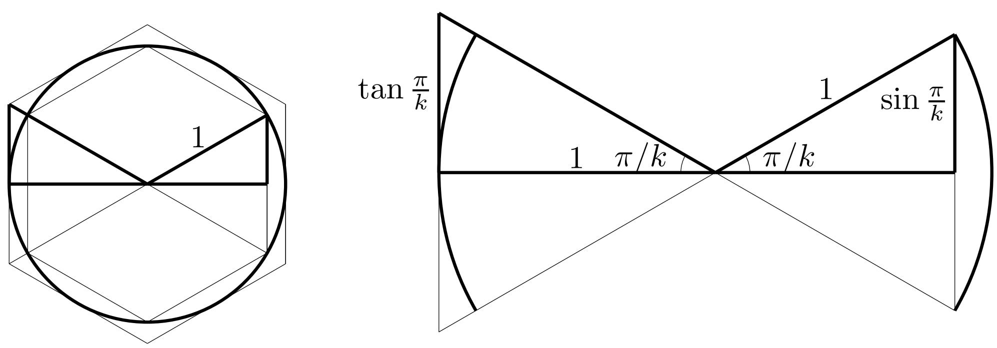
Archimedes' (Approximate) Computation of π

Archimedes (287–212 BC) managed to show that

$$\tfrac{223}{71} < \pi < \tfrac{22}{7}$$

Here is how he did it, but using modern terminology. As you're reading this through, remember that Archimedes did not have access to trignometric functions. He used plain geometry instead. Let k be any positive integer and draw a circle of radius 1 and then inscribe and circumscribe regular polygons with k sides, as in the figure on the left below. Denote by $2B_k$ and $2A_k$ the perimeters of the inscribed and circumscribed polygons,



respectively. The two perimeters and the circumference, $2\pi \times 1$, of the circle are related by

$$2B_k < 2\pi < 2A_k \qquad \text{or} \qquad B_k < \pi < A_k$$

Each side of a regular polygon of k sides subtends an angle of $\frac{2\pi}{k}$. So, using the figure on the right above (which is just an enlargement of part of the figure on the left above) we see that each side of the inscribed polygon has length $2 \times \sin\frac{\pi}{k}$ and each side of the circumscribed polygon has length $2 \times \tan\frac{\pi}{k}$ so that

$$2B_k = 2k\sin\tfrac{\pi}{k} \qquad 2A_k = 2k\tan\tfrac{\pi}{k} \qquad \text{or} \qquad B_k = k\sin\tfrac{\pi}{k} \qquad A_k = k\tan\tfrac{\pi}{k}$$

To compute some A_k's and B_k's, Archimedes used

(1) $A_3 = 3\tan\frac{\pi}{3} = 3\sqrt{3}$ and $B_3 = 3\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}$

(2) $A_{2k} = \frac{2A_kB_k}{A_k+B_k}$

Proof: By the double angle formulae, $\sin(2\theta) = 2\sin\theta\cos\theta$ and $\cos(2\theta) = 2\cos^2\theta - 1$, with $\theta = \frac{\pi}{2k}$,

$$A_{2k} = \frac{2k\sin\frac{\pi}{2k}}{\cos\frac{\pi}{2k}} = \frac{2k\sin\frac{\pi}{2k}\cos\frac{\pi}{2k}}{\cos^2\frac{\pi}{2k}} = \frac{k\sin\frac{\pi}{k}}{\frac{1}{2}\big(\cos\frac{\pi}{k}+1\big)} = \frac{1}{\frac{1}{2}\big(\frac{\cos\frac{\pi}{k}}{k\sin\frac{\pi}{k}} + \frac{1}{k\sin\frac{\pi}{k}}\big)}$$

$$= \frac{1}{\frac{1}{2}\big(\frac{1}{A_k}+\frac{1}{B_k}\big)} = \frac{1}{\frac{B_k+A_k}{2A_kB_k}} = \frac{2A_kB_k}{A_k+B_k}$$

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(3) $B_{2k}^2 = A_{2k}B_k$

Proof: Again using the double angle formula $\sin(2\theta) = 2\sin\theta\cos\theta$ with $\theta = \frac{\pi}{2k}$,

$$B_{2k}^2 = 4k^2\sin^2\tfrac{\pi}{2k} = 2k^2\frac{\sin\frac{\pi}{2k}}{\cos\frac{\pi}{2k}}\ 2\sin\tfrac{\pi}{2k}\cos\tfrac{\pi}{2k} = 2k\frac{\sin\frac{\pi}{2k}}{\cos\frac{\pi}{2k}}\ k\sin\tfrac{\pi}{k} = A_{2k}B_k$$

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Observe that if we know A_k and B_k, then property (2) gives A_{2k} and property (3) gives B_{2k}. So the idea is to now start with A_3, B_3 from (1), then compute A_6 using (2), then compute B_6 using (3), then compute A_{12} using (2), then compute B_{12} using (3) and so on. Here goes:

$$\begin{aligned}
A_3 &= 3\sqrt{3} \qquad B_3 = \tfrac{3\sqrt{3}}{2} \\
A_6 &= \frac{2A_3B_3}{A_3+B_3} = \frac{3\times3\times3}{\frac{3}{2}\times3\sqrt{3}} = 2\sqrt{3} \approx 3.4641 \\
B_6 &= \sqrt{A_6B_3} = \sqrt{2\sqrt{3}\times\tfrac{3\sqrt{3}}{2}} = 3 \\
A_{12} &= \frac{2A_6B_6}{A_6+B_6} = \frac{4\times3\sqrt{3}}{2\sqrt{3}+3} = \frac{12}{\sqrt{3}+2} \approx 3.2154 \\
B_{12} &= \sqrt{A_{12}B_6} = \sqrt{\frac{36}{\sqrt{3}+2}} \approx 3.1058 \\
A_{24} &= \frac{2A_{12}B_{12}}{A_{12}+B_{12}} \approx \frac{2\times3.2154\times3.1058}{3.2154+3.1058} \approx 3.1596 \\
B_{24} &= \sqrt{A_{24}B_{12}} = \sqrt{3.1596\times3.1058} \approx 3.1326 \\
A_{48} &= \frac{2A_{24}B_{24}}{A_{24}+B_{24}} \approx \frac{2\times3.1596\times3.1326}{3.1596+3.1326} \approx 3.1461 \\
B_{48} &= \sqrt{A_{48}B_{24}} = \sqrt{3.1461\times3.1326} \approx 3.1393 \\
A_{96} &= \frac{2A_{48}B_{48}}{A_{48}+B_{48}} \approx \frac{2\times3.1461\times3.1393}{3.1461+3.1393} \approx 3.1427 \\
B_{96} &= \sqrt{A_{96}B_{48}} = \sqrt{3.1427\times3.1393} \approx 3.1410
\end{aligned}$$

Archimedes stopped here — except that he did not have access to decimal numbers, so he just used fractions. Also Archimedes used $\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$. He did not say how he arrived at these inequalities. (Of course it is easy to check that they are true by observing that $265^2 < 3\times153^2$ and $1351^2 > 3\times780^2$.)