Warnings

These notes highlight number of common, but serious, first year calculus errors.

Warning 1. The formula

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

is valid only under the hypothesis $\lim_{x\to a} g(x) \neq 0$.

Discussion. In some cases, when $\lim_{x\to a} g(x) = 0$, the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ does not exist at all. But in other cases it does exist and when it does, it can take any value at all. Here are two simple examples. In both examples $g(x) = x - a$, which has the property that $\lim_{x\to a} g(x) = 0$. In the first example, $\lim_{x\to a}\frac{f(x)}{g(x)}$ does not exist at all (because $\frac{f(x)}{g(x)}$ is huge and positive for $x - a$ small and positive and huge and negative for $x - a$ small and negative). In the second it does exist and takes the value 123.45.

$f(x) = 1$ $\qquad g(x) = x - a \quad \Rightarrow \lim_{x\to a}\frac{f(x)}{g(x)}$ does not exist

$f(x) = 123.45(x-a)$ $\qquad g(x) = x - a \quad \Rightarrow \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{123.45(x-a)}{x-a} = \lim_{x\to a} 123.45 = 123.45$

Warning 2. Let a and b be two real numbers with $a < b$. Then $ac < bc$ **only** if $c > 0$. When $c < 0$ we have $ac > bc$ and when $c = 0$, we have $ac = bc$.

Discussion. Here is an example with $a = 1$ and $b = 2$.

- For $c = 3$, $ac = (1)(3) = 3 < bc = (2)(3) = 6$.
- For $c = -3$, $ac = (1)(-3) = -3 > bc = (2)(-3) = -6$.
- For $c = 0$, $ac = (1)(0) = 0 = bc = (2)(0) = 0$.

A related inequality is that $a^2 < b^2$ **if** $0 < a < b$. If $a < b < 0$, then $a^2 > b^2$.

Warning 3. L'Hôpital's rule,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

is not valid if the limit $\lim_{x\to a} g(x)$ exists and is nonzero.

Discussion. Here is an example with $a = 0$. If $f(x) = 3x$ and $g(x) = 4 + 5x$, then

$$\begin{aligned}
\lim_{x\to 0} \tfrac{f(x)}{g(x)} &= \lim_{x\to 0} \tfrac{3x}{4+5x} &&= \tfrac{3\times 0}{4+5\times 0} = 0 \\
\lim_{x\to 0} \tfrac{f'(x)}{g'(x)} &= \lim_{x\to 0} \tfrac{3}{5} &&= \tfrac{3}{5}
\end{aligned}$$

Warning 4. $\frac{d}{dx}[f(a)]$ is **not** the same as $f'(a)$.

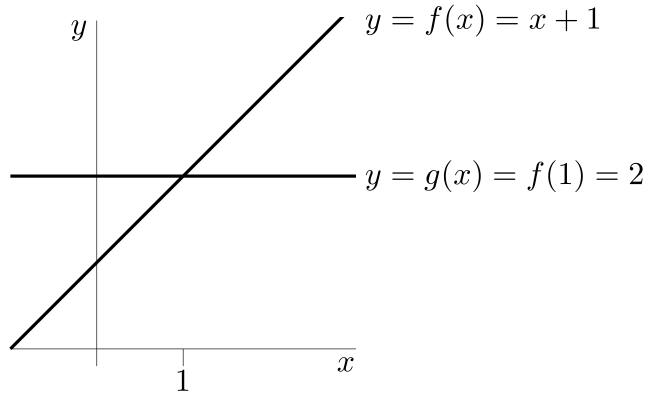
Discussion. The expression $\frac{d}{dx}[f(a)]$ is, by definition, the derivative of the constant function $g(x) = f(a)$. In other words, $\frac{d}{dx}[f(a)]$ means

- first evaluate $f(x)$ at $x = a$, to get a constant, $f(a)$.
- Then differentiate the constant to get zero.

On the other hand, $f'(a)$ means

- first find the derivative $f'(x)$.
- Then evaluate the derivative at $x = a$. Typically, this is not zero.

Here is an example. Let $f(x) = x + 1$ and $a = 1$. Then $f'(x) = 1$ for all x. In particular, $f'(1) = 1$. But if we first evaluate $f(x)$ at $x = 1$, we get $f(1) = 2$. The derivative of the constant function $g(x) = 2$ is $g'(x) = 0$.



The two functions $f(x)$ and $f(1)$ (which I am also calling $g(x)$ here) both take the value 1 when $x = 1$, but they have different slopes and hence have different derivatives: $f'(1) = 1$ but $\frac{d}{dx}[f(1)] = \frac{d}{dx}1 = 0$.

Warning 5. The usual formula for the derivative of $\sin x$, namely $\frac{d}{dx}\sin x = \cos x$, is valid only if x is measured in radians.

Discussion. Any derivative measures the instantaneous rate of change of one quantity per unit change of a second quantity. If you change the units of either quantity, the numerical value of the derivative changes. For example, consider the average rate of change of $\sin x$ as x moves from 0 to a right angle. If x is measured in radians,

$$\text{average rate of change} = \frac{\text{change in } \sin x}{\text{change in } x} = \frac{1-0}{\frac{\pi}{2}-0} = \frac{2}{\pi}$$

because $\sin x$ changes from 0 to 1 as x changes from 0 radians to $\frac{\pi}{2}$ radians. But, if x is measured in degrees,

$$\text{average rate of change} = \frac{\text{change in } \sin x}{\text{change in } x} = \frac{1-0}{90-0} = \frac{1}{90}$$

The two answers are different even though x moved from 0 to a right angle in both cases.

Here is a sample problem in which one gets the wrong answer by expressing an angle in the wrong units. Suppose that you wish to compute, approximately, the sin of two degrees, but that you can't just use your calculator. (Pretend that you are the person designing the software used by the calculator.) Recall that

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

gives an approximate value for $f(x)$ for all x close to x_0. Apply this approximation to $f(x) = \sin(x)$ with $x_0 = 0$. Then

$$f(x) = \sin x \qquad f(x_0) = f(0) = 0$$
$$f'(x) = \cos x \qquad f'(x_0) = f'(0) = 1$$

so that $\sin x = f(x) \approx f(x_0) + f'(x_0)(x - x_0) = 0 + 1(x - 0) = x$. If we apply this with $x = 2$ we get

$$\sin 2 \approx 2$$

which is both wrong and ridiculous, as $\sin x$ is never bigger than one in magnitude. The problem arose because we used $\frac{d}{dx}\sin x = \cos x$ in the computation

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = 0 + 1(x - 0) = x$$

thereby implicitly assuming that x is in radians. We should have first rewritten $2^\circ = 2\frac{\pi}{180} = 0.0349066$ radians and then applied

$$\sin(2^\circ) = \sin(0.0349066\,\text{rad}) \approx [f(x_0)+f'(x_0)(x-x_0)]_{x=0.0349066} = x\big|_{x=0.0349066} = 0.0349066$$

For comparison purposes, the exact value of sin of 2 degrees is 0.0348995 to 7 decimal places. Our rather simple minded approximation scheme gave an error of $0.0349066 - 0.0348995 = 0.0000071$, which is a relative error of $100\frac{0.0000071}{0.0348995}\% = 0.021\%$.

Warning 6. $\frac{d}{dx}a^x$ is not the same as $\frac{d}{dx}x^a$.

Discussion. $\frac{d}{dx}f(x)$ is the rate of change of $f(x)$ per unit change in x. In $\frac{d}{dx}a^x$, the exponent changes as x changes, **with the base, a, remaining fixed**. In $\frac{d}{dx}x^a$, the base changes as x changes, **with the exponent, a, remaining fixed**. As a concrete example, suppose $a = 4$. As x changes from 4 to 5,

- $a^x = 4^x$ changes from $4^4 = 256$ to $4^5 = 1024$
  - $\Rightarrow$ the average rate of change of $a^x = 4^x$ from $x = 4$ to $x = 5$ is $\frac{1024-256}{5-4} = 768$
- $x^a = x^4$ changes from $4^4 = 256$ to $5^4 = 625$
  - $\Rightarrow$ the average rate of change of $x^a = x^4$ from $x = 4$ to $x = 5$ is $\frac{625-256}{5-4} = 369$

The correct formulae for the instantaneous rates of change of a^x and x^a are

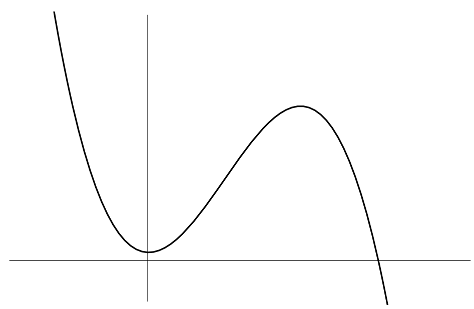
$$\frac{d}{dx}a^x = \ln a\ a^x \qquad \frac{d}{dx}x^a = ax^{a-1}$$

Warning 7. The slope of the curve $y = f(x)$ at $x = a$ is not $f'(x)$. It is $f'(a)$.

Discussion. Consider the problem "Find the tangent line to the curve $y = x^2$ at the point $(1, 1)$". The tangent line is a straight line. Any straight line has only one slope. In this case the slope of the tangent line is the same as the slope of $y = x^2$ at $x = 1$, which is $\frac{d}{dx}x^2\big|_{x=1} = 2x\big|_{x=1} = 2$. So the tangent line is $y - 1 = 2(x - 1)$. If you neglect to evaluate $\frac{d}{dx}x^2 = 2x$ at $x = 1$, you get $y - 1 = (2x)(x - 1)$ instead. This is not a straight line at all, because as x changes the quantity you put in for the slope $(2x)$ also changes. As you move along a straight line, the slope is not allowed to change.

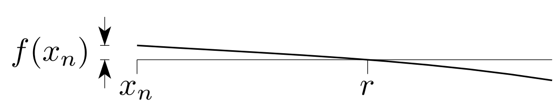
Warning 8. Suppose that you solve $f(x) = 0$ approximately by some numerical procedure, like Newton's method. Suppose further that you want the root accurate to three decimal places. So you iterate until the first three (or four or five or six, $\cdots$) decimal places stabilize. While this is strongly suggestive that you have three decimal place accuracy, it is **not conclusive**. Also, if you substitute your guess into f and find that the result is zero to three (or four or five or six, $\cdots$) decimal places, you still **cannot conclude definitively** that your root is accurate to three decimal places.

Discussion. It is perfectly possible for Newton's method to behave very nicely for, for example, fifty iterations and look like it is converging to some value and then suddenly go crazy on the fifty first iteration. For example, if the graph of f looks like



and you start with $x_1 < 0$ (or even x_1 positive but not too big), Newton's method will start off looking like it is going to converge to zero, even though there is no root near $x = 0$.

The above example also shows that it is possible for $f(0)$ to be extremely small even if f has no root near $x = 0$. Even if $f(x)$ does have a root r near your guess x_n, you cannot tell how big $|x_n - r|$, the error in your guess, is just from how close $f(x_n)$ is to zero. If the graph of f looks like



then $|f(x_n)|$ is a lot smaller than $|x_n - r|$. If the slope of the graph is small enough it is perfectly possible to have $|f(x_n)| = 10^{-10}$ with $|x_n - r| = 1$.

Assuming that $f(x)$ is continuous, a sure way to test if $x_n = 1.234$ is accurate to four decimal places is to check if $f(1.2335)$ and $f(1.2345)$ are of opposite sign. If so f has to be zero somewhere between $x = 1.2335$ and $x = 1.2345$.

Warning 9. If c is a constant $\int_a^b c\,dx = c\,x\Big|_a^b = c(b-a)$ **not** $\frac{c^2}{2}\Big|_a^b = \frac{b^2-a^2}{2}$.

Discussion. The integral $\int_a^b c\,dx$ represents the area between $y=0$ and $y=c$ with $a \le x \le b$. Because the height c is constant, this is a rectangle of width $b-a$ and height c and consequently of area $c(b-a)$.

Warning 10. The square root $\sqrt{\cos^2 x}$ is $|\cos x|$ **not** $\cos x$.

Discussion. Suppose that you are walking along a path and that at time t you are at $x(t) = \sin^3 t$ and $y(t) = \cos^3 t$ and that t runs from 0 to 2π. Then your velocity in the x direction is $x'(t) = 3\cos t \sin^2 t$ and your velocity in the y direction is $y'(t) = -3\sin t\cos^2 t$. Your speed is

$$\begin{aligned}\sqrt{x'(t)^2 + y'(t)^2} &= \sqrt{9\cos^2 t\sin^4 t + 9\cos^4 t\sin^2 t} = 3\sqrt{\cos^2 t\sin^2 t(\sin^2 t + \cos^2 t)}\\ &= 3\sqrt{\cos^2 t\sin^2 t}\end{aligned}$$

Speed is distance traveled per unit time. It is always positive. In particular the $\sqrt{\cdots}$ refers to the positive square root. The total distance traveled is

$$\int_0^{2\pi} \text{speed}\,dt = \int_0^{2\pi} 3\sqrt{\cos^2 t\sin^2 t}\,dt = 3\int_0^{2\pi} \big|\cos t\sin t\big|\,dt = \tfrac{3}{2}\int_0^{2\pi} \big|\sin(2t)\big|\,dt$$

If we had forgotten the absolute value signs, we would now have that the total distance traveled is $\frac{3}{2}\int_0^{2\pi}\sin(2t)\,dt = 0$, which is ridiculous! The real answer is

$$\begin{aligned}\tfrac{3}{2}\int_0^{2\pi} \big|\sin(2t)\big|\,dt &= 4\tfrac{3}{2}\int_0^{\pi/2} \big|\sin(2t)\big|\,dt \qquad \text{(Graph the integrand to see this!)}\\ &= 6\int_0^{\pi/2}\sin(2t)\,dt = 6\,\Big[-\tfrac{1}{2}\cos(2t)\Big]_0^{\pi/2} = 6\end{aligned}$$

Warning 11. The computation

$$\int_{-1}^{1}\frac{1}{x^2}\,dx = \frac{x^{-1}}{-1}\Big|_{-1}^{1} = \frac{1}{-1} - \frac{-1}{-1} = -2$$

is **wrong**.

Discussion. In fact, the answer is ridiculous. The integrand $\frac{1}{x^2} > 0$, so the integral has to be positive. The flaw in the argument is that the Fundamental Theorem of Calculus, which says that if $F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a)$ is applicable only when $F'(x)$ exists and equals $f(x)$ for all $a \le x \le b$. In this case $F'(x) = \frac{1}{x^2}$ does not exist for $x = 0$. The given integral is improper. The correct computation is

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \lim_{t\to 0+} \int_{-1}^{-t} \frac{1}{x^2}\,dx + \lim_{t\to 0+} \int_{t}^{1} \frac{1}{x^2}\,dx = \lim_{t\to 0+} \frac{x^{-1}}{-1}\Big|_{-1}^{-t} + \lim_{t\to 0+} \frac{x^{-1}}{-1}\Big|_{t}^{1}$$
$$= \lim_{t\to 0+} \left[\frac{-1/t}{-1} - \frac{-1}{-1}\right] + \lim_{t\to 0+} \left[\frac{1}{-1} - \frac{1/t}{-1}\right] = \lim_{t\to 0+} \left[\frac{2}{t} - 2\right] = \infty$$

Warning 12. The indefinite integral $\int \frac{1}{5-3x}\,dx$ is **NOT** $\ln|5-3x| + C$.

Discussion. Any time you think you know an indefinite integral, you can always check your guess by differentiating it. In this case, when $5 - 3x > 0$ so that $\ln|5-3x| = \ln(5-3x)$,

$$\tfrac{d}{dx}\ln(5-3x) = \tfrac{1}{5-3x}\tfrac{d}{dx}(5-3x) = -3\tfrac{1}{5-3x} \ne \tfrac{1}{5-3x}$$

by the chain rule. We can get the correct answer by substituting $y = 5 - 3x$, $dy = -3\,dx$, $dx = \frac{dy}{-3}$

$$\int \tfrac{1}{5-3x}\,dx = \int \tfrac{1}{y}\,\tfrac{dy}{-3} = -\tfrac{1}{3}\int \tfrac{dy}{y} = -\tfrac{1}{3}\ln|y| + C = -\tfrac{1}{3}\ln|5-3x| + C$$

Warning 13. The indefinite integral $\int \frac{1}{x^{1/2}+x^{3/2}}\,dx$ is **NOT** $\frac{x^{1/2}}{1/2} + \frac{x^{-1/2}}{-1/2} + C$.

Discussion. The **incorrect** computation leading to the answer $\frac{x^{1/2}}{1/2} + \frac{x^{-1/2}}{-1/2} + C$ is

$$\int \frac{1}{x^{1/2}+x^{3/2}}\,dx = \int \left[x^{-1/2} + x^{-3/2}\right]dx = \int x^{-1/2}\,dx + \int x^{-3/2}\,dx = \frac{x^{1/2}}{1/2} + \frac{x^{-1/2}}{-1/2} + C$$

The mistake occurred in the very first step: $\frac{1}{x^{1/2}+x^{3/2}}$ is not equal to $\frac{1}{x^{1/2}} + \frac{1}{x^{3/2}}$. For example, when $x = 4$, $x^{1/2} = 2$ and $x^{3/2} = 8$, so saying $\frac{1}{x^{1/2}+x^{3/2}} = \frac{1}{x^{1/2}} + \frac{1}{x^{3/2}}$ is the same as saying $\frac{1}{2+8} = \frac{1}{2} + \frac{1}{8}$, which is absurd.