Differentiation Rules

Statement

Suppose that $f(x)$ and $g(x)$ are both differentiable at x_0.

a) Let $S(x) = af(x) + bg(x)$, with a and b constants. Then $S(x)$ is differentiable at x_0 and $S'(x_0) = af'(x_0) + bg'(x_0)$. That is,

$$\tfrac{d}{dx}[af(x) + bg(x)]\Big|_{x=x_0} = af'(x_0) + bg'(x_0)$$

Setting $a = b = 1$ and $b = 0$ gives the two special cases

$$\tfrac{d}{dx}[f(x) + g(x)]\Big|_{x=x_0} = f'(x_0) + g'(x_0) \qquad \tfrac{d}{dx}[af(x)]\Big|_{x=x_0} = af'(x_0)$$

b) (Product Rule) Let $P(x) = f(x)g(x)$. Then $P(x)$ is differentiable at x_0 and $P'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$. That is,

$$\tfrac{d}{dx}[f(x)g(x)]\Big|_{x=x_0} = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

c) (Quotient Rule) Let $Q(x) = \frac{f(x)}{g(x)}$. Then $Q(x)$ is differentiable at x_0 and $Q'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g(x_0)^2}$, provided $g(x_0) \neq 0$. That is,

$$\frac{d}{dx}\Big[\frac{f(x)}{g(x)}\Big]\Big|_{x=x_0} = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Setting $f(x)$ to the constant function $f(x) = 1$ gives the special case

$$\frac{d}{dx}\Big[\frac{1}{g(x)}\Big]\Big|_{x=x_0} = -\frac{g'(x_0)}{g(x_0)^2}$$

Derivation

Define

$$F(x_0, h) = \frac{f(x_0 + h) - f(x_0)}{h} \qquad G(x_0, h) = \frac{g(x_0 + h) - g(x_0)}{h}$$

Observe that, cross multiplying by h,

$$f(x_0 + h) = f(x_0) + hF(x_0, h) \qquad g(x_0 + h) = g(x_0) + hG(x_0, h)$$

By the definition of the derivative

$$\lim_{h\to 0} F(x_0,h) = f'(x_0) \qquad\qquad \lim_{h\to 0} G(x_0,h) = g'(x_0)$$

a)

$$\begin{aligned} S(x_0+h) - S(x_0) &= af(x_0+h) + bg(x_0+h) - af(x_0) - bg(x_0) \\ &= a[f(x_0) + hF(x_0,h)] + b[g(x_0) + hG(x_0,h)] - af(x_0) - bg(x_0) \\ &= ahF(x_0,h) + bhG(x_0,h) \end{aligned}$$

By definition

$$\begin{aligned} S'(x_0) &= \lim_{h\to 0} \tfrac{S(x_0+h)-S(x_0)}{h} = \lim_{h\to 0} \tfrac{ahF(x_0,h)+bhG(x_0,h)}{h} = \lim_{h\to 0}[aF(x_0,h) + bG(x_0,h)] \\ &= af'(x_0) + bg'(x_0) \end{aligned}$$

b)

$$\begin{aligned} P(x_0+h) - P(x_0) &= f(x_0+h)g(x_0+h) - f(x_0)g(x_0) \\ &= [f(x_0) + hF(x_0,h)][g(x_0) + hG(x_0,h)] - f(x_0)g(x_0) \\ &= hF(x_0,h)g(x_0) + f(x_0)hG(x_0,h) + h^2F(x_0,h)G(x_0,h) \end{aligned}$$

By definition

$$\begin{aligned} P'(x_0) &= \lim_{h\to 0} \tfrac{P(x_0+h)-P(x_0)}{h} = \lim_{h\to 0} \tfrac{hF(x_0,h)g(x_0)+hf(x_0)G(x_0,h)+h^2F(x_0,h)G(x_0,h)}{h} \\ &= \lim_{h\to 0}[F(x_0,h)g(x_0) + f(x_0)G(x_0,h) + hF(x_0,h)G(x_0,h)] \\ &= f'(x_0)g(x_0) + f(x_0)g'(x_0) + 0f'(x_0)g'(x_0) \\ &= f'(x_0)g(x_0) + f(x_0)g'(x_0) \end{aligned}$$

c) By definition,

$$Q'(x_0) = \lim_{h\to 0} \tfrac{Q(x_0+h)-Q(x_0)}{h} = \lim_{h\to 0} \tfrac{1}{h}\left[\tfrac{f(x_0+h)}{g(x_0+h)} - \tfrac{f(x_0)}{g(x_0)}\right] = \lim_{h\to 0} \tfrac{1}{h}\tfrac{f(x_0+h)g(x_0)-f(x_0)g(x_0+h)}{g(x_0)g(x_0+h)}$$

The numerator

$$\begin{aligned} f(x_0+h)g(x_0) - f(x_0)g(x_0+h) &= \big[f(x_0) + hF(x_0,h)\big]g(x_0) - f(x_0)\big[g(x_0) + hG(x_0,h)\big] \\ &= hF(x_0,h)g(x_0) - f(x_0)hG(x_0,h) \end{aligned}$$

Dividing by $hg(x_0)g(x_0+h)$

$$\tfrac{1}{h}\tfrac{f(x_0+h)g(x_0)-f(x_0)g(x_0+h)}{g(x_0)g(x_0+h)} = \tfrac{hF(x_0,h)g(x_0)-hf(x_0)G(x_0,h)}{hg(x_0)g(x_0+h)} = \tfrac{F(x_0,h)g(x_0)-f(x_0)G(x_0,h)}{g(x_0)[g(x_0)+hG(x_0,h)]}$$

Hence

$$\begin{aligned} Q'(x_0) &= \lim_{h\to 0} \tfrac{1}{h}\tfrac{f(x_0+h)g(x_0)-f(x_0)g(x_0+h)}{g(x_0)g(x_0+h)} = \lim_{h\to 0} \tfrac{F(x_0,h)g(x_0)-f(x_0)G(x_0,h)}{g(x_0)[g(x_0)+hG(x_0,h)]} \\ &= \tfrac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g(x_0)^2} \end{aligned}$$

as desired. ■