A Delta–Epsilon Example.

Problem: Let $\varepsilon > 0$. Find a $\delta > 0$ such that $\big|\cos\big(2\pi x - \sin(x-1)\big) - 1\big| < \varepsilon$ for all $|x-1| < \delta$.

Solution: Define $f(x) = \cos\big(2\pi x - \sin(x-1)\big)$. We are given some number $\varepsilon > 0$. We have to find a $\delta > 0$ such that $\big|f(x) - f(1)\big| < \varepsilon$ for all $|x-1| < \delta$. We wish, in the end, to write an argument of the form

$$\begin{aligned} &\text{Set } \delta = \cdots. \text{ If } |x-1| < \delta \text{ then} \\ &\big|f(x) - f(1)\big| \le \cdots \\ &\qquad\vdots \\ &\qquad < \varepsilon \end{aligned}$$

However at this stage, we still do not know what δ to pick. So I like to start by writing out an argument of the above form, but leaving the choice of δ blank.

$$\begin{aligned} &\text{Set } \delta = \quad. \text{ If } |x-1| < \delta \text{ then} \\ \big|f(x) - f(1)\big| &= \big|f'(z)\,(x-1)\big| && \text{for some } z \text{ between } x \text{ and } 1, \\ & && \text{by the Mean–Value Theorem} \\ &= \big|-\sin\big(2\pi z - \sin(z-1)\big)\big\{2\pi - \cos(x-1)\big\}\,(x-1)\big| \\ &\le \big|\big\{2\pi - \cos(x-1)\big\}\,(x-1)\big| && \text{since } \big|\sin\big(2\pi z - \sin(z-1)\big)\big| \le 1 \\ &= \big|2\pi - \cos(x-1)\big|\ |x-1| \\ &\le \big(2\pi + 1\big)\ |x-1| && \text{since } -1 \le \cos(x-1) \le 1 \end{aligned}$$

We would now like to terminate the string of inequalities with $< \varepsilon$. But for that to be true we need $\big(2\pi + 1\big)\ |x-1| < \varepsilon$. That is, we need $|x-1| < \frac{\varepsilon}{2\pi+1}$. This tells us to choose $\delta = \frac{\varepsilon}{2\pi+1}$. We may now δ and give the full argument.

$$\begin{aligned} &\text{Set } \boxed{\delta = \tfrac{\varepsilon}{2\pi+1}}. \text{ If } |x-1| < \delta \text{ then} \\ \big|f(x) - f(1)\big| &= \big|f'(z)\,(x-1)\big| && \text{for some } z \text{ between } x \text{ and } 1, \\ &= \big|-\sin\big(2\pi z - \sin(z-1)\big)\big\{2\pi - \cos(x-1)\big\}\,(x-1)\big| \\ &\le \big|\big\{2\pi - \cos(x-1)\big\}\,(x-1)\big| && \text{since } \big|\sin\big(2\pi z - \sin(z-1)\big)\big| \le 1 \\ &= \big|2\pi - \cos(x-1)\big|\ |x-1| \\ &\le \big(2\pi + 1\big)\ |x-1| && \text{since } -1 \le \cos(x-1) \le 1 \\ &< \varepsilon && \text{since } |x-1| < \delta = \tfrac{\varepsilon}{2\pi+1} \end{aligned}$$