The Binomial Theorem

In these notes we prove the binomial theorem, which says that for any integer $n \ge 1$,

$$(x+y)^n = \sum_{\ell=0}^{n} \binom{n}{\ell} x^\ell y^{n-\ell} = \sum_{\substack{\ell,m\ge 0\\ \ell+m=n}} \binom{\ell+m}{\ell} x^\ell y^m \qquad \text{where } \binom{n}{\ell} = \tfrac{n!}{\ell!(n-\ell)!} \qquad (\mathrm{B_n})$$

Here $n!$ (read "n factorial") means $1\times 2\times 3\times\cdots\times n$ so that, for example,

$$\begin{aligned}\binom{n}{1} = \binom{n}{n-1} &= \frac{n!}{1!(n-1)!} = \frac{1\times2\times3\times\cdots\times(n-1)\times n}{1\times2\times3\times\cdots\times(n-1)} = n\\ \binom{n}{2} = \binom{n}{n-2} &= \frac{n!}{2!(n-2)!} = \frac{1\times2\times3\times\cdots\times(n-2)\times(n-1)\times n}{(1\times2)\times(1\times2\times3\times\cdots\times(n-2))} = \frac{n(n-1)}{2}\end{aligned}$$

By convention $0! = 1$ so that $\binom{n}{0} = \binom{n}{n} = \frac{n!}{0!n!} = 1$. As special cases of the Binomial Theorem, we have

$$\begin{aligned} n=1 &\quad (x+y)^1 = x+y\\ n=2 &\quad (x+y)^2 = x^2+2xy+y^2\\ n=3 &\quad (x+y)^3 = x^3+3x^2y+3xy^2+y^3\end{aligned}$$

Proof of the Binomial Theorem: The proof is by induction on n. First we check that, when $n=1$,

$$\begin{aligned}\sum_{\ell=0}^{n} \tfrac{n!}{\ell!(n-\ell)!} x^\ell y^{n-\ell} &= \tfrac{n!}{\ell!(n-\ell)!} x^\ell y^{n-\ell}\Big|_{\substack{n=1\\ \ell=0}} + \tfrac{n!}{\ell!(n-\ell)!} x^\ell y^{n-\ell}\Big|_{\substack{n=1\\ \ell=1}} = \tfrac{1!}{0!1!} x^0y^1 + \tfrac{1!}{1!0!}x^1y^0\\ &= x+y\end{aligned}$$

so that $(\mathrm{B_n})$ is correct for $n=1$. To complete the proof we have to show that, for any integer $n\ge 2$, $(\mathrm{B_n})$ is a consequence of $(\mathrm{B_{n-1}})$. So pick any integer $n\ge 2$ and assume that

$$(x+y)^{n-1} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^\ell y^{n-1-\ell}$$

is true. Now compute

$$(x+y)^n = (x+y)^{n-1}(x+y) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^{\ell+1} y^{n-1-\ell} + \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^\ell y^{n-\ell}$$

The second sum has the same powers of x and y, namely $x^\ell y^{n-\ell}$, as appear in $(\mathrm{B_n})$. The make the powers of x and y in the first sum, namely $x^{\ell+1}y^{n-1-\ell}$ look more like those of $(\mathrm{B_n})$, we make the change of summation variable from ℓ to $\tilde\ell = \ell+1$. The first sum

$$\sum_{\ell=0}^{n-1}\tbinom{n-1}{\ell}x^{\ell+1}y^{n-1-\ell} = \sum_{\tilde\ell=1}^{n}\tbinom{n-1}{\tilde\ell-1}x^{\tilde\ell}y^{n-\tilde\ell}$$

As $\tilde\ell$ is just a dummy summation variable, we may call it anything we like. In particular, we may rename $\tilde\ell$ back to ℓ. So we now have

$$\begin{aligned}(x+y)^n &= \sum_{\ell=1}^{n}\tbinom{n-1}{\ell-1}x^\ell y^{n-\ell} + \sum_{\ell=0}^{n-1}\tbinom{n-1}{\ell}x^\ell y^{n-\ell}\\ &= \tbinom{n-1}{\ell-1}x^\ell y^{n-\ell}\Big|_{\ell=n} + \tbinom{n-1}{\ell}x^\ell y^{n-\ell}\big|_{\ell=0} + \sum_{\ell=1}^{n-1}\Big[\tbinom{n-1}{\ell-1}+\tbinom{n-1}{\ell}\Big]x^\ell y^{n-\ell}\end{aligned}$$

Recalling that $n! = n\ (n-1)!$ we have

$$\begin{aligned}\tbinom{n}{\ell} &= \tfrac{n!}{\ell!(n-\ell)!} = \tfrac{n(n-1)!}{\ell(\ell-1)!\ (n-\ell)!} &&= \tfrac{n}{\ell}\tbinom{n-1}{\ell-1}\\ \tbinom{n}{\ell} &= \tfrac{n!}{\ell!(n-\ell)!} = \tfrac{n(n-1)!}{\ell!\ (n-\ell)(n-\ell-1)!} &&= \tfrac{n}{n-\ell}\tbinom{n-1}{\ell}\end{aligned}$$

So

$$\begin{aligned}(x+y)^n &= \tbinom{n-1}{n-1}x^n + \tbinom{n-1}{0}y^n + \sum_{\ell=1}^{n-1}\Big[\tbinom{n-1}{\ell-1}+\tbinom{n-1}{\ell}\Big]x^\ell y^{n-\ell}\\ &= x^n + y^n + \sum_{\ell=1}^{n-1}\tbinom{n}{\ell}\Big[\tfrac{\ell}{n}+\tfrac{n-\ell}{n}\Big]x^\ell y^{n-\ell}\\ &= x^n + y^n + \sum_{\ell=1}^{n-1}\tbinom{n}{\ell}x^\ell y^{n-\ell}\\ &= \sum_{\ell=0}^{n}\tbinom{n}{\ell}x^\ell y^{n-\ell}\end{aligned}$$

as desired.

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