Derivatives of Exponentials

Fix any $a > 0$. The definition of the derivative of a^x is

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h} = \lim_{h\to 0} a^x\frac{a^h-1}{h} = a^x \lim_{h\to 0}\frac{a^h-1}{h} = C(a)\ a^x$$

where we are using $C(a)$ to denote the coefficient $\lim\limits_{h\to 0}\frac{a^h-1}{h}$ that appears in the derivative. This coefficient does not depend on x. So, at this stage, we know that $\frac{d}{dx}a^x$ is a^x times some fixed constant $C(a)$. We can learn more about $C(a)$ by just writing $a^h = \big(10^{\log_{10} a}\big)^h = 10^{h\log_{10} a}$:

$$C(a) = \lim_{h\to 0}\frac{a^h-1}{h} = \lim_{h\to 0}\frac{10^{h\log_{10} a}-1}{h} \overset{h'=h\log_{10} a}{=} \lim_{h'\to 0}\frac{10^{h'}-1}{h'/\log_{10} a} = \log_{10} a \lim_{h'\to 0}\frac{10^{h'}-1}{h'}$$
$$= C(10)\log_{10} a$$

So we now know

$$\frac{d}{dx}a^x = C(10)\ (\log_{10} a)\ a^x$$

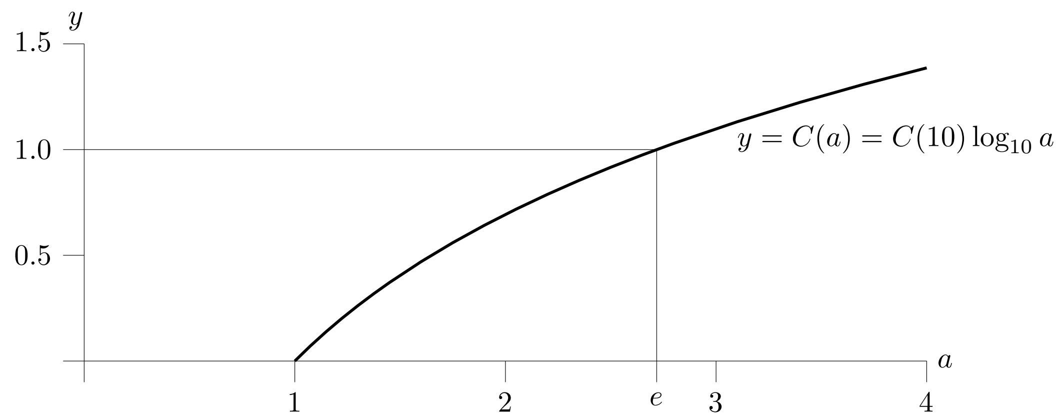
We will get a formula for $C(10)$ later in these notes. For now, we just try to get an idea of what $C(10)$ looks like by computing $\frac{10^h-1}{h}$ for various values of a and various small values of h. Here is a table of such values.

h	$\frac{10^h-1}{h}$
0.1	2.5893
0.01	2.3293
0.001	2.3052
0.0001	2.3028
0.00001	2.3026
0.000001	2.3026
0.0000001	2.3026

So it looks like $C(10) = 2.3026$, to four decimal places. In any event, recall that

- $\log_{10} a\big|_{a=1} = 0$ so that $C(a)\big|_{a=1} = 0$ (This is to be expected — when $a = 1$, $\frac{d}{dx}a^x = \frac{d}{dx}1 = 0$.)
- $\log_{10} a$ increases as a increases, and hence $C(a)$ increases as a increases
- $\log_{10} a$ tends to $+\infty$ as $a \to \infty$, and hence $C(a)$ tends to $+\infty$ as $a \to \infty$

Consequently there is exactly one value of a for which $C(a) = 1$. See the figure below. The value of a for which $C(a) = C(10)\log_{10} a = 1$ is given the name e. That is, e is defined by the condition



$C(e) = C(10)\log_{10} e = 1$, or equivalently, by the condition that $\boxed{\frac{d}{dx}e^x = e^x}$. From our previous numerical experiment, it looks like

$$2.3026\log_{10} e \approx 1 \implies \log_{10} e \approx \tfrac{1}{2.3026} \implies e \approx 10^{1/2.3026} \approx 2.7183$$

We shall find a much better way to determine e, to any desired degree of accuracy, shortly.

The Taylor Expansion of e^x

Let $f(x) = e^x$. Then

$$\begin{array}{lllllll} f(x) = e^x & \Rightarrow & f'(x) = e^x & \Rightarrow & f''(x) = e^x & \cdots \\ f(0) = e^0 = 1 & \Rightarrow & f'(0) = e^0 = 1 & \Rightarrow & f''(0) = e^0 = 1 & \cdots \end{array}$$

Recall that, for any positive integer n,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \tfrac{1}{n!}f^{(n)}(x_0)(x - x_0)^n + \tfrac{1}{(n+1)!}f^{(n+1)}(c)(x - x_0)^{n+1}$$

for some c between x_0 and x. Applying this with $f(x) = e^x$ and $x_0 = 0$, and using that $f^{(m)}(x_0) = e^{x_0} = e^0 = 1$ for all m,

$$e^x = f(x) = 1 + x + \cdots + \tfrac{x^n}{n!} + \tfrac{1}{(n+1)!}e^c x^{n+1}$$

for some c between 0 and x.

I claim that, for any fixed x, the error term $\frac{1}{(n+1)!}e^c x^{n+1}$ alway goes to zero as n goes to infinity. To see this, first observe that e^c increases as c increases, so e^c is necessarily between e^0 and e^x, for all n. So to show that the error term $\frac{1}{(n+1)!}e^c x^{n+1}$ alway goes to zero as n goes to infinity, I just have to show that $\varepsilon_n = \frac{|x|^{n+1}}{(n+1)!}$ alway goes to zero as n goes to infinity. Now note that

$$\varepsilon_{n+1} = \tfrac{|x|^{n+2}}{(n+2)!} = \tfrac{|x|^{n+1}}{(n+1)!}\tfrac{|x|}{(n+2)} = \tfrac{|x|}{(n+2)}\varepsilon_n$$

Once n gets bigger than, for example, $2|x|$, we have $\varepsilon_{n+1} = \frac{|x|}{(n+2)}\varepsilon_n < \frac{1}{2}\varepsilon_n$. That is, increasing n by decreases ε_n by a factor of at least 2. So ε_n must tend to zero as n tends to infinity.

Because, for any fixed x, the error term $\frac{1}{(n+1)!}e^c x^{n+1}$ alway goes to zero as n goes to infinity, we have, exactly,

$$e^x = \lim_{n\to\infty} \left[1 + x + \cdots + \tfrac{x^n}{n!}\right]$$

This limit is generally written

$$e^x = 1 + x + \cdots + \tfrac{x^n}{n!} + \cdots$$

or

$$e^x = \sum_{\ell=0}^{\infty} \tfrac{x^\ell}{\ell!}$$

In fact one may take $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ as the definition of e^x. If we set $x = 1$ we get

$$\begin{aligned}
e &= e^x\big|_{x=1} = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots\big|_{x=1} \\
&= 1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \tfrac{1}{4!} + \tfrac{1}{5!} + \tfrac{1}{6!} + \tfrac{1}{7!} + \tfrac{1}{8!} + \tfrac{1}{9!} + \cdots \\
&= 1 + 1 + 0.5 + 0.1\dot{6} + 0.041\dot{6} + 0.008\dot{3} + 0.0013\dot{8} + 0.00019841 + 0.00002480 + 0.00000276 + \cdots \\
&= 2.71828182846
\end{aligned}$$

and, since e was defined by $1 = C(e) = C(10)\log_{10} e$,

$$C(10) = \frac{1}{\log_{10} e} = \frac{\log_{10} 10}{\log_{10} e} = \ln 10 = 2.30258509299$$

and $C(a) = C(10)\log_{10} a = \frac{\log_{10} a}{\log_{10} e} = \ln a$ and

$$\tfrac{d}{dx} a^x = C(a)\ a^x = (\ln a)\ a^x$$

I do not have this derivative memorised. Every time I need it, I use

$$a^x = \left(e^{\ln a}\right)^x = e^{x\ln a} \implies \tfrac{d}{dx} a^x = (\ln a)\ e^{x\ln a} = (\ln a)\ a^x$$

by the chain rule.