Inequalities

Every real number a is designated as being either positive ($a > 0$) or zero ($a = 0$) or negative ($a < 0$). By definition $a > b$ if and only if $a - b > 0$ and $a < b$ if and only if $a - b < 0$. Hence, given any two real numbers a and b,

$$\text{either} \qquad a > b \qquad \text{or} \qquad a = b \qquad \text{or} \qquad a < b$$

By definition $a \le b$ if either $a = b$ or $a < b$, and $a \ge b$ if either $a = b$ or $a > b$.

Properties of Inequalities

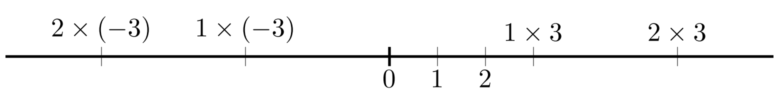
Let a, b and c be real numbers.

1) If $a < b$, then $a + c < b + c$ and $a - c < b - c$.

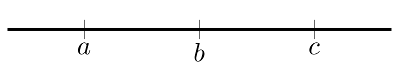
2) If $a < b$ and $c > 0$ then $ac < bc$. But if $a < b$ and $c < 0$ then $ac > bc$. *Note* that mulitplication by a negative number flips the inequality.

   Proof: If $a < b$ and $c > 0$, then, by definition, $b - a > 0$ and $c > 0$. But the product of any two positive numbers is positive, so $bc - ac = c(b - a) > 0$, which implies $ac < bc$. On the other hand, the product of a positive number and a negative number is a negative number. So if $a < b$ and $c < 0$, we have $b - a > 0$ and $c < 0$ implying $bc - ac = c(b - a) < 0$ so that $ac > bc$.

   Example: Here is an example with $a = 1$ and $b = 2$.
   - For $c = 3$, $ac = (1)(3) = 3 < bc = (2)(3) = 6$.
   - For $c = -3$, $ac = (1)(-3) = -3 > bc = (2)(-3) = -6$.
   - For $c = 0$, $ac = (1)(0) = 0 = bc = (2)(0) = 0$.



3) If $a < b$ and $b < c$, then $a < c$.

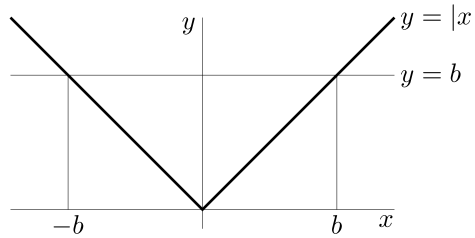


4) If $a > 0$, then $\frac{1}{a} > 0$. If $0 < a < b$, then $\frac{1}{b} < \frac{1}{a}$.
5) If $0 \le a < b$, then $a^2 < b^2$.

   Proof: This is an easy consequence of Properties (2) and (3). By Property (2) with $c = a$, $a^2 \le ab$. By Property (2) with $c = b$, $ab < b^2$. By Property (3), $a^2 \le ab < b^2$.
6) Recall that the absolute value, $|a|$, of a real number a is defined by

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a \le 0 \end{cases}$$

   For example $|2| = |-2| = 2$. We have $|a| < b$ if and only if $-b < a < b$. This is most easily seen by looking at the graph $y = |x|$, below. To determine which values of x obey, $|x| < b$, we have to determine which points on $y = |x|$ have $y < b$. That is, we have to determine which points on $y = |x|$ lie below the line $y = b$. The point $(x, |x|)$ on the graph $y = |x|$ lies below the line $y = b$ if and only if x is between $-b$ and b.



Example 1 Find all real numbers x obeying $|\sqrt{x} - 3| < 10^{-6}$.

Solution.

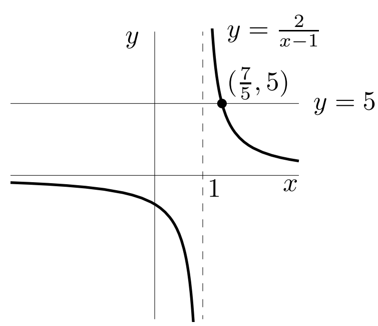
$$\begin{aligned} |\sqrt{x} - 3| < 10^{-6} &\iff -10^{-6} < \sqrt{x} - 3 < 10^{-6} && \text{by Property (6)} \\ &\iff 3 - 10^{-6} < \sqrt{x} < 3 + 10^{-6} && \text{by Property (1) with } c = 3 \\ &\iff (3 - 10^{-6})^2 < x < (3 + 10^{-6})^2 && \text{by Property (5)} \end{aligned}$$

Example 2 Find all real numbers x obeying $\frac{2}{x-1} \ge 5$.

Solution. If $x - 1 < 0$, then $\frac{2}{x-1} < 0$ while $5 > 0$. So no x's with $x - 1 < 0$, that is $x < 1$, are allowed.
If $x - 1 = 0$, that is $x = 1$, then $\frac{2}{x-1}$ is not defined. In this case, we say the inequality $\frac{2}{x-1} \ge 5$ is not satisified.
If $x - 1 > 0$, that is $x > 1$,

$$\begin{aligned} \tfrac{2}{x-1} \ge 5 &\iff 2 \ge 5(x-1) && \text{by Property (2) with } c = x - 1 > 0 \\ &\iff \tfrac{2}{5} \ge x - 1 && \text{by Property (2) with } c = \tfrac{1}{5} \\ &\iff \tfrac{7}{5} \ge x && \text{by Property (1) with } c = 1 \end{aligned}$$

So the allowed x's are $\boxed{1 < x \le \frac{7}{5}}$. We can also see this from the graph below.



The interval of all real numbers x that obey $1 < x \le \frac{7}{5}$ is denoted $\big(1, \frac{7}{5}\big]$. The round bracket in "$\big(1$" signifies that 1 is not included in the interval while the square bracket in "$\frac{7}{5}\big]$" signifies that $\frac{7}{5}$ is included in the interval.

Example 3 Find all real numbers x obeying $-x^2 + 3x + 4 > 0$.

Solution. Factoring

$$-x^2 + 3x + 4 = -(x^2 - 3x - 4) = -(x - 4)(x + 1)$$

If the two factors $(x - 4)$ and $(x + 1)$ have the same sign (either both positive or both negative) then $(x-4)(x+1)$ will be positive and $-x^2+3x+4 = -(x-4)(x+1)$ will be negative. Hence we need the two factors $(x-4)$ and $(x+1)$ to have opposite sign. That is, we need $\boxed{-1 < x < 4}$. This is consistent with the graph below.