The Chain Rule

Question

You are walking. Your position at time t is $g(t)$. Your are walking in an environment in which the air temperature depends on position. The temperature at position x is $f(x)$. What instantaneous rate of change of temperature do you feel at time t?

Because your position at time t is $x = g(t)$, the temperature you feel at time t is $F(t) = f\big(g(t)\big)$. The instantaneous rate of change of temperature that you feel is $F'(t)$. We have a complicated function $F(t)$, constructed from two simple functions, $g(t)$ and $f(x)$. We wish to compute the derivative, $F'(t)$, of the complicated function in terms of the derivatives, $g'(t)$ and $f'(x)$, of the two simple functions. This is exactly what the chain rule does.

The Chain Rule

If $g(t)$ is differentiable at t_0 and $f(x)$ is differentiable at $x_0 = g(t_0)$ then $F(t) = f\big(g(t)\big)$ is differentiable at t_0 and

$$F'(t_0) = f'\big(g(t_0)\big)g'(t_0)$$

Special Cases

a) If $f(x) = x^n$, then $f'(x) = nx^{n-1}$, $F(t) = f\big(g(t)\big) = g(t)^n$ and $f'\big(g(t)\big)g'(t) = ng(t)^{n-1}g'(t)$. So

$$\tfrac{d}{dt}g(t)^n = n\,g(t)^{n-1}\,g'(t)$$

b) If $f(x) = \sin x$, then $f'(x) = \cos x$, $F(t) = f\big(g(t)\big) = \sin\big(g(t)\big)$ and $f'\big(g(t)\big)g'(t) = \cos\big(g(t)\big)g'(t)$. So

$$\tfrac{d}{dt}\sin\big(g(t)\big) = \cos\big(g(t)\big)\,g'(t)$$

Similarly

$$\tfrac{d}{dt}\cos\big(g(t)\big) = -\sin\big(g(t)\big)\,g'(t)$$

Units

In the question posed above, t has units of seconds, $g(t)$ has units of meters, x has units of meters and $f(x)$ has units of degrees. Consequently, $F(t) = f\big(g(t)\big)$ has units of degrees, $F'(t)$ has units $\frac{\text{degrees}}{\text{second}}$, $f'(x)$ has units $\frac{\text{degrees}}{\text{meter}}$ and $g'(t)$ has units $\frac{\text{meters}}{\text{second}}$. Thus $f'\big(g(t)\big)g'(t)$ has units $\frac{\text{degrees}}{\text{meter}} \times \frac{\text{meters}}{\text{second}} = \frac{\text{degrees}}{\text{second}}$ which is the same as the units of $F'(t)$. This of course does not prove that $F'(t)$ and $f'\big(g(t)\big)g'(t)$ are the same. But it does provide a consistency check.

Derivation of the Chain Rule

Write $x_0 = g(t_0)$. We are told that

$$f'(x_0) = \lim_{H\to 0} \frac{f(x_0+H) - f(x_0)}{H} \qquad \text{and} \qquad g'(t_0) = \lim_{h\to 0} \frac{g(t_0+h) - g(t_0)}{h}$$

In particular, if we define

$$\varphi(H) = \frac{f(x_0+H) - f(x_0)}{H}$$

then we know that

$$f(x_0+H) = f(x_0) + H\varphi(H) \qquad \text{and} \qquad \lim_{H\to 0} \varphi(H) = f'(x_0)$$

Our goal is to evaluate

$$F'(t_0) = \lim_{h\to 0} \frac{F(t_0+h) - F(t_0)}{h} = \lim_{h\to 0} \frac{f\big(g(t_0+h)\big) - f\big(g(t_0)\big)}{h}$$

Now $f\big(g(t_0)\big) = f(x_0)$ and we can turn $f\big(g(t_0+h)\big)$ into $f(x_0+H)$ by writing

$$g(t_0+h) = g(t_0) + H(h) \qquad \text{with} \qquad H(h) = g(t_0+h) - g(t_0)$$

So

$$\begin{aligned} F'(t_0) &= \lim_{h\to 0} \frac{f\big(g(t_0+h)\big) - f\big(g(t_0)\big)}{h} \\ &= \lim_{h\to 0} \frac{f\big(x_0 + H(h)\big) - f(x_0)}{H(h)} \frac{H(h)}{h} \\ &= \lim_{h\to 0} \varphi\big(H(h)\big)\ \frac{g(t_0+h) - g(t_0)}{h} \\ &= \lim_{h\to 0} \varphi\big(H(h)\big)\ \lim_{h\to 0} \frac{g(t_0+h) - g(t_0)}{h} \\ &= f'(x_0)\ g'(t_0) \end{aligned}$$

because $H(h)$ tends to 0 as h tends to 0. This is the chain rule.