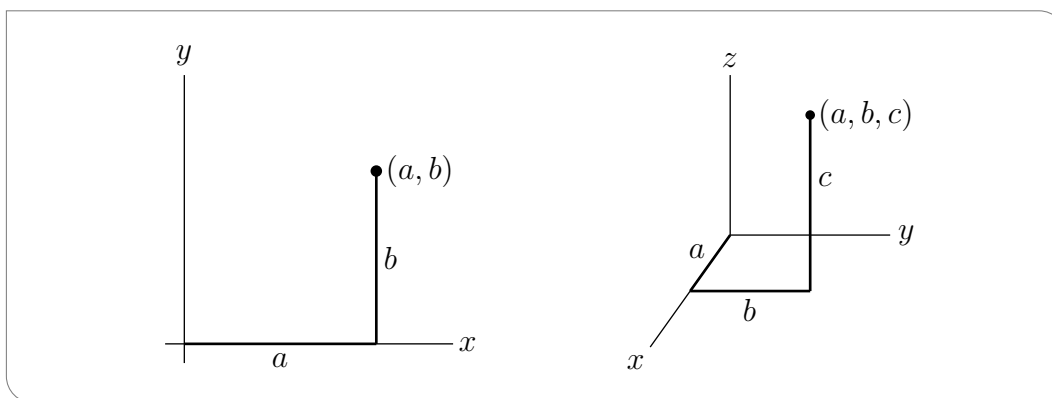


Vectors and Planes

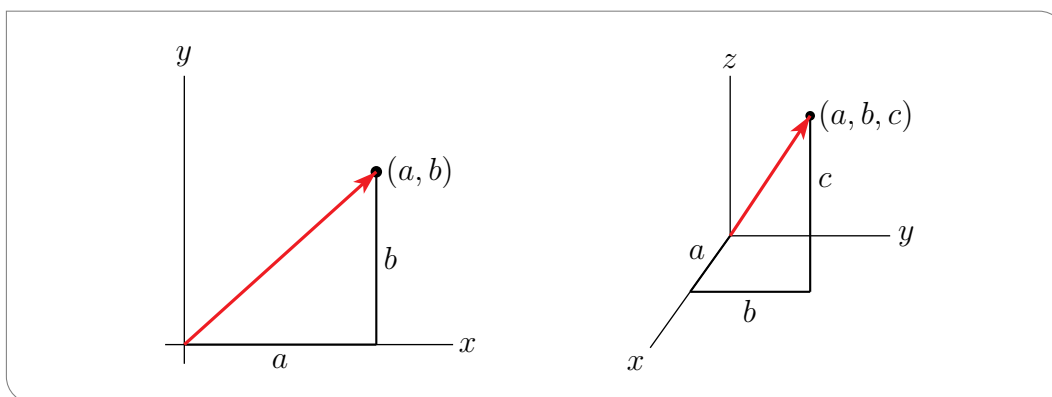
1. Points and Vectors

Each point in two dimensions may be labeled by two coordinates (a, b) which specify the position of the point in some units with respect to some axes as in the figure on the left below. If $a > 0$ then the point is to the right of the y -axis and if $a < 0$ then the point is to the left of the y -axis. The distance from (a, b) to the y -axis is $|a|$ and $|b|$ is the distance from (a, b) to the x -axis. In particular, the equation of the y -axis is $x = 0$ and the equation of the x -axis is $y = 0$. Similarly, each point in three dimensions may be labeled by three coordinates (a, b, c) . To get from the origin, $(0, 0, 0)$, to (a, b, c) you walk along the x -axis a units (think of the x -axis as pointing horizontally straight out of the page), then parallel to the y -axis b units and then vertically parallel to the z -axis c units. The distance from (a, b, c) to the xy -plane is $|c|$ and the equation of the xy -plane is $z = 0$. The set of all points in two dimensions



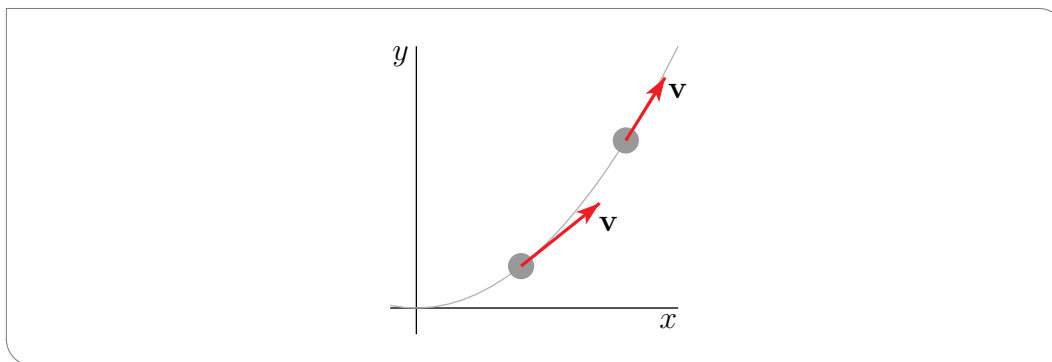
is denoted \mathbb{R}^2 and the set of all points in three dimensions is denoted \mathbb{R}^3 . The distance from the point (x, y, z) to the point (x', y', z') is $\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$ so that the equation of the sphere centered on $(1, 2, 3)$ with radius 4 is $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 16$.

A *vector* is a quantity which has both a direction and a magnitude, like a velocity. If you are moving, the magnitude (length) of your velocity vector is your speed (distance travelled per unit time) and the direction of your velocity vector is your direction of motion. To specify a vector in three dimensions you have to give three components, just as for a point. To draw the vector with components a, b, c you can draw an arrow from the point $(0, 0, 0)$ to the point (a, b, c) . Similarly, to specify a vector in two dimensions you have to give two

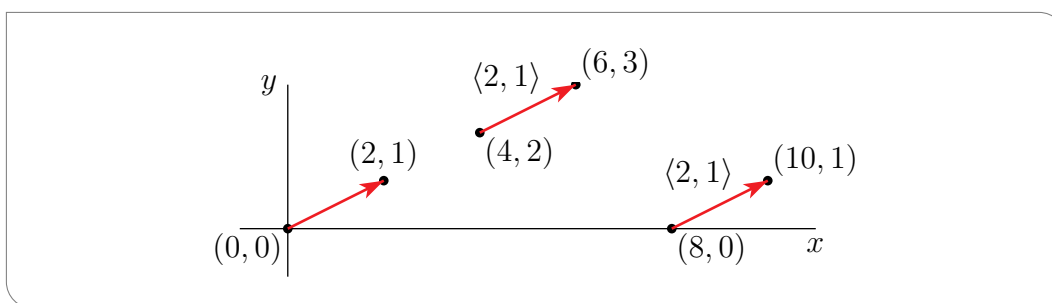


components and to draw the vector with components a , b you can draw an arrow from the point $(0, 0)$ to the point (a, b) .

There are many situations in which it is preferable to draw a vector with its tail at some point other than the origin. For example, it is natural to draw the velocity vector of a moving particle with the tail of the velocity vector at the position of the particle, whether or not the particle is at the origin. The sketch below shows a moving particle and its velocity vector at two different times.



To distinguish between the components of a vector and the coordinates of the point at its head, when its tail is at some point other than the origin, we shall use angle brackets rather than round brackets around the components of a vector. For example, here is the two-dimensional vector $\langle 2, 1 \rangle$ drawn in three different positions. In each case, when the tail is at the point (u, v) the head is at $(2 + u, 1 + v)$. We warn you that, out in the real world, no one uses notation that distinguishes between components of a vector and the coordinates of its head — usually round brackets are used for both. It is up to you to keep straight which is being referred to.

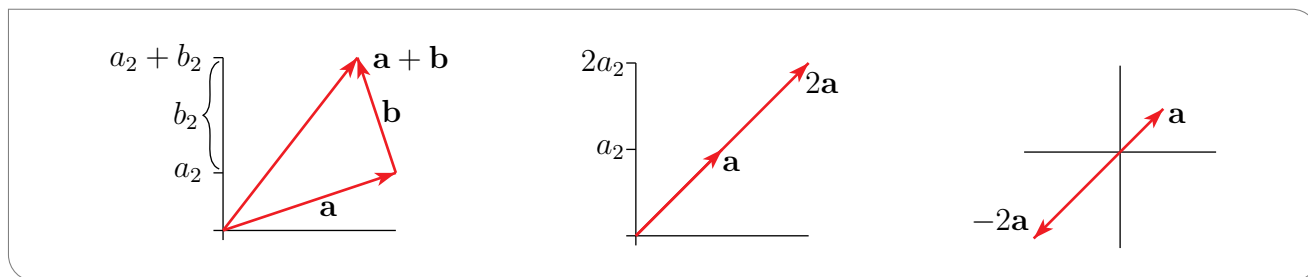


2. Addition of Vectors and Multiplication of a Vector by a Number

These two operations have the obvious definitions

$$\begin{aligned} \mathbf{a} = \langle a_1, a_2 \rangle, \mathbf{b} = \langle b_1, b_2 \rangle &\implies \mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \\ \mathbf{a} = \langle a_1, a_2 \rangle, s \text{ a number} &\implies s\mathbf{a} = \langle sa_1, sa_2 \rangle \end{aligned}$$

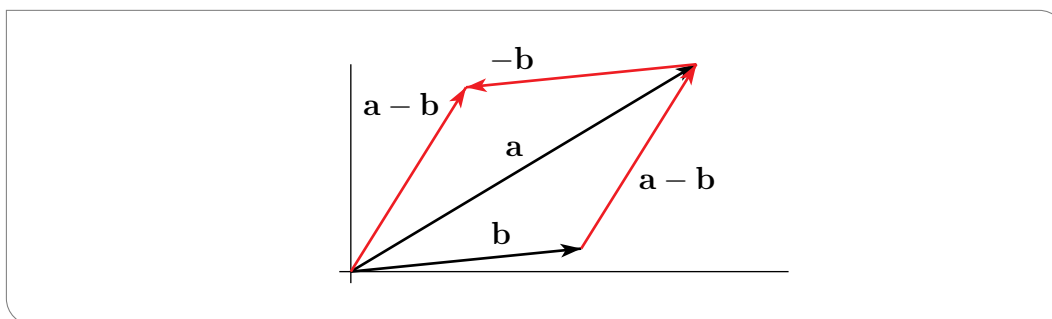
and similarly in three dimensions. Pictorially, you add \mathbf{b} to \mathbf{a} by drawing \mathbf{b} with its tail at the head of \mathbf{a} and then drawing a vector from the tail of \mathbf{a} to the head of \mathbf{b} , as in the figure on the left below. To draw $s\mathbf{a}$, you just change \mathbf{a} 's length by the (signed) factor s , as in the other two figures below.



These operations rarely cause any problems, because they inherit from the real numbers the properties of addition and multiplication that you are used to. Using $\mathbf{0}$ to denote the vector all of whose components are zero and $-\mathbf{a}$ to denote the vector each of whose components is the negative of the corresponding component of \mathbf{a} (so that $-\langle a_1, a_2 \rangle = \langle -a_1, -a_2 \rangle$)

- | | |
|---|--|
| 1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ | 2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ |
| 3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$ | 4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ |
| 5. $s(\mathbf{a} + \mathbf{b}) = s\mathbf{a} + s\mathbf{b}$ | 6. $(s + t)\mathbf{a} = s\mathbf{a} + t\mathbf{a}$ |
| 7. $(st)\mathbf{a} = s(t\mathbf{a})$ | 8. $1\mathbf{a} = \mathbf{a}$ |

To subtract \mathbf{b} from \mathbf{a} pictorially, you may add $-\mathbf{b}$ (which is drawn by reversing the direction of \mathbf{b}) to \mathbf{a} . Alternatively, if you draw \mathbf{a} and \mathbf{b} with their tails at a common point, then $\mathbf{a} - \mathbf{b}$ is the vector from the head of \mathbf{b} to the head of \mathbf{a} . That is, $\mathbf{a} - \mathbf{b}$ is the vector you must add to \mathbf{b} in order to get \mathbf{a} .



Example 1

For example, if

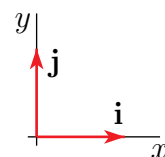
$$\mathbf{a} = \langle 1, 2, 3 \rangle \quad \mathbf{b} = \langle 3, 2, 1 \rangle \quad \mathbf{c} = \langle 1, 0, 1 \rangle$$

then

$$\begin{aligned} 2\mathbf{a} - \mathbf{b} + 3\mathbf{c} &= 2\langle 1, 2, 3 \rangle - \langle 3, 2, 1 \rangle + 3\langle 1, 0, 1 \rangle \\ &= \langle 2, 4, 6 \rangle + \langle -3, -2, -1 \rangle + \langle 3, 0, 3 \rangle \\ &= \langle 2 - 3 + 3, 4 - 2 + 0, 6 - 1 + 3 \rangle \\ &= \langle 2, 2, 8 \rangle \end{aligned}$$

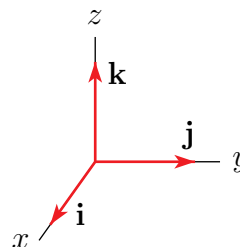
Example 1

There are some vectors that occur sufficiently commonly that they are given special names. One is the vector $\mathbf{0}$. Some others are the “standard basis vectors in two dimensions”

$$\mathbf{i} = \langle 1, 0 \rangle \quad \mathbf{j} = \langle 0, 1 \rangle$$


and the “standard basis vectors in three dimensions”

$$\mathbf{i} = \langle 1, 0, 0 \rangle \quad \mathbf{j} = \langle 0, 1, 0 \rangle \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$



Some people rename \mathbf{i} , \mathbf{j} and \mathbf{k} to \mathbf{e}_1 , \mathbf{e}_2 and \mathbf{e}_3 respectively. Using the above properties we have, for all vectors,

$$\langle a_1, a_2 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j} \quad \langle a_1, a_2, a_3 \rangle = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

A sum of numbers times vectors, like $a_1 \mathbf{i} + a_2 \mathbf{j}$ is called a linear combination of the vectors. Thus all vectors can be expressed as linear combinations of the standard basis vectors. The standard basis vectors are unit vectors, meaning that they are of length one, where the length of a vector is defined by

$$\begin{aligned} \mathbf{a} = \langle a_1, a_2 \rangle &\implies |\mathbf{a}| = \sqrt{a_1^2 + a_2^2} \\ \mathbf{a} = \langle a_1, a_2, a_3 \rangle &\implies |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \end{aligned}$$

3. The Dot Product

There is more than one type of product used with vectors, One is multiplication by a scalar, which we have already seen. A second is the *dot product*, which is defined by

$$\begin{aligned} \mathbf{a} = \langle a_1, a_2 \rangle, \quad \mathbf{b} = \langle b_1, b_2 \rangle &\implies \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 \\ \mathbf{a} = \langle a_1, a_2, a_3 \rangle, \quad \mathbf{b} = \langle b_1, b_2, b_3 \rangle &\implies \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned}$$

in two and three dimensions respectively. The properties of the dot product are as follows:

0. \mathbf{a}, \mathbf{b} are vectors and $\mathbf{a} \cdot \mathbf{b}$ is a number
1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$, $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$
4. $(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$
6. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$ where θ is the angle between \mathbf{a} and \mathbf{b}
7. $\mathbf{a} \cdot \mathbf{b} = 0 \iff \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a} \perp \mathbf{b}$

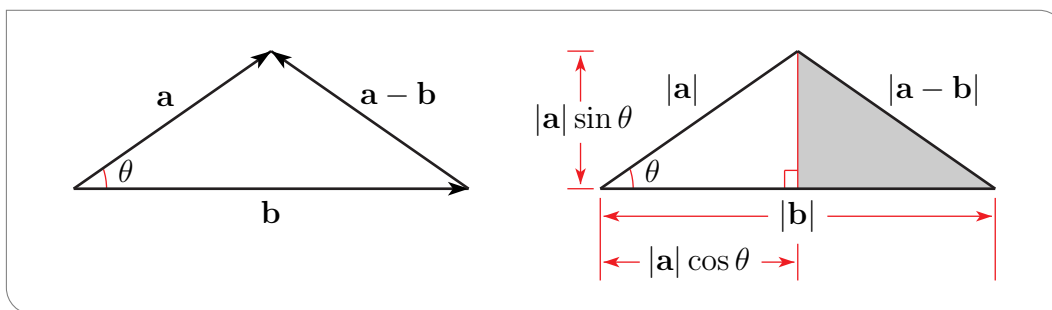
Proof. Properties 0 through 5 are almost immediate consequences of the definition. For example, for property 3 in dimension 2,

$$\begin{aligned}\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2 \rangle \cdot \langle b_1 + c_1, b_2 + c_2 \rangle \\ &= a_1(b_1 + c_1) + a_2(b_2 + c_2) = a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 \\ \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} &= \langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle + \langle a_1, a_2 \rangle \cdot \langle c_1, c_2 \rangle \\ &= a_1b_1 + a_2b_2 + a_1c_1 + a_2c_2\end{aligned}$$

Property 6 is sufficiently important that it is often used as the definition of dot product. It is not at all an obvious consequence of the definition. To verify it, we just write $|\mathbf{a} - \mathbf{b}|^2$ in two different ways. The first expresses $|\mathbf{a} - \mathbf{b}|^2$ in terms of $\mathbf{a} \cdot \mathbf{b}$. It is

$$\begin{aligned}|\mathbf{a} - \mathbf{b}|^2 &\stackrel{1}{=} (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &\stackrel{3}{=} \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &\stackrel{1,2}{=} |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b}\end{aligned}$$

Here, $\stackrel{1}{=}$, for example, means that the equality is a consequence of property 1. The second way we write $|\mathbf{a} - \mathbf{b}|^2$ involves $\cos \theta$ and follows from the cosine law. Just in case you don't remember the cosine law, we prove it along the way. To derive the cosine law, apply Pythagoras to the shaded triangle in the right hand figure of



That triangle is a right triangle whose hypotenuse has length $|\mathbf{a} - \mathbf{b}|$ and whose other two sides have lengths $(|\mathbf{b}| - |\mathbf{a}| \cos \theta)$ and $|\mathbf{a}| \sin \theta$. So Pythagoras gives

$$\begin{aligned}|\mathbf{a} - \mathbf{b}|^2 &= (|\mathbf{b}| - |\mathbf{a}| \cos \theta)^2 + (|\mathbf{a}| \sin \theta)^2 \\ &= |\mathbf{b}|^2 - 2|\mathbf{a}| |\mathbf{b}| \cos \theta + |\mathbf{a}|^2 \cos^2 \theta + |\mathbf{a}|^2 \sin^2 \theta \\ &= |\mathbf{b}|^2 - 2|\mathbf{a}| |\mathbf{b}| \cos \theta + |\mathbf{a}|^2\end{aligned}$$

(That's the cosine law. Observe that, when $\theta = \frac{\pi}{2}$, this reduces to, (surprise!) Pythagoras' theorem.) Setting the two expressions for $|\mathbf{a} - \mathbf{b}|^2$ equal to each other,

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a} \cdot \mathbf{b} = |\mathbf{b}|^2 - 2|\mathbf{a}| |\mathbf{b}| \cos \theta + |\mathbf{a}|^2$$

cancelling the $|\mathbf{a}|^2$ and $|\mathbf{b}|^2$ common to both sides

$$-2\mathbf{a} \cdot \mathbf{b} = -2|\mathbf{a}| |\mathbf{b}| \cos \theta$$

and dividing by -2 gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

which is property 6.

Property 7 follows directly from property 6: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$ is zero if and only if at least one of the three factors $|\mathbf{a}|$, $|\mathbf{b}|$, $\cos \theta$ is zero. The first factor is zero if and only if $\mathbf{a} = \mathbf{0}$. The second factor is zero if and only if $\mathbf{b} = \mathbf{0}$. The third factor is zero if and only if $\theta = \pm\frac{\pi}{2} + 2k\pi$, for some integer k , which in turn is true if and only if \mathbf{a} and \mathbf{b} are mutually perpendicular. \square

Because of Property 7, the dot product can be used to test whether or not two vectors are perpendicular to each other. That is, whether or not the angle between the two vectors is 90° . Another name for “perpendicular” is “orthogonal”. Testing for orthogonality is one of the main uses of the dot product.

Example 2

Consider the three vectors

$$\mathbf{a} = \langle 1, 1, 0 \rangle \quad \mathbf{b} = \langle 1, 0, 1 \rangle \quad \mathbf{c} = \langle 1, -1, -1 \rangle$$

The dot products

$$\mathbf{a} \cdot \mathbf{b} = \langle 1, 1, 0 \rangle \cdot \langle 1, 0, 1 \rangle = 1 \times 1 + 1 \times 0 + 0 \times 1 = 1$$

$$\mathbf{a} \cdot \mathbf{c} = \langle 1, 1, 0 \rangle \cdot \langle 1, -1, -1 \rangle = 1 \times 1 + 1 \times (-1) + 0 \times (-1) = 0$$

$$\mathbf{b} \cdot \mathbf{c} = \langle 1, 0, 1 \rangle \cdot \langle 1, -1, -1 \rangle = 1 \times 1 + 0 \times (-1) + 1 \times (-1) = 0$$

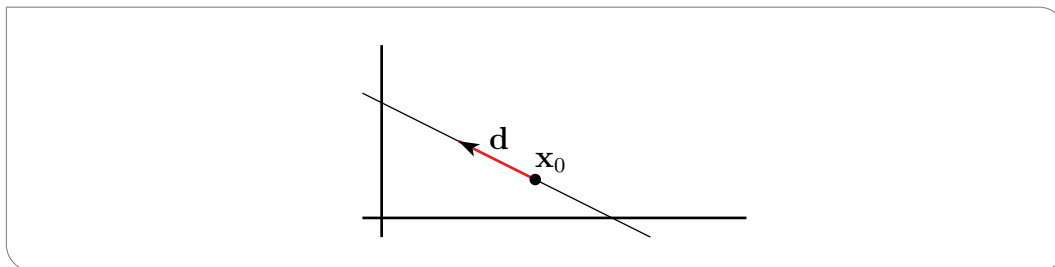
tell us that \mathbf{c} is perpendicular to both \mathbf{a} and \mathbf{b} . Since both $|\mathbf{a}| = |\mathbf{b}| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$ the first dot product tells us that the angle, θ , between \mathbf{a} and \mathbf{b} obeys

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{1}{2} \implies \theta = \frac{\pi}{3}$$

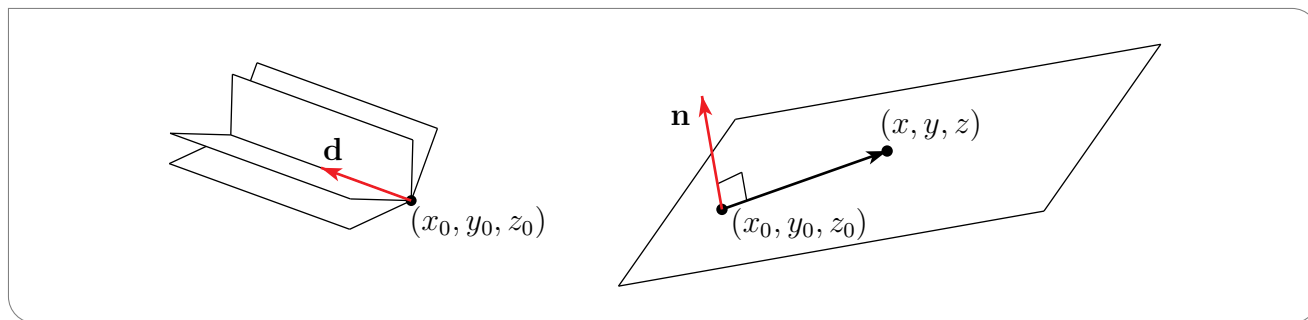
Example 2

4. Equations of Planes in Three Dimensions

A line in two or three dimensions can be specified by giving one point \mathbf{x}_0 on the line and one vector \mathbf{d} whose direction is parallel to the line. But specifying one point (x_0, y_0, z_0) on a



plane and a vector \mathbf{d} parallel to the plane does not uniquely determine the plane, because it is free to rotate about \mathbf{d} . On the other hand, giving one point on the plane and one vector



$\mathbf{n} = \langle n_x, n_y, n_z \rangle$ whose direction is perpendicular to that of the plane does uniquely determine the plane. If (x, y, z) is any point on the line then the vector $\langle x - x_0, y - y_0, z - z_0 \rangle$, whose tail is at (x_0, y_0, z_0) and whose head is at (x, y, z) , lies entirely inside the plane and so must be perpendicular to \mathbf{n} . That is,

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Writing out in components

$$n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) = 0 \quad \text{or} \quad n_x x + n_y y + n_z z = n_x x_0 + n_y y_0 + n_z z_0$$

Note that the coefficients n_x, n_y, n_z of x, y and z in the equation of the plane are the components of a vector $\langle n_x, n_y, n_z \rangle$ perpendicular to the plane. The vector \mathbf{n} is often called a normal vector for the plane.

Example 3

We have just seen that if we write the equation of a plane in the standard form $ax + by + cz = d$, then it is easy to read off a normal vector for the plane. It is just $\langle a, b, c \rangle$. So for example the planes

$$P : x + 2y + 3z = 4 \quad \text{and} \quad P' : 3x + 6y + 9z = 7$$

have normal vectors $\mathbf{n} = \langle 1, 2, 3 \rangle$ and $\mathbf{n}' = \langle 3, 6, 9 \rangle$, respectively. Since $\mathbf{n}' = 3\mathbf{n}$, the two normal vectors \mathbf{n} and \mathbf{n}' are parallel to each other. This tells us that the planes P and P' are parallel to each other.

When the normal vectors of two planes are perpendicular to each other, we say that the planes are perpendicular to each other. For example the planes

$$P : x + 2y + 3z = 4 \quad \text{and} \quad P'' : 2x - y = 7$$

have normal vectors $\mathbf{n} = \langle 1, 2, 3 \rangle$ and $\mathbf{n}'' = \langle 2, -1, 0 \rangle$, respectively. Since

$$\mathbf{n} \cdot \mathbf{n}'' = 1 \times 2 + 2 \times (-1) + 3 \times 0 = 0$$

the normal vectors \mathbf{n} and \mathbf{n}'' are mutually perpendicular, so the corresponding planes P and P'' are perpendicular to each other.

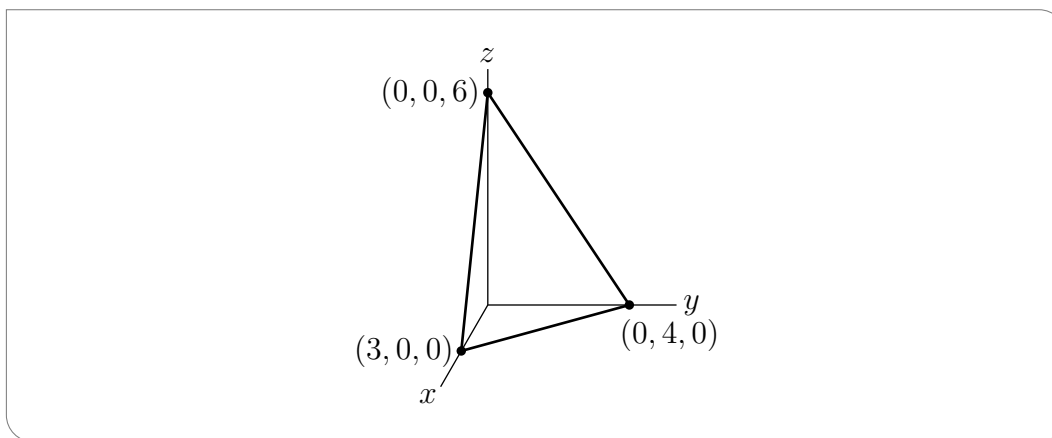
Example 3

Example 4

In this example, we'll sketch the plane

$$P : 4x + 3y + 2z = 12$$

A good way to prepare for sketching a plane is to find the intersection points of the plane with the x -, y - and z -axes. For example, any point on the x axis must be of the form $(x, 0, 0)$. For $(x, 0, 0)$ to also be on P we need $x = 12/4 = 3$. So P intersects the x -axis at $(3, 0, 0)$. Similarly, P intersects the y -axis at $(0, 4, 0)$ and the z -axis at $(0, 0, 6)$. Now plot the points $(3, 0, 0)$, $(0, 4, 0)$ and $(0, 0, 6)$. P is the plane through these three points. Often a



visually effective way to sketch a surface in three dimensions is to

- only sketch the part of the surface in the first octant. That is, the part with $x \geq 0$, $y \geq 0$ and $z \geq 0$.
- To do so, sketch the curve of intersection of the surface with the part of the xy -plane in the first octant and,
- similarly, sketch the curve of intersection of the surface with the part of the xz -plane in the first octant and the curve of intersection of the surface with the part of the yz -plane in the first octant.

That's what we'll do. The intersection of the plane P with the xy -plane is the straight line through the two points $(3, 0, 0)$ and $(0, 4, 0)$. So the part of that intersection in the first octant is the line segment from $(3, 0, 0)$ to $(0, 4, 0)$. Similarly the part of the intersection of P with the xz -plane that is in the first octant is the line segment from $(3, 0, 0)$ to $(0, 0, 6)$ and the part of the intersection of P with the yz -plane that is in the first octant is the line segment from $(0, 4, 0)$ to $(0, 0, 6)$. So we just have to sketch the three line segments joining three axis intercepts $(3, 0, 0)$, $(0, 4, 0)$ and $(0, 0, 6)$. That's it.

Example 4

Example 5

In this example we'll compute the distance between the planes

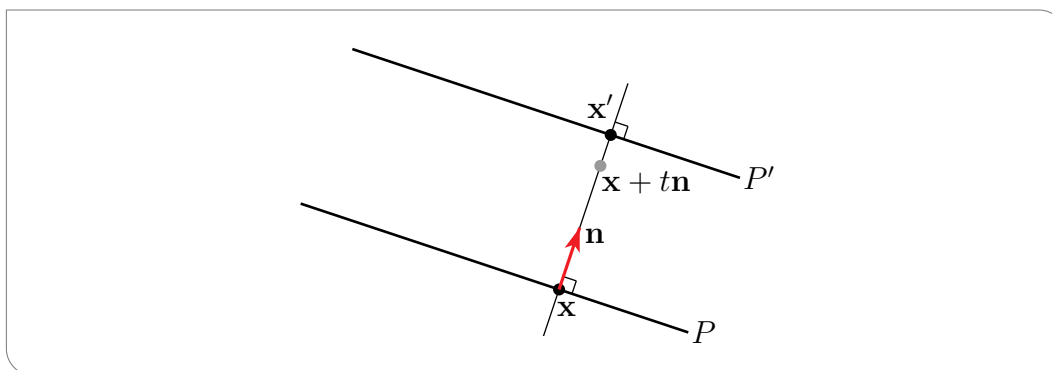
$$P : x + 2y + 2z = 1 \quad \text{and} \quad P' : 2x + 4y + 4z = 11$$

By the “distance between the planes P and P' ” we mean the shortest distance between any pair of points \mathbf{x} and \mathbf{x}' with \mathbf{x} in P and \mathbf{x}' in P' . First observe that the normal vectors

$$\mathbf{n} = \langle 1, 2, 2 \rangle \quad \text{and} \quad \mathbf{n}' = \langle 2, 4, 4 \rangle = 2\mathbf{n}$$

are parallel to each other. So the planes P and P' are parallel to each other. If they had not been parallel, they would have intersected and the distance between them would have been zero. Our strategy for finding the distance will be to

- first find a point \mathbf{x} on P and then
- start walking away from P in the direction of the normal vector \mathbf{n} and
- keep walking until we hit P' . Call the point on P' that we hit \mathbf{x}' . Then the desired distance is the distance between \mathbf{x} and \mathbf{x}' . From the figure below it does indeed look like distance between \mathbf{x} and \mathbf{x}' is the shortest distance between any pair of points with one point on P and one point on P' . This is in fact true, though we won't prove it.



Now let's find a point on P . The plane P is given by a single equation, namely $x + 2y + 2z = 1$, in the three unknowns, x, y, z . The easiest way to find one solution to this equation is to assign two of the unknowns the value zero and then solve for the third unknown. For example, if we set $y = z = 0$, then the equation reduces to $x = 1$. So we may take $\mathbf{x} = \langle 1, 0, 0 \rangle$.

Now imagine that we start walking, and that we start at time $t = 0$ at \mathbf{x} and walk in the direction \mathbf{n} . Then at time t we might be at

$$\mathbf{x} + t\mathbf{n} = \langle 1, 0, 0 \rangle + t \langle 1, 2, 2 \rangle = \langle 1 + t, 2t, 2t \rangle$$

We hit the second plane P' at exactly the time t for which $\langle 1 + t, 2t, 2t \rangle$ satisfies the equation for P' , which is $2x + 4y + 4z = 11$. So we are on P' at the unique time t obeying

$$2(1 + t) + 4(2t) + 4(2t) = 11 \iff 18t = 9 \iff t = \frac{1}{2}$$

So the point on P' which is closest to \mathbf{x} is

$$\mathbf{x}' = [\mathbf{x} + t\mathbf{n}]_{t=1/2} = \langle 1 + t, 2t, 2t \rangle \Big|_{t=1/2} = \langle 3/2, 1, 1 \rangle$$

and the distance from P to P' is the distance from \mathbf{x} to \mathbf{x}' which is

$$\sqrt{(1 - 1/2)^2 + (0 - 1)^2 + (0 - 1)^2} = \sqrt{9/4} = 3/2$$

