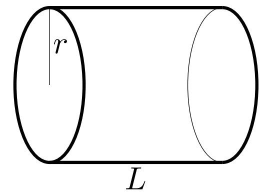
The Fuel Tank

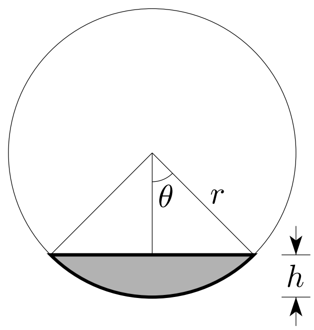
Question

Consider a cylindrical fuel tank of radius r and length L, that is lying on its side. Suppose that fuel is being pumped into the tank at a rate q. At what rate is the fuel level rising?



Solution

Here is an end view of the tank. The shaded part of the circle is filled with fuel.

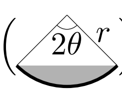
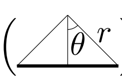


Let us denote by $V(t)$ the volume of fuel in the tank at time t and by $h(t)$ the fuel level at time t. We have been told that $V'(t) = q$ and have been asked to determine $h'(t)$. It is possible to do so by finding a formula relating $V(t)$ and $h(t)$. But it is computationally easier to first find a formula relating V and the angle θ shown in the end view. Once we know $\theta'(t)$, we can easily obtain $h'(t)$ by differentiating $h = r - r\cos\theta$.

The volume of fuel is L times the cross–sectional area filled by the fuel. That is,

$V = L \times$ Area()

While we do not have a canned formula for the area of a chord of a circle like this, it is easy to express the area of the chord in terms of two areas that we can compute.

$V = L \times$ Area() $= L \times \Big[$ Area() $-$ Area() $\Big]$ (1)

The piece of pie on the right hand side of formula (1) is the fraction $\frac{2\theta}{2\pi}$ of the full circle, so its area is $\frac{2\theta}{2\pi}\pi r^2 = \theta r^2$. The triangle on the right hand side of formula (1) has height $r\cos\theta$ and base $2r\sin\theta$ and hence has area $\frac{1}{2}(r\cos\theta)(2r\sin\theta) = r^2\sin\theta\cos\theta$. Subbing these two areas into formula (1) gives

$$V(t) = L \times \Big[\theta r^2 - r^2\sin\theta\cos\theta\Big] = Lr^2\big[\theta(t) - \sin\theta(t)\cos\theta(t)\big] \tag{2}$$

Since

$$\begin{aligned}
\tfrac{d}{dt}\sin\theta(t)\cos\theta(t) &= \cos\theta(t)\tfrac{d}{dt}\sin\theta(t) + \sin\theta(t)\tfrac{d}{dt}\cos\theta(t) \\
&= \cos\theta(t)\big[\theta'(t)\cos\theta(t)\big] + \sin\theta(t)\big[-\theta'(t)\sin\theta(t)\big] \\
&= \theta'(t)\big[\cos^2\theta(t) - \sin^2\theta(t)\big]
\end{aligned}$$

differentiating both sides of formula (2) with respect to t gives

$$\begin{aligned} V'(t) &= Lr^2\big[\theta'(t) - \theta'(t)\cos^2\theta(t) + \theta'(t)\sin^2\theta(t)\big] = Lr^2\big[1 - \cos^2\theta(t) + \sin^2\theta(t)\big]\theta'(t) \\ &= 2\theta'(t)Lr^2\sin^2\theta(t) \end{aligned}$$

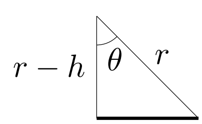
In the last step, I just subbed in $1 = \sin^2\theta + \cos^2\theta$. We can now easily solve for $\theta'(t)$.

$$\theta'(t) = \frac{V'(t)}{2Lr^2\sin^2\theta(t)} = \frac{q}{2Lr^2\sin^2\theta(t)}$$

Since $h(t) = r - r\cos\theta(t)$

$$h'(t) = r\theta'(t)\sin\theta(t) = \frac{qr\sin\theta(t)}{2Lr^2\sin^2\theta(t)} = \frac{q}{2Lr\sin\theta(t)}$$

We have now found $h'(t)$. It is more useful to express the answer in terms of h, rather than θ. From the triangle



and Pythagorous we have

$$\sin\theta = \frac{\sqrt{r^2 - (r-h)^2}}{r} = \frac{\sqrt{2rh - h^2}}{r}$$

and hence

$$h'(t) = \boxed{\frac{q}{2L\sqrt{2rh(t) - h(t)^2}}}$$