# A Little Logic

## "There Exists" and "For All"

The symbol $\exists$ is read "there exists" and the symbol $\forall$ is read "for all" (or "for each" or "for every", if it reads better). Let $S(\varepsilon)$ be a statement that contains the parameter $\varepsilon$. For example, $S(\varepsilon)$ might be " $5 < \varepsilon$". Then

- the statement "$\exists\ \varepsilon > 0$ such that $S(\varepsilon)$" is true if there exists at least one $\varepsilon > 0$ such that $S(\varepsilon)$ is true and
- the statement "$\forall\ \varepsilon > 0 \quad S(\varepsilon)$" is true if $S(\varepsilon)$ is true whenever $\varepsilon > 0$.

On the other hand

- the statement "$\exists\ \varepsilon > 0$ such that $S(\varepsilon)$" is false when $S(\varepsilon)$ is false for every $\varepsilon > 0$ and
- the statement "$\forall\ \varepsilon > 0 \quad S(\varepsilon)$" is false when there exists at least one $\varepsilon > 0$ for which $S(\varepsilon)$ is false.

In symbols,

- the statement "$\exists\ \varepsilon > 0$ such that $S(\varepsilon)$" is false when "$\forall \varepsilon > 0 \quad S(\varepsilon)$ is false" and
- the statement "$\forall\ \varepsilon > 0 \quad S(\varepsilon)$" is false when "$\exists\ \varepsilon > 0$ such that $S(\varepsilon)$ is false".

**Example 1** Let $S(\varepsilon)$ be the statement " $5 < \varepsilon$". Then

- the statement "$\exists\ \varepsilon > 0$ such that $S(\varepsilon)$" is true since there does indeed exist an $\varepsilon > 0$, for example $\varepsilon = 6$, such that $S(\varepsilon)$ is true.
- On the other hand, the statement "$\forall\ \varepsilon > 0 \quad S(\varepsilon)$" is false since there is at least one $\varepsilon > 0$, for example $\varepsilon = 4$, such that $S(\varepsilon)$ is is false.

Let $S(\delta, \varepsilon)$ be a statement that contains the two parameters $\delta$ and $\varepsilon$. For example, $S(\delta, \varepsilon)$ might be "if $|x| < \delta$ then $x^2 < \varepsilon$". Define the statement $U$ to be

$$\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \text{ such that } S(\delta, \varepsilon)$$

To analyse $U$, define, for each $\varepsilon > 0$, the statement $T(\varepsilon)$ to be "$\exists\, \delta > 0$ such that $S(\delta, \varepsilon)$". Then $U$ is the statement "$\forall\, \varepsilon > 0 \ \ T(\varepsilon)$" and

- $U$ is true if $T(\varepsilon)$ is true for every $\varepsilon > 0$.
- Given any fixed $\varepsilon_0 > 0$, $T(\varepsilon_0)$ is true if there exists at least one $\delta > 0$ such that $S(\delta, \varepsilon_0)$ is true.
- So, all together, $U$ is true if for each $\varepsilon > 0$, there exists at least one $\delta > 0$ (which may depend on $\varepsilon$) such that $S(\delta, \varepsilon)$ is true.

On the other hand

- $U$ is false if $T(\varepsilon)$ is false for at least one $\varepsilon > 0$.
- Given any fixed $\varepsilon_0 > 0$, $T(\varepsilon_0)$ is false if there does not exist at least one $\delta > 0$ such that $S(\delta, \varepsilon_0)$ is true. That is, if $S(\delta, \varepsilon_0)$ is false for all $\delta > 0$.
- So, all together, $U$ is false if there exists at least one $\varepsilon > 0$, such that $S(\delta, \varepsilon)$ is false for all $\delta > 0$. That is, $U$ is false if the statement

$$\exists\, \varepsilon > 0 \text{ such that } \forall\, \delta > 0\ S(\delta, \varepsilon) \text{ is false}$$

is true.

**Example 2** In this example, we will always assume that $\delta > 0$ and $\varepsilon > 0$. Let $U$ be the statement

$$\forall\, \varepsilon > 0\ \ \exists\, \delta > 0 \ \text{ such that } \ \text{ if } |x| < \delta \text{ then } x^2 < \varepsilon$$

We wish to decide whether or not this statement is true. To do so, we split it up into bite sized pieces, working from right to left. Precisely, we let (see $(*)$ below)

- $S(\delta, \varepsilon)$ be the statement "if $|x| < \delta$ then $x^2 < \varepsilon$", and
- $T(\varepsilon)$ be the statement "$\exists\, \delta > 0$ such that $S(\delta, \varepsilon)$" or

$$\exists\, \delta > 0 \ \text{ such that } \ \text{ if } |x| < \delta \text{ then } x^2 < \varepsilon$$

- Then $U$ is the statement "$\forall\, \varepsilon > 0\ \ T(\varepsilon)$".

$$\overbrace{\forall\, \varepsilon > 0\ \ \overbrace{\exists\, \delta > 0 \ \text{ such that } \ \overbrace{\text{if } |x| < \delta \text{ then } x^2 < \varepsilon}^{S(\delta,\varepsilon)}}^{T(\varepsilon)}}^{U} \qquad (*)$$

We analyze $U$ in three steps.

- *Step 1:* We find all $\delta$'s and $\varepsilon$'s for which $S(\delta, \varepsilon)$ is true. For example, $S(\delta, \varepsilon)$ is true when $\delta = 2$ and $\varepsilon = 2^2 = 4$. That is $S(2, 4)$ is true. On the other hand $S(2, 3)$ is false, because, for example $x = \frac{7}{4} < 2$ but $x^2 = \frac{49}{16} > 3$. In general, as $x$ runs over the interval $-\delta < x < \delta$, $x^2$ covers the set $[0, \delta^2)$. So $S(\delta, \varepsilon)$ is true if and only if the interval $[0, \delta^2)$ is contained in the interval $[0, \varepsilon)$, which is the case if and only if $\delta^2 \le \varepsilon$. So, $S(\delta, \varepsilon)$ is true if and only if $\delta^2 \le \varepsilon$.
- *Step 2:* We find all $\varepsilon$'s for which $T(\varepsilon)$ is true. For example, $T(4)$ is true because when $\varepsilon = 4$, we may choose $\delta = 2$ and then $S(\delta = 2, \varepsilon = 4)$ is true. In fact, $T(\varepsilon)$ is true for every $\varepsilon > 0$, because we may choose $\delta = \sqrt{\varepsilon}$ and then $S\big(\sqrt{\varepsilon}, \varepsilon\big)$ is true.
- *Step 3:* We just conclude that $U$ is true since, as we have just seen, $T(\varepsilon)$ is true for all $\varepsilon > 0$.

**Example 3** In this example, we will again assume that $\delta > 0$ and $\varepsilon > 0$. This time let $U$ be the statement

$$\forall\,\varepsilon > 0 \;\; \exists\,\delta > 0 \;\text{ such that } \;\; \text{if } |x| < \delta \text{ then } 1 + x^2 < \varepsilon$$

We wish to decide whether or not this statement is true. To do so, we again split it up into bite sized pieces, working from right to left. Precisely, we let (see (∗∗) below)

- $S(\delta, \varepsilon)$ be the statement "if $|x| < \delta$ then $1 + x^2 < \varepsilon$", and
- $T(\varepsilon)$ be the statement "$\exists\,\delta > 0$ such that $S(\delta, \varepsilon)$" or

$$\exists\,\delta > 0 \;\text{ such that } \;\; \text{if } |x| < \delta \text{ then } 1 + x^2 < \varepsilon$$

- Then $U$ is the statement "$\forall\,\varepsilon > 0 \;\; T(\varepsilon)$".

$$\overbrace{\forall\,\varepsilon > 0 \;\; \overbrace{\exists\,\delta > 0 \;\text{ such that } \;\; \overbrace{\text{if } |x| < \delta \text{ then } 1 + x^2 < \varepsilon}^{S(\delta,\varepsilon)}}^{T(\varepsilon)}}^{U} \qquad (**)$$

We analyze $U$ using the same three steps as in Example 2.

- *Step 1:* We find all $\delta$'s and $\varepsilon$'s for which $S(\delta, \varepsilon)$ is true. When $x$ runs over the interval $-\delta < x < \delta$, $1 + x^2$ covers the set $[1, 1 + \delta^2)$. Hence $S(\delta, \varepsilon)$ is true if and only if the interval $[1, 1 + \delta^2)$ is contained in the interval $[0, \varepsilon)$, and that is true if and only if $\varepsilon \ge 1 + \delta^2$.
- *Step 2:* We find all $\varepsilon$'s for which $T(\varepsilon)$ is true. Because $S(\delta, \varepsilon)$ is true if and only if $\varepsilon \ge 1 + \delta^2$, the statement $T(\varepsilon)$ is equivalent to "$\exists\,\delta > 0$ such that $\varepsilon \ge 1 + \delta^2$" which is true if and only if $\varepsilon > 1$. (If $\varepsilon > 1$, we may choose $\delta = \sqrt{\varepsilon - 1}$. If $\varepsilon < 1$, no $\delta$ works since $1 + \delta^2$ is always at least 1. If $\varepsilon = 1$, the only $\delta$ which could work is $\delta = 0$, and it does not satisfy the condition $\delta > 0$.)
- *Step 3:* We just conclude that $U$ is false since, as we have just seen, $T(\varepsilon)$ is false for at least one $\varepsilon > 0$. For example $T\big(\frac{1}{2}\big)$ is false.

## Converse, Inverse, Contrapositive

Let $S_1$ and $S_2$ be statements. For example $S_1$ might be "$x$ is a rational number" and $S_2$ might be "$x$ is a real number". Define the statement $T$ to be "If $S_1$ is true then $S_2$ is true.". Then

- the converse of $T$ is the statement "If $S_2$ is true then $S_1$ is true.",
- the inverse of $T$ is the statement "If $S_1$ is false then $S_2$ is false." and
- the contrapositive of $T$ is the statement "If $S_2$ is false then $S_1$ is false."

If the statement $T$ is true, then

- the converse of $T$ need not be true,
- the inverse of $T$ need not be true and
- the contrapositive of $T$ is necessarily true.

**Example 4** Let $S_1$ be the statement "$x$ is a rational number" and $S_2$ be the statement "$x$ is a real number". Then

- $T$ is the statement "If $x$ is a rational number then $x$ is a real number." and is true,
- the converse of $T$ is the statement "If $x$ is a real number then $x$ is a rational number." and is false,
- the inverse of $T$ is the statement "If $x$ is not a rational number then $x$ is not a real number." and is false, and
- the contrapositive of $T$ is the statement "If $x$ is not a real number then $x$ is not a rational number." and is true.