1}{4!} - \frac{1}{2\times 3!})x^4 + x^4 O(x^2)}{x^4[1 + O(|x|)]^4} && \text{by Remark 18, part 2.} \\
&= \lim_{x\to 0}\frac{(\frac{1}{4!} - \frac{1}{2\times 3!}) + O(x^2)}{[1 + O(|x|)]^4} \\
&= \frac{1}{4!} - \frac{1}{2\times 3!} && \text{by Remark 18 part 1.} \\
&= \frac{1}{3!}\Big(\frac{1}{4} - \frac{1}{2}\Big) = -\frac{1}{4!}
\end{aligned}
$$

Example 20