Logistic Growth

Logistic growth is a simple model for predicting the size $P(t)$ of a population as a function of the time t.

In the most naive model of population growth, each couple produces β offspring (for some constant β) and then dies. Thus over the course of one generation $\beta\frac{P(t)}{2}$ children are produced and $P(t)$ parents die so that the size of the population grows from $P(t)$ to $P(t+t_g) = P(t) + \beta\frac{P(t)}{2} - P(t) = \frac{\beta}{2}P(t)$, where t_g denotes the lifespan of one generation. The rate of change of the size of the population per unit time is $\frac{P(t+t_g)-P(t)}{t_g} = bP(t)$ where $b = \frac{\beta-2}{2t_g}$ is the net birthrate per member of the population per unit time. If we approximate $\frac{P(t+t_g)-P(t)}{t_g} \approx \frac{dP}{dt}(t)$ we get the differential equation

$$P'(t) = bP(t)$$

Logistic growth adds one more wrinkle to this model. It assumes that the population only has access to limited resources. As the size of the population grows the amount of food available to each member decreases. This in turn causes the net birth rate b to decrease. In the logistic growth model $b = b_0\left(1 - \frac{P}{K}\right)$, where K is called the carrying capacity of the environment, so that

$$P'(t) = b_0\left(1 - \frac{P(t)}{K}\right)P(t)$$

We can learn quite a bit about the behaviour of solutions to differential equations like this, without ever finding formulae for the solutions, just by watching the sign of $P'(t)$. For concreteness we'll look at solutions of the differential equation

$$\frac{dP}{dt}(t) = \big(6000 - 3P(t)\big)P(t)$$

We'll sketch the graphs of four functions $P(t)$ that obey this equation.

- For the first function, $P(0) = 0$.
- For the second function, $P(0) = 1000$.
- For the third function, $P(0) = 2000$.
- For the fourth function, $P(0) = 3000$.

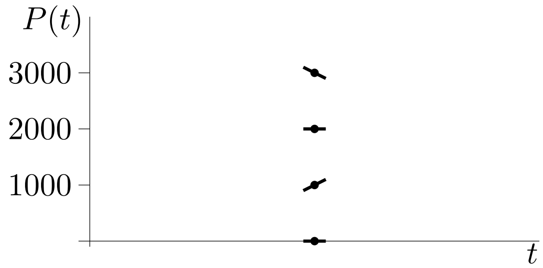
The sketchs will be based on the observation that

$$\frac{dP}{dt}(t)\begin{cases} = 0 & \text{if } P(t) = 0 \\ > 0 & \text{if } 0 < P(t) < 2000 \\ = 0 & \text{if } P(t) = 2000 \\ < 0 & \text{if } P(t) > 2000 \end{cases}$$

Thus if $P(t)$ is some function that obeys $\frac{dP}{dt}(t) = \big(6000 - 3P(t)\big)P(t)$, then as the graph of $P(t)$ passes through $\big(t, P(t)\big)$

$$\text{the graph has} \begin{cases} \text{slope zero, i.e. is horizontal, if } P(t) = 0 \\ \text{positive slope , i.e. is increasing, if } 0 < P(t) < 2000 \\ \text{slope zero, i.e. is horizontal, if } P(t) = 2000 \\ \text{negative slope , i.e. is decreasing, if } 0 < P(t) < 2000 \end{cases}$$

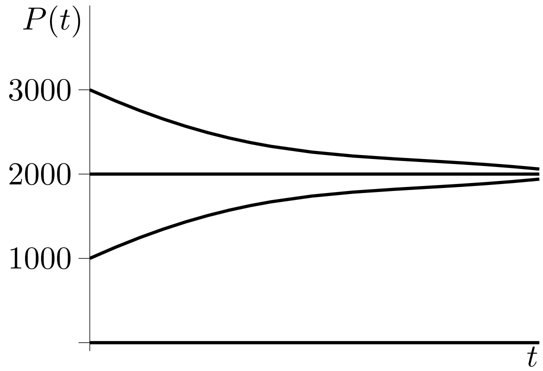
as illustrated in the figure



So

- If $P(0) = 0$, the graph starts out horizontally. In other words, as t starts to increase, $P(t)$ remains at zero, so the slope of the graph remains at zero. The population size remains zero for all time. As a check, observe that the function $P(t) = 0$ obeys $\frac{dP}{dt}(t) = \big(6000 - 3P(t)\big)P(t)$ for all t.
- Similarly, if $P(0) = 2000$, the graph again starts out horizontally. So $P(t)$ remains at 2000 and the slope remains at zero. The population size remains 2000 for all time. Again, the function $P(t) = 2000$ obeys $\frac{dP}{dt}(t) = \big(6000 - 3P(t)\big)P(t)$ for all t.
- If $P(0) = 1000$, the graph starts out with positive slope. So $P(t)$ increases with t. As $P(t)$ increases towards 2000, the slope $(6000 - 3P(t)\big)P(t)$, while remaining positive, gets closer and closer to zero. As the graph approachs height 2000, it becomes more and more horizontal. The graph cannot actually cross from below 2000 to above 2000, because to do so, it would have to have positive slope for some value of P above 2000. which is not allowed.
- If $P(0) = 3000$, the graph starts out with negative slope. So $P(t)$ decreases with t. As $P(t)$ decreases towards 2000, the slope $(6000 - 3P(t)\big)P(t)$, while remaining negative, gets closer and closer to zero. As the graph approachs height 2000, it becomes more and more horizontal. The graph cannot actually cross from above 2000 to below 2000, because to do so, it would have to have negative slope for some value of P below 2000. which is not allowed.

These curves are sketched in the figure below. We conclude that for any initial population size $P(0)$, except $P(0) = 0$, the population size approachs 2000 as $t \to \infty$.


P(t)
3000
2000
1000
t