The Exponent Tower

Let $x > 0$ and $a > 0$. In these notes, we consider the sequence

$$x_1 = x, \quad x_2 = x^x = x^{x_1}, \quad x_3 = x^{x^x} = x^{x_2}, \quad x_4 = x^{x^{x^x}} = x^{x_3}, \ \cdots, \ x_{n+1} = x^{x_n}, \ \cdots$$

We determine for which choices of x the limit of this sequence, which we denote $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$, exists and equals a. We shall show that

- if $0.066 \approx \frac{1}{e^e} \le x \le e^{1/e} \approx 1.44$, then the limit $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ exists and
- if $\frac{1}{e} \le a \le e$, then there is exactly one $x > 0$ for which $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$, exists and equals a and that $x = a^{1/a}$ and
- if $x > e^{1/e}$ or if $0 < x < \frac{1}{e^e}$, then the limit $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ does not exist and
- if $a > e$ or if $0 < a < \frac{1}{e}$, then there is no $x > 0$ for which $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$, exists and equals a.

Step 1. In this step we show that if the limit $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ exists and $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = a$, then $x = a^{1/a}$.

Proof: Fix any $x > 0$. We are assuming that the limit $\lim_{n\to\infty} x_n$ exists and that $\lim_{n\to\infty} x_n = a$. So taking the limit, as $n \to \infty$, of $x_{n+1} = x^{x_n}$ gives

$$a = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x^{x_n} = x^a$$

So $x^a = a$. Taking the a^{th} root of both sides gives $x = a^{1/a}$. ■

Step 2. In this step we fix any $x > 0$ and solve $x = a^{1/a}$ for $a > 0$. We show that if

- if $x > e^{1/e}$, then there is no $a > 0$ that obeys $x = a^{1/a}$ and
- if $x = e^{1/e}$, then there is exactly one $a > 0$ obeying $x = a^{1/a}$, namely $a = e$, and
- if $1 < x < e^{1/e}$, the there are exactly two a's obeying $x = a^{1/a}$, and
- if $0 < x \le 1$, then there is exactly one $a > 0$ obeying $x = a^{1/a}$.

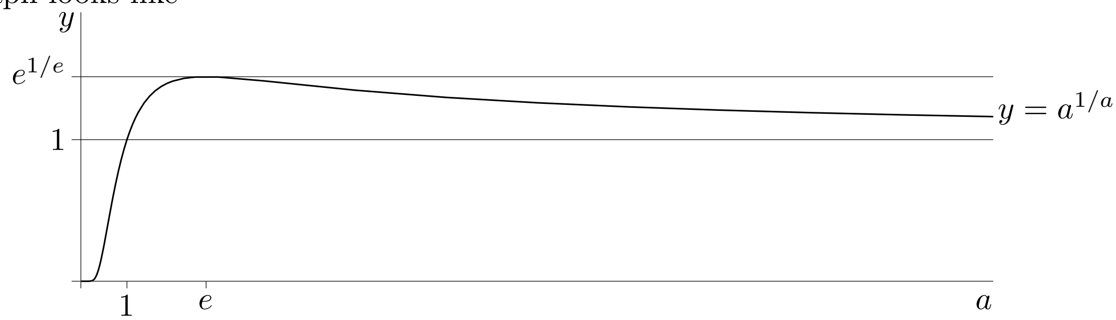
Proof: Observe that

$$\frac{d}{da}a^{1/a} = \frac{d}{da}e^{\frac{1}{a}\ln a} = e^{\frac{1}{a}\ln a}\frac{d}{da}\frac{\ln a}{a} = e^{\frac{1}{a}\ln a}\Big[\frac{1-\ln a}{a^2}\Big] \quad \begin{cases} > 0 & \text{if } 0 < a < e \\ = 0 & \text{if } a = e \\ < 0 & \text{if } a > e \end{cases}$$

So the graph of $y = a^{1/a}$ against a is increasing for $a < e$ and decreasing for $a > e$. Since

$$\lim_{a\to 0} a^{1/a} = \lim_{a\to 0} e^{\frac{1}{a}\ln a} = e^{-\infty} = 0 \qquad \lim_{a\to\infty} a^{1/a} = \lim_{a\to\infty} e^{\frac{1}{a}\ln a} = e^0 = 1$$

the graph looks like



(The graph of $y = a^{1/a}$ against a does tend asymptotically to the horizontal line $y = 1$ as $a \to \infty$. It just does so very slowly.) So the horizontal line $y = x$ crosses the curve $y = a^{1/a}$

- exactly once if $0 < x \le 1$ or $x = e^{1/e}$
- exactly twice if $1 < x < e^{1/e}$
- never if $x > e^{1/e}$

■

Step 3. In this step we fix any $x > 1$ and show that

- the sequence $x_1, x_2, x_3, \cdots$ is increasing. That is, $x_{n+1} > x_n$ for all $n \in \mathbb{N}$.
- If, in addition $x = a^{1/a}$, then $x_n < a$ for all $n \in \mathbb{N}$

Proof: The proof that $x_{n+1} > x_n$ is by induction on n. Note that, for $x > 1$, x^y is strictly increasing with y. So $x_2 = x^x > x^1 = x = x_1$ and if $x_n > x_{n-1}$, then $x_{n+1} = x^{x_n} > x^{x_{n-1}} = x_n$.

The proof that, if $x = a^{1/a}$ for some $a > 1$, then $x_n < a$ for all n, is once again by induction on n. Note again that x^y is strictly increasing with y. First $x_1 = a^{1/a} < a^1 = a$. Then, if $x_n < a$ for some $n \in \mathbb{N}$, we have

$$x_{n+1} = x^{x_n} < x^a = \left(a^{\frac{1}{a}}\right)^a = a^{\frac{1}{a}a} = a$$

■

Step 4. In this step we fix any $x \ge 1$ and finish this case off, showing that

- if $1 \le x \le e^{1/e}$, then the limit $\lim\limits_{n\to\infty} x_n$ exists and takes a value $1 \le a \le e$ and
- if $x > e^{1/e}$, then the limit $\lim\limits_{n\to\infty} x_n$ does not exist.

Proof: If $x = 1$, then every $x_n = 1$ and it is obvious that $\lim_{n\to\infty} x_n$ exists and equals 1. So assume that $x > 1$. Since the sequence $x_1, x_2, x_3, \cdots$ is increasing, it either converges to some number a or it diverges to ∞.

If $x > e^{1/e}$ the sequence must diverge, since if it were to converge to some a, then x would be $a^{1/a}$. But we saw, in Step 2, that no number $x > e^{1/e}$ can be of the form $a^{1/a}$ for any $a > 0$.

So fix any $1 < x \le e^{1/e}$. Then by Step 2, there is an $a > 0$ with $x = a^{1/a}$. In fact, looking at the graph in Step 2, we may always choose $1 < a \le e$. Do so. (For $1 < x < e^{1/e}$ there are two a's obeying $x = a^{1/a}$. We are choosing the smaller of the two.) By Step 3, the sequence $x_1, x_2, x_3, \cdots$ is both increasing and bounded above by a. So it must converge to some number $a' \le a$. By Step 1, we must have $x = (a')^{1/a'}$. Since a is the smallest number with this property, a' must be a. ■

Step 5. In this step, fix $0 < x < 1$ and write $x = a^{1/a}$ with $0 < a < 1$. (There is exactly one such a.) We show that

- the sequence $x_2, x_4, x_6, \cdots$ converges to some $B \ge a$ and
- the sequence $x_1, x_3, x_5, \cdots$ converges to some $b \le a$
- $b = x^{x^b}$ and $B = x^{x^B}$

Proof: Note that $x^a = \left(a^{\frac{1}{a}}\right)^a = a^{\frac{1}{a}a} = a$. Since x^y is strictly decreasing with y, we have $x = a^{1/a} < a^1 = a$ and

$$\begin{aligned}
x_1 &= x \in (0, a) \\
x_2 &= x^{x_1} \in \big(x^a, 1\big) = \big(a, 1\big) \\
x_3 &= x^{x_2} \in \big(x, x^a\big) = \big(x_1, a\big) \\
x_4 &= x^{x_3} \in \big(x^a, x^{x_1}\big) = \big(a, x_2\big) \\
x_5 &= x^{x_4} \in \big(x^{x_2}, x^a\big) = \big(x_3, a\big) \\
x_6 &= x^{x_5} \in \big(x^a, x^{x_3}\big) = \big(a, x_4\big)
\end{aligned}$$

and so on. From this we see that

$$x_1 < x_3 < x_5 < \cdots < a \qquad x_2 > x_4 > x_6 > \cdots > a$$

so the sequence $\{x_{2n+1}\}_{n\in\mathbb{N}}$ must converge to some $b \le a$ and that the sequence $\{x_{2n}\}_{n\in\mathbb{N}}$ must converge to some $B \ge a$. Furthermore

$$\begin{aligned}
x_{2n+1} &= x^{x_{2n}} & &\overset{n\to\infty}{\Longrightarrow} & b &= x^B \\
x_{2n+1} &= x^{x^{x_{2n-1}}} & &\overset{n\to\infty}{\Longrightarrow} & b &= x^{x^b} \\
x_{2n} &= x^{x_{2n-1}} & &\overset{n\to\infty}{\Longrightarrow} & B &= x^b \\
x_{2n} &= x^{x^{x_{2n-2}}} & &\overset{n\to\infty}{\Longrightarrow} & B &= x^{x^B}
\end{aligned}$$

■

We now know, as a consequence of Step 5, when $0 < x < 1$, there are only two possibilities:

- either $b = a = B$, in which case the sequence $x_1, x_2, x_3, \cdots$ converges to a or
- either $b < a < B$, in which case the sequence $x_1, x_2, x_3, \cdots$ diverges. This case is only possible of the equation $c = x^{x^c}$ has at least three distinct solutions (namely a, b and B).

Step 6. We next finish off the case $\frac{1}{e^e} \le x < 1$, which corresponds to $\frac{1}{e} \le a < 1$, by showing that, in this case, the equation $c = x^{x^c}$ has exactly one solution $0 < c < 1$.

Proof: Fix any $0 < x < 1$ and define a by $x = a^{1/a}$. First observe that for any real number c we always have $x^c > 0$ and hence $0 < x^{x^c} < 1$. Hence any c obeying $c = x^{x^c}$ must also obey $0 < c < 1$. The equation $c = x^{x^c}$ has at least one solution, namely $c = a$, since $x^a = a$ so that $x^{x^a} = x^a = a$. To test if it has other solutions we define the function $f(c) = c - x^{x^c}$ and see what we can learn about it from its derivative. Recalling that $\frac{d}{dy} x^y = \frac{d}{dy} e^{(\ln x)y} = (\ln x)\, e^{(\ln x)y} = (\ln x)\, x^y$, we have

$$f'(c) = 1 - \tfrac{d}{dc} x^{x^c} = 1 - (\ln x) x^{x^c} \tfrac{d}{dc} x^c = 1 - (\ln x)^2 x^{x^c} x^c$$

Write $y = x^c$. Then

$$\tfrac{d}{dy}(y x^y) = x^y + y(\ln x) x^y = x^y(1 + y \ln x) = x^y(1 - y|\ln x|)$$

So $y x^y$ increases as y increases for $y < \frac{1}{|\ln x|}$ and decreases as y increases for $y > \frac{1}{|\ln x|}$ and the maximum value of $y x^y$ is

$$y x^y \Big|_{y=\frac{1}{|\ln x|}} = \frac{1}{|\ln x|} x^{\frac{1}{|\ln x|}} = \frac{1}{|\ln x|} e^{\frac{\ln x}{|\ln x|}} = \frac{1}{e|\ln x|}$$

So the maximum value of $(\ln x)^2 x^{x^c} x^c$ is

$$(\ln x)^2 x^{x^c} x^c \Big|_{x^c=\frac{1}{|\ln x|}} = (\ln x)^2 x^y y \Big|_{y=\frac{1}{|\ln x|}} = \frac{|\ln x|}{e} = |\ln x^{1/e}|$$

Recall that

$$\lim_{c\to 0} x^c = 1 \qquad \lim_{c\to 0} x^{x^c} = x \qquad \lim_{c\to 1} x^c = x \qquad \lim_{c\to 1} x^{x^c} = x^x$$

At this stage we know the following properties of $f'(c)$:

- $f'(0) = 1 - x(\ln x)^2 > 0$ (since $x(\ln x)^2 \le x(\ln x)^2\big|_{x=e^{-2}} = \left(\frac{2}{e}\right)^2 < 1$)
- $f'(c)$ decreases as c increases until $x^c = \frac{1}{|\ln x|}$
- $f'(c)$ bottoms out at $1 - |\ln x^{1/e}|$ when $x^c = \frac{1}{|\ln x|}$
- $f'(c)$ then increases as c increases until $c = 1$
- $f'(1) = 1 - x x^x (\ln x)^2 > 0$ (since $x^x < 1$ and, as above, $x(\ln x)^2 < 1$)

So as long as $|\ln x^{1/e}| < 1$, i.e. $x^{1/e} > \frac{1}{e}$, i.e. $x > \frac{1}{e^e}$, we have $f'(c) > 0$ for all $0 < c < 1$. If $|\ln x^{1/e}| = 1$, i.e. $x = \frac{1}{e^e}$, then $f'(c) > 0$ except for a single value of c where $f'(c) = 0$. In both cases the function $f(c)$ is strictly increasing and the equation $c = x^{x^c}$ has $c = a$ as its only solution. ■

Step 7. We finally finish off the case $0 < x < \frac{1}{e^e}$, showing that for such x's, the sequence $x_1, x_2, x_3, \cdots$ diverges.

Proof: If the sequence were to converge, the limit would have to be the a determined by $x = a^{1/a}$. So for large n, x_n would have to be very near a. So x_n would have to be of the form $a(1+\xi_n)$ with ξ_n close to zero. In terms of these new ξ_n variables, the recursion rule $x_{n+1} = x^{x_n}$ becomes

$$a(1+\xi_{n+1}) = x_{n+1} = x^{x_n} = a^{\frac{1}{a}\ a(1+\xi_n)} = a^{1+\xi_n} = a\,a^{\xi_n}$$

or

$$\xi_{n+1} = a^{\xi_n} - 1 = e^{\xi_n \ln a} - 1$$

Using the Taylor expansion $e^z = 1 + z + \frac{1}{2}e^c z^2$, for some c between 0 and z,

$$\xi_{n+1} = (\ln a)\xi_n + \tfrac{1}{2}e^c[(\ln a)\xi_n]^2 = \left[1 + \tfrac{1}{2}e^c(\ln a)\xi_n\right]\ (\ln a)\,\xi_n$$

Now recall that we are considering the case $0 < x < \frac{1}{e^e}$, which corresponds to $0 < a < \frac{1}{e}$ or $\ln a < -1$. If the sequence were to converge, ξ_n would have to tend to zero as $n \to \infty$, which would force $\frac{1}{2}e^c(\ln a)\xi_n$ to tend to zero too (since c has to be between 0 and $\xi_n \ln a$) and $\left[1 + \frac{1}{2}e^c(\ln a)\xi_n\right](\ln a)$ to tend to $\ln a$. In particular, $\left|\left[1 + \frac{1}{2}e^c(\ln a)\xi_n\right](\ln a)\right|$ would have to be bigger than 1 for all large enough n and we would have to have

$$|\xi_{n+1}| > |\xi_n|$$

for all large enough n. This prevents ξ_n from tending to zero. (The above argument is called a (linear) stability analysis.) ■

