Limits

Notation.

- $\mathbb{N}$ is the set $\{1, 2, 3, \cdots\}$ of all natural numbers
- $\mathbb{R}$ is the set of all real numbers
- $\forall$ is read "for all"
- $\exists$ is read "there exists"
- $\in$ is read "element of"
- $\notin$ is read "not an element of"
- $\{\ A \bigm| B\ \}$ is read "the set of all A such that B"
- If S is a set and T is a subset of S, then $S \setminus T$ is $\{\ x \in S \bigm| x \notin T\ \}$, the set S with the elements of T removed. For example, $\mathbb{R} \setminus \{a\} = \{\ x \in \mathbb{R} \bigm| x \neq a\ \}$.
- If S and T are sets, then $f : S \to T$ means that f is a function which assigns to each element of S an element of T.
- $[a, b] = \{\ x \in \mathbb{R} \bigm| a \le x \le b\ \}$
  $(a, b] = \{\ x \in \mathbb{R} \bigm| a < x \le b\ \}$
  $[a, b) = \{\ x \in \mathbb{R} \bigm| a \le x < b\ \}$
  $(a, b) = \{\ x \in \mathbb{R} \bigm| a < x < b\ \}$

Roughly speaking, $\lim\limits_{x\to a} f(x) = L$ means that $f(x)$ approachs L as x approachs a. Here is the precise definition of limit.

Definition 1 (Limit) Let $a, L \in \mathbb{R}$ and $f : \mathbb{R} \setminus \{a\} \to \mathbb{R}$. Then $\lim\limits_{x\to a} f(x) = L$ if

$$\forall\, \varepsilon > 0\ \ \exists\, \delta > 0\ \text{ such that } \quad \big|f(x) - L\big| < \varepsilon \text{ whenever } 0 < |x - a| < \delta$$

Remark 2

(a) Here is what that definition of "$\lim\limits_{x\to a} f(x) = L$" says. Suppose you have a magic microscope whose magnification can be set as high as you like. Suppose that when the magnification is set to $\frac{1}{\varepsilon}$, you can only see those points whose distance from L is less than ε. The definition says that no matter how high you set the magnification, (i.e. no matter how small you set $\varepsilon > 0$), you will be able to see $f(x)$ whenever x is close enough to a. (If the distance from x to a is less than δ, then you will certainly see $f(x)$.)

(b) Definition 1, of $\lim\limits_{x\to a} f(x)$, is set up so that the function $f(x)$ is never evaluated at $x = a$. Indeed $f(x)$ need not even be defined at $x = a$. This is exactly what happens in the definition of the derivative $h'(a) = \lim\limits_{x\to a} \frac{h(x)-h(a)}{x-a}$. (In this case $f(x) = \frac{h(x)-h(a)}{x-a}$.)

Example 3 In Example 2 of the notes "A Little Logic" we saw that the statement

$$\forall\,\varepsilon > 0 \;\; \exists\,\delta > 0 \;\text{ such that }\; x^2 < \varepsilon \text{ whenever } |x| < \delta$$

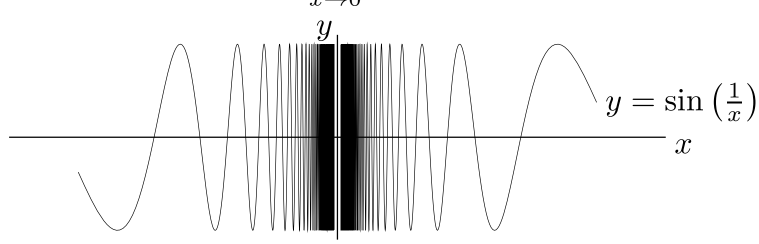
is true. This statement implies that the statement

$$\forall\,\varepsilon > 0 \;\; \exists\,\delta > 0 \;\text{ such that }\; \big|f(x) - L\big| < \varepsilon \text{ whenever } 0 < |x - a| < \delta \tag{1}$$

is true when $f(x) = x^2$, $L = 0$ and $a = 0$. Of course (1) is exactly the definition of $\lim\limits_{x\to a} f(x) = L$ in Definition 1, so

$$\lim_{x\to 0} x^2 = 0$$

Example 4 In this example, we consider $\lim\limits_{x\to 0} \sin\frac{1}{x}$. From the graph



we would guess that $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist. So we fix any real number L and show that $\lim\limits_{x\to 0} \sin\frac{1}{x}$ cannot be L. To do so, let U be the statement

$$\forall\,\varepsilon > 0 \;\; \exists\,\delta > 0 \;\text{ such that }\; |\sin\tfrac{1}{x} - L| < \varepsilon \text{ whenever } 0 < |x| < \delta$$

We wish to show that U is false. To do so, we split it up into bite sized pieces, working from right to left, just as we did in the notes "A Little Logic". Precisely, we let (see (2) below)

- $S(\delta,\varepsilon)$ be the statement "$|\sin\frac{1}{x} - L| < \varepsilon$ whenever $0 < |x| < \delta$", and
- $T(\varepsilon)$ be the statement "$\exists\,\delta > 0$ such that $S(\delta,\varepsilon)$" or

$$\exists\,\delta > 0 \;\text{ such that }\; |\sin\tfrac{1}{x} - L| < \varepsilon \text{ whenever } 0 < |x| < \delta$$

- Then U is the statement "$\forall\,\varepsilon > 0 \;\; T(\varepsilon)$".

$$\overbrace{\forall\,\varepsilon > 0 \;\; \overbrace{\exists\,\delta > 0 \;\text{ such that }\; \overbrace{|\sin\tfrac{1}{x} - L| < \varepsilon \text{ whenever } 0 < |x| < \delta}^{S(\delta,\varepsilon)}}^{T(\varepsilon)}}^{U} \tag{2}$$

We analyze U using the same three steps as in Example 2 of "A Little Logic".

- *Step 1:* We find all δ's and ε's for which $S(\delta,\varepsilon)$ is true. Fix any $\varepsilon > 0$ and any $\delta > 0$. The statement $S(\delta,\varepsilon)$ is true if all values of $\sin\frac{1}{x}$, with $0 < |x| < \delta$, lie in the interval $(L-\varepsilon, L+\varepsilon)$. As x runs over the interval $(0,\delta)$, (so that, in particular, $0 < |x| < \delta$) $\frac{1}{x}$ covers the set $\left(\frac{1}{\delta},\infty\right)$. This contains many intervals of length 2π and hence many periods of sin. So, as x runs over the interval $(0,\delta)$, $\sin\frac{1}{x}$ covers all of $[-1,1]$. So $S(\delta,\varepsilon)$ is true if and only if the interval $[-1,1]$ is contained in the interval $(L-\varepsilon, L+\varepsilon)$. In particular, when $\varepsilon < 1$, the interval $(L-\varepsilon, L+\varepsilon)$, which has length 2ε, is shorter than $[-1,1]$ and cannot contain it, so that $S(\delta,\varepsilon)$ is false.
- *Step 2:* Because $S(\delta,\varepsilon)$is false for all $\delta > 0$ when $\varepsilon < 1$, $T(\varepsilon)$ is false for all $\varepsilon < 1$.
- *Step 3:* We conclude that U is false since, as we have just seen, $T(\varepsilon)$ is false for at least one $\varepsilon > 0$. For example $T\left(\frac{1}{2}\right)$ is false.

In conclusion, $\sin\frac{1}{x}$ has no limit as $x \to 0$.

Example 5 Consider $\lim\limits_{x\to 2}\frac{1}{x}$. We would of course expect that $\lim\limits_{x\to 2}\frac{1}{x} = \frac{1}{2}$. In this example we verify directly, using Definition 1, that this is the case. In other words, we verify that the statement

$$\forall\,\varepsilon > 0\ \ \exists\,\delta > 0\ \text{ such that }\quad \left|\tfrac{1}{x} - \tfrac{1}{2}\right| < \varepsilon \text{ whenever } 0 < |x-2| < \delta$$

is true. To do so, it suffices for us to fix any $\varepsilon > 0$ and then find a $\delta > 0$ such that $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$ for all $|x-2| < \delta$ and that's what I'll do.

I shall pick a δ that is smaller than 1. Then, if $|x-2| < \delta$, we have $|x-2| < 1$ so that $1 < x < 3$ and

$$\left|\tfrac{1}{x} - \tfrac{1}{2}\right| = \left|\tfrac{2-x}{2x}\right| = \tfrac{|2-x|}{2|x|} < \tfrac{|2-x|}{2\times 1}$$

since $x > 1$. As

$$\tfrac{|2-x|}{2\times 1} < \varepsilon \qquad \text{if} \qquad |2-x| < 2\varepsilon$$

we have that

$$|2-x| < \min\{1, 2\varepsilon\} \quad \Rightarrow \quad \left|\tfrac{1}{x} - \tfrac{1}{2}\right| < \varepsilon$$

Hence $\delta = \min\{1, 2\varepsilon\}$ does the trick.

Example 6 Again consider $\lim\limits_{x\to 2}\frac{1}{x}$. But this time suppose that someone claims that $\lim\limits_{x\to 2}\frac{1}{x} = \frac{1}{3}$. We want to disprove the claim. That is, we wish to show that the statement

$$\forall\,\varepsilon > 0\ \ \exists\,\delta > 0\ \text{ such that }\quad \left|\tfrac{1}{x} - \tfrac{1}{3}\right| < \varepsilon \text{ whenever } 0 < |x-2| < \delta \tag{3}$$

is false. To do so, it suffices to find one "bad" ε for which the statement "$\left|\frac{1}{x} - \frac{1}{3}\right| < \varepsilon$ whenever $0 < |x-2| < \delta$" is false for all $\delta > 0$. To guess a "bad" ε observe that when x is very close to 2, we have $\left|\frac{1}{x}-\frac{1}{3}\right| \approx \left|\frac{1}{2}-\frac{1}{3}\right| = \frac{1}{6}$. So I pick $\varepsilon = \frac{1}{10}$ (any other $\varepsilon < \frac{1}{6}$ would work too) and I shall show that, for this ε there is no $\delta > 0$ such that $\left|\frac{1}{x}-\frac{1}{3}\right| < \frac{1}{10}$ for all $0 < |x-2| < \delta$.

I'll now show that all x's "sufficiently close" to 2 obey $\left|\frac{1}{x}-\frac{1}{3}\right| > \frac{1}{10}$. For "sufficiently close", let's try "distance less than 0.01". (There's nothing magical about the number 0.01. If it doesn't work, we'll just try again with a smaller distance.) For all x obeying $|x-2| < 0.01$ we have $1.99 < x < 2.01$ and hence

$$\left|\frac{1}{x}-\frac{1}{3}\right| = \left|\frac{3-x}{3x}\right| \geq \frac{3-2.01}{3\times 2.01} = \frac{0.99}{6.03} > \frac{1}{10} = \varepsilon$$

No matter what $\delta > 0$ we pick, there will be some x's obeying $|x-2| < \delta$ that also obey $|x-2| < 0.01$ and hence that also obey $\left|\frac{1}{x}-\frac{1}{3}\right| > \frac{1}{10} = \varepsilon$. So (3) is false.

Example 7 In this example, we fix any real number θ and show that

$$\lim_{h\to 0} \sin(\theta + h) = \sin\theta$$

by verifying directly that the statement

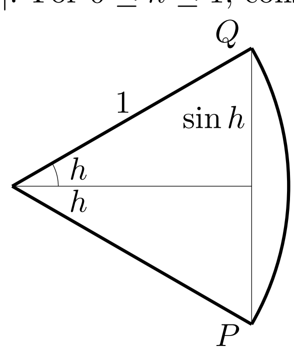
$$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \;\text{ such that }\quad \big|\sin(\theta+h) - \sin\theta\big| < \varepsilon \text{ whenever } 0 < |h| < \delta$$

is true. To do so, it suffices for us to fix any $\varepsilon > 0$ and then find a $\delta > 0$ such that $\big|\sin(\theta+h) - \sin\theta\big| < \varepsilon$ for all $|h| < \delta$ and that's what we'll do.

We shall use the fact that

$$|\sin h| \leq |h| \tag{4}$$

for all h, provided the angle h is given in radians. First, we verify this fact. If $|h| \geq 1$, (4) is obvious because $|\sin h| \leq 1 \leq |h|$. For $0 \leq h \leq 1$, consider the figure



The arc from P to Q is part of a circle of radius one. Because the arc subtends the angle $2h$, it is the fraction $\frac{2h}{2\pi}$ of the circle and so has length $\frac{2h}{2\pi} \times 2\pi = 2h$. The straight line from P to Q has length $2\sin h$. Because the straight line from P to Q is shorter than the arc from P to Q, we have $2\sin h \leq 2h$. For $-1 \leq h < 0$, $\sin h$ is negative so that

$$|\sin h| = -\sin h = \sin(-h)$$

But $-h$ is between 0 and 1, so we already know that $\sin(-h) \le -h = |h|$. This completes the verification of (3).

Now back to the main problem. By the trig identities $\sin(a+b) = \sin a \cos b + \cos a \sin b$ and $\cos 2a = 1 - 2\sin^2 a$,

$$\begin{aligned} \sin(\theta+h) - \sin\theta &= \sin\theta\cos h + \cos\theta \sin h - \sin\theta \\ &= \big(\cos h - 1\big)\sin\theta + \cos\theta\sin h \\ &= -2\sin^2\tfrac{h}{2}\sin\theta + \cos\theta\sin h \end{aligned}$$

Since $|\sin h| \le |h|$, $\big|\sin\frac{h}{2}\big| \le \big|\frac{h}{2}\big|$, $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$

$$\begin{aligned} \big|\sin(\theta+h) - \sin\theta\big| &\le 2\big|\tfrac{h}{2}\big|^2 \times 1 + 1 \times |h| \\ &= |h| + \tfrac{1}{2}|h|^2 \end{aligned}$$

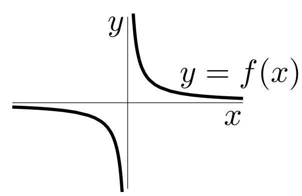
If we pick $\delta < 1$, then $|h| < \delta < 1$ implies $|h|^2 = |h|\,|h| < |h|$ and

$$\big|\sin(\theta+h) - \sin\theta\big| \le |h| + \tfrac{1}{2}|h| = \tfrac{3}{2}|h| < \varepsilon \text{ if } |h| < \tfrac{2}{3}\varepsilon$$

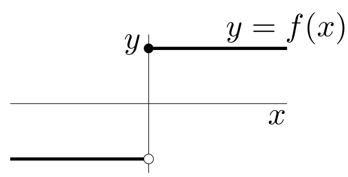
Hence $\delta = \min\big\{1, \frac{2}{3}\varepsilon\big\}$ does the trick.

Example 8 We saw, in Example 4, that $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist. Here are two more $f(x)$'s for which the limit $\lim\limits_{x\to 0} f(x)$ does not exist.

$$f(x) = \frac{1}{x}$$



$$f(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$$



Definition 9 (Limit Variations)

(a) Let $a, L \in \mathbb{R}$, and $f : \mathbb{R} \setminus \{a\} \to \mathbb{R}$. Then

$\lim\limits_{x\to a+} f(x) = L$ if and only if

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \;\; \text{such that} \quad \big|f(x) - L\big| < \varepsilon \text{ whenever } a < x < a + \delta$$

$\lim\limits_{x\to a-} f(x) = L$ if and only if

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \;\; \text{such that} \quad \big|f(x) - L\big| < \varepsilon \text{ whenever } a - \delta < x < a$$

$\lim_{x\to a} f(x) = \infty$ if and only if

$$\forall\, Y > 0 \;\; \exists\, \delta > 0 \;\text{ such that }\quad f(x) > Y \text{ whenever } 0 < |x-a| < \delta$$

$\lim_{x\to a} f(x) = -\infty$ if and only if

$$\forall\, Y > 0 \;\; \exists\, \delta > 0 \;\text{ such that }\quad f(x) < -Y \text{ whenever } 0 < |x-a| < \delta$$

$\lim_{x\to \infty} f(x) = L$ if and only if

$$\forall\, \varepsilon > 0 \;\; \exists\, X > 0 \;\text{ such that }\quad \big|f(x) - L\big| < \varepsilon \text{ whenever } x > X$$

$\lim_{x\to -\infty} f(x) = L$ if and only if

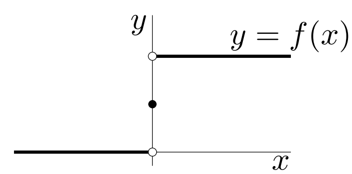
$$\forall\, \varepsilon > 0 \;\; \exists\, X > 0 \;\text{ such that }\quad \big|f(x) - L\big| < \varepsilon \text{ whenever } x < -X$$

(b) Let $f : \mathbb{R} \to \mathbb{R}$. Then f is continuous at $a \in \mathbb{R}$ if $\lim_{x\to a} f(x) = f(a)$ and f is continous on $\mathbb{R}$ if it is continuous at every $a \in \mathbb{R}$.

Remark 10

(a) On a handwaving level, $\lim_{x\to a+} f(x) = L$ means that $f(x)$ approachs L as x approachs a from the right and $\lim_{x\to a-} f(x) = L$ means that $f(x)$ approachs L as x approachs a from the left. For example

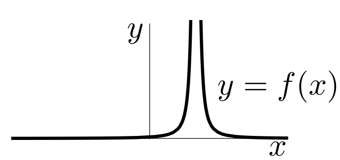
$$f(x) = \left\{ \begin{matrix} 1 & \text{if } x > 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{if } x < 0 \end{matrix} \right\} \qquad \lim_{x\to 0+} f(x) = 1 \qquad \lim_{x\to 0-} f(x) = 0$$



Of course "$\lim_{x\to a} f(x) = L$" is equivalent to "$\lim_{x\to a+} f(x) = L$ and $\lim_{x\to a-} f(x) = L$".

(b) On a handwaving level, $\lim_{x\to a} f(x) = \infty$ means that, as x approachs a, $f(x)$ eventually gets (and remains) larger than any possible positive number. We say "$f(x)$ tends to infinity". For example

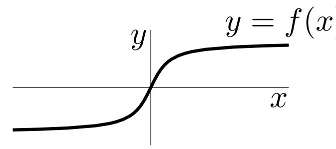
$$f(x) = \tfrac{1}{(x-1)^2} \qquad \lim_{x\to 1} f(x) = \infty$$



(c) On a handwaving level, $\lim_{x\to \infty} f(x) = L$ means that, as x tends to infinity (i.e. gets and remains bigger than any possible positive number), $f(x)$ tends to L and $\lim_{x\to -\infty} f(x) = L$

means that, as x tends to minus infinity (i.e. gets and remains more negative than any possible negative number), $f(x)$ tends to L. For example

$$f(x) = \arctan x \qquad \lim_{x\to\infty} f(x) = \tfrac{\pi}{2} \qquad \lim_{x\to-\infty} f(x) = -\tfrac{\pi}{2}$$



(By $\arctan x$, we mean the unique $\theta \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$ that obeys $\tan\theta = x$.)

As we have seen, evaluating limits by directly verifying Definition 1 gets ugly and hard very quickly. It is much more efficient to have a list of simple limits that we already know together with a toolbox that allows us to build complicated limits out of simple ones. As examples of known simple limits, it is trivial that, for any $a, b \in \mathbb{R}$,

$$\lim_{x\to a} b = b \qquad \lim_{x\to a} x = a$$

and we have already seen, in Example 7, that, for any $a \in \mathbb{R}$,

$$\lim_{x\to a} \sin x = \sin a$$

The following theorem provides a toolbox.

Theorem 11 *Let $a, b \in \mathbb{R}$, $F, G, \Gamma \in \mathbb{R}$ and*

$$f, g : \mathbb{R} \setminus \{a\} \to \mathbb{R} \qquad X : \mathbb{R} \setminus \{b\} \to \mathbb{R} \setminus \{a\} \qquad \gamma : \mathbb{R} \to \mathbb{R}$$

Assume that

$$\lim_{x\to a} f(x) = F \qquad \lim_{x\to a} g(x) = G \qquad \lim_{y\to b} X(y) = a \qquad \lim_{t\to F} \gamma(t) = \gamma(F) = \Gamma$$

Then

(a) $\displaystyle\lim_{x\to a} \big[f(x) + g(x)\big] = F + G$

(b) $\displaystyle\lim_{x\to a} f(x)g(x) = FG$

(c) $\displaystyle\lim_{x\to a} \tfrac{f(x)}{g(x)} = \tfrac{F}{G}$ *if* $G \neq 0$

(d) $\displaystyle\lim_{y\to b} f\big(X(y)\big) = F$

(e) $\displaystyle\lim_{x\to a} \gamma\big(f(x)\big) = \Gamma$

Proof: Note that the ε and δ in "$\forall\,\varepsilon>0\ \exists\,\delta>0$ such that $S(\delta,\varepsilon)$" are dummy variables, just as x is a dummy variable in $\int_0^1 x\,dx$. You may replace ε and δ by whatever symbols you like. The hypotheses of this theorem say that

$$\forall\,\varepsilon_f>0\ \ \exists\,\delta_f>0\ \text{ such that } |f(x)-F|<\varepsilon_f \text{ whenever } 0<|x-a|<\delta_f \tag{5}$$

$$\forall\,\varepsilon_g>0\ \ \exists\,\delta_g>0\ \text{ such that } |g(x)-G|<\varepsilon_g \text{ whenever } 0<|x-a|<\delta_g \tag{6}$$

$$\forall\,\varepsilon_X>0\ \ \exists\,\delta_X>0\ \text{ such that } |X(y)-a|<\varepsilon_X \text{ whenever } 0<|y-b|<\delta_X \tag{7}$$

$$\forall\,\varepsilon_\gamma>0\ \ \exists\,\delta_\gamma>0\ \text{ such that } |\gamma(t)-\Gamma|<\varepsilon_\gamma \text{ whenever } 0<|t-F|<\delta_\gamma \tag{8}$$

(a) We are to prove that

$$\forall\,\varepsilon>0\ \ \exists\,\delta>0\ \text{ such that } |f(x)+g(x)-F-G|<\varepsilon \text{ whenever } 0<|x-a|<\delta$$

So pick any $\varepsilon>0$. We must prove that there is a $\delta>0$ such that

$$|f(x)+g(x)-F-G|<\varepsilon \text{ whenever } 0<|x-a|<\delta$$

Observe that

$$|f(x)+g(x)-F-G|=\big|[f(x)-F]+[g(x)-G]\big|\le|f(x)-F|+|g(x)-G|$$

Set $\varepsilon_1=\frac{\varepsilon}{2}$ and $\varepsilon_2=\frac{\varepsilon}{2}$. By (5) with $\varepsilon_f=\varepsilon_1$ and (6) with $\varepsilon_g=\varepsilon_2$,

$$\exists\,\delta_1>0\ \text{ such that } |f(x)-F|<\varepsilon_1 \text{ whenever } 0<|x-a|<\delta_1$$

$$\exists\,\delta_2>0\ \text{ such that } |g(x)-G|<\varepsilon_2 \text{ whenever } 0<|x-a|<\delta_2$$

Choose $\delta=\min\{\delta_1,\delta_2\}$. Then whenever $0<|x-a|<\delta$ we also have $0<|x-a|<\delta_1$ and $0<|x-a|<\delta_2$ so that

$$|f(x)+g(x)-F-G|\le|f(x)-F|+|g(x)-G|<\varepsilon_1+\varepsilon_2=\varepsilon$$

(b) We are to prove that

$$\forall\,\varepsilon>0\ \ \exists\,\delta>0\ \text{ such that } |f(x)g(x)-FG|<\varepsilon \text{ whenever } 0<|x-a|<\delta$$

So pick any $\varepsilon>0$. We must prove that there is a $\delta>0$ such that

$$|f(x)g(x)-FG|<\varepsilon \text{ whenever } 0<|x-a|<\delta$$

Observe that

$$|f(x)g(x) - FG| = \big|[[f(x) - F]g(x) + F[g(x) - G]\big| \le |f(x) - F|\,|g(x)| + |F|\,|g(x) - G|$$

Set $\varepsilon_1 = \frac{\varepsilon}{2(|G|+1)}$ and $\varepsilon_2 = \frac{\varepsilon}{2(|F|+1)}$. By hypothesis

$$\begin{aligned}
&\exists\,\delta_1 > 0 \ \text{ such that } |f(x) - F| < \varepsilon_1 \text{ whenever } 0 < |x-a| < \delta_1\\
&\exists\,\delta_2 > 0 \ \text{ such that } |g(x) - G| < \varepsilon_2 \text{ whenever } 0 < |x-a| < \delta_2\\
&\exists\,\delta_3 > 0 \ \text{ such that } |g(x) - G| < 1 \text{ whenever } 0 < |x-a| < \delta_3
\end{aligned}$$

Choose $\delta = \min\big\{\delta_1, \delta_2, \delta_3\big\}$. Then whenever $0 < |x-a| < \delta$ we also have $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$ and $0 < |x-a| < \delta_3$ so that

$$\begin{aligned}
|f(x)g(x) - FG| &\le |f(x) - F|\,|g(x)| + |F|\,|g(x) - G|\\
&< \varepsilon_1\,|g(x)| + [|F| + 1]\,\varepsilon_2\\
&= \varepsilon_1\,|g(x) - G + G| + [|F| + 1]\,\varepsilon_2\\
&\le \varepsilon_1\,|g(x) - G| + \varepsilon_1|G| + [|F| + 1]\,\varepsilon_2\\
&\le \varepsilon_1\,\big[1 + |G|\big] + [|F| + 1]\,\varepsilon_2\\
&= \tfrac{\varepsilon}{2(|G|+1)}\,\big[1 + |G|\big] + [|F| + 1]\,\tfrac{\varepsilon}{2(|F|+1)}\\
&= \varepsilon
\end{aligned}$$

(c) We are to prove that

$$\forall\,\varepsilon > 0\ \ \exists\,\delta > 0 \ \text{ such that } \big|\tfrac{f(x)}{g(x)} - \tfrac{F}{G}\big| < \varepsilon \text{ whenever } 0 < |x-a| < \delta$$

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$\big|\tfrac{f(x)}{g(x)} - \tfrac{F}{G}\big| < \varepsilon \text{ whenever } 0 < |x-a| < \delta$$

Set $\varepsilon_1 = \frac{1}{6}|G|\varepsilon$ and $\varepsilon_2 = \frac{G^2}{6(|F|+1)}\varepsilon$. By (5) with $\varepsilon_f = \varepsilon_1$, (6) with $\varepsilon_g = \varepsilon_2$ and (6) with $\varepsilon_g = \frac{1}{2}|G|$,

$$\begin{aligned}
&\exists\,\delta_1 > 0 \ \text{ such that } |f(x) - F| < \varepsilon_1 \text{ whenever } 0 < |x-a| < \delta_1\\
&\exists\,\delta_2 > 0 \ \text{ such that } |g(x) - G| < \varepsilon_2 \text{ whenever } 0 < |x-a| < \delta_2\\
&\exists\,\delta_3 > 0 \ \text{ such that } |g(x) - G| < \tfrac{1}{2}|G| \text{ whenever } 0 < |x-a| < \delta_3
\end{aligned}$$

Choose $\delta = \min\big\{\delta_1, \delta_2, \delta_3\big\}$. Then whenever $0 < |x-a| < \delta$ we also have $0 < |x-a| < \delta_1$ and $0 < |x-a| < \delta_2$ and $0 < |x-a| < \delta_3$ so that

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{F}{G}\right| &= \frac{|f(x)G - Fg(x)|}{|g(x)G|} = \frac{|\{f(x)-F\}G - F\{g(x)-G\}|}{|g(x)G|} \\
&\le \frac{|f(x)-F|\,|G| + |F|\,|g(x)-G|}{|g(x)|\,|G|} \\
&\le \frac{\varepsilon_1\,|G| + |F|\,\varepsilon_2}{\frac{1}{2}|G|\,|G|} \quad \text{since } |g(x)| = \big|g(x)-G+G\big| \ge |G| - |g(x)-G| \ge \tfrac{1}{2}|G| \\
&= \tfrac{1}{6}|G|\varepsilon\,\tfrac{|G|}{G^2/2} + \tfrac{|F|}{G^2/2}\tfrac{G^2}{6(|F|+1)}\varepsilon = \tfrac{\varepsilon}{3} + \tfrac{1}{3}\tfrac{|F|}{|F|+1}\varepsilon \\
&< \varepsilon
\end{aligned}$$

(d) We are to prove that

$$\forall\,\varepsilon > 0\ \ \exists\,\delta > 0\ \text{ such that } \big|f\big(X(y)\big) - F\big| < \varepsilon \text{ whenever } 0 < |y-b| < \delta$$

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$\big|f\big(X(y)\big) - F\big| < \varepsilon \text{ whenever } 0 < |y-b| < \delta$$

By (5) with $\varepsilon_f = \varepsilon$

$$\exists\,\delta_f > 0\ \text{ such that } |f(x) - F| < \varepsilon \text{ whenever } 0 < |x-a| < \delta_f$$

and (7) with $\varepsilon_X = \delta_f$,

$$\exists\,\delta_X > 0\ \text{ such that } |X(y) - a| < \delta_f \text{ whenever } 0 < |y-b| < \delta_X$$

Choosing $\delta = \delta_X$, we have

$$0 < |y-b| < \delta = \delta_X \implies 0 < |X(y) - a| < \delta_f \implies \big|f\big(X(y)\big) - F\big| < \varepsilon$$

(e) has essentially the same proof as part (d). We are to prove that

$$\forall\,\varepsilon > 0\ \ \exists\,\delta > 0\ \text{ such that } \big|\gamma\big(f(x)\big) - \Gamma\big| < \varepsilon \text{ whenever } 0 < |x-a| < \delta$$

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$\big|\gamma\big(f(x)\big) - \Gamma\big| < \varepsilon \text{ whenever } 0 < |x-a| < \delta$$

By (8) with $\varepsilon_\gamma = \varepsilon$ and the hypothesis that $\gamma(F) = \Gamma$

$$\exists\,\delta_\gamma > 0\ \text{ such that } |\gamma(t) - \Gamma| < \varepsilon \text{ whenever } |t - F| < \delta_\gamma$$

By (5) with $\varepsilon_f = \delta_\gamma$

$$\exists\,\delta_f > 0\ \text{ such that } |f(x) - F| < \delta_\gamma \text{ whenever } 0 < |x-a| < \delta_f$$

Choosing $\delta = \delta_f$, we have

$$0 < |x-a| < \delta = \delta_f \implies \big|f(x) - F\big| < \delta_\gamma \implies \big|\gamma\big(f(x)\big) - \Gamma\big| < \varepsilon$$

■

Example 12 As a typical application of Theorem 11, we compute

$$\lim_{x\to 2} \frac{x+\sin\frac{\pi x}{8}}{x^4+1}$$

Here " $\overset{a}{=}$ " means that Theorem 11.a justifies that equality.

$$\begin{aligned}
\lim_{x\to 2}\big(x+\sin\tfrac{\pi x}{8}\big) &\overset{a}{=} \lim_{x\to 2} x + \lim_{x\to 2}\sin\tfrac{\pi x}{8} \\
&\overset{e}{=} \lim_{x\to 2} x + \sin\Big(\lim_{x\to 2}\tfrac{\pi x}{8}\Big) \qquad \text{(by Example 7)} \\
&= 2+\sin\tfrac{\pi}{4} \\
&= 2+\tfrac{1}{\sqrt{2}} \\
\lim_{x\to 2}\big(x^4+1\big) &\overset{a}{=} \lim_{x\to 2} x^4 + \lim_{x\to 2} 1 \\
&\overset{b}{=} \Big(\lim_{x\to 2} x\Big)\Big(\lim_{x\to 2} x\Big)\Big(\lim_{x\to 2} x\Big)\Big(\lim_{x\to 2} x\Big) + 1 \\
&= 2^4+1 \\
\lim_{x\to 2}\frac{x+\sin\frac{\pi x}{8}}{x^4+1} &\overset{c}{=} \frac{\lim_{x\to 2}(x+\sin\frac{\pi x}{8})}{\lim_{x\to 2}(x^4+1)} \\
&= \frac{2+\frac{1}{\sqrt{2}}}{17}
\end{aligned}$$