Error Formulae for Taylor Polynomial Approximations

Let

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \tfrac{1}{n!}f^{(n)}(x_0)(x - x_0)^n$$

be the Taylor polynomial of degree n for the function $f(x)$ and expansion point x_0. Using this polynomial to approximate $f(x)$ introduces an error

$$E_n(x) = f(x) - P_n(x)$$

We shall now prove that

$$E_n(x) = \tfrac{1}{(n+1)!}f^{(n+1)}(c)(x - x_0)^{n+1} \qquad (1_\mathrm{n})$$

for some c between x_0 and x and that

$$E_n(x) = \tfrac{1}{n!}\int_{x_0}^{x}(x - t)^n f^{(n+1)}(t)\ dt \qquad (2_\mathrm{n})$$

Neither (2_n) nor the proofs are part of the official course. It rarely necessary, or even possible, to evaluate $E_n(x)$ exactly. It is usually sufficient to find a number M such that $\left|f^{(n+1)}(c)\right| \le M$ for all c between x_0 and the x of interest. Both (1_n) and (2_n) then imply that $|E_n(x)| \le \frac{1}{(n+1)!}M|x - x_0|^{n+1}$.

Both (1_n) and (2_n) are easily proven in the special case $n = 0$. When $n = 0$, (1_n) and (2_n) are the statements that

$$f(x) - f(x_0) = f'(c)(x - x_0) \qquad (1_0)$$

for some c between x_0 and x and that

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(t)\ dt \qquad (2_0)$$

So (1_0) is just a restatement of the mean–value theorem and (2_0) is just a restatement of part of the fundamental theorem of calculus.

To prove (1_n) with $n \ge 1$, we need the following small generalization of the mean–value theorem.

Theorem (Generalized Mean–Value Theorem) *Let the functions $F(x)$ and $G(x)$ both be defined and continuous on $a \le x \le b$ and both be differentiable on $a < x < b$. Furthermore, suppose that $G'(x) \ne 0$ for all $a < x < b$. Then, there is a number c obeying $a < c < b$ such that*

$$\tfrac{F(b)-F(a)}{G(b)-G(a)} = \tfrac{F'(c)}{G'(c)}$$

Proof: Define

$$h(x) = \big[F(b) - F(a)\big]\big[G(x) - G(a)\big] - \big[F(x) - F(a)\big]\big[G(b) - G(a)\big]$$

Observe that $h(a) = h(b) = 0$. So, by the mean–value theorem, there is a number c obeying $a < c < b$ such that

$$0 = h'(c) = \big[F(b) - F(a)\big]G'(c) - F'(c)\big[G(b) - G(a)\big]$$

As $G(a) \neq G(b)$ (otherwise the mean–value theorem would imply the existence of an $a < x < b$ obeying $G'(x) = 0$), we may divide by $G'(c)\big[G(b) - G(a)\big]$ which gives the desired result. ■

Proof of ($\mathbf{1_n}$): To prove (1_1), that is (1_n) for $n = 1$, simply apply the generalized mean–value theorem with $F(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$, $G(x) = (x - x_0)^2$, $a = x_0$ and $b = x$. Then $F(a) = G(a) = 0$, so that

$$\tfrac{F(b)}{G(b)} = \tfrac{F'(\tilde{c})}{G'(\tilde{c})} \;\Rightarrow\; \tfrac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2} = \tfrac{f'(\tilde{c})-f'(x_0)}{2(\tilde{c}-x_0)}$$

for some $\tilde{c}$ between x_0 and x. By the mean–value theorem (the standard one, but with $f(x)$ replaced by $f'(x)$), $\frac{f'(\tilde{c})-f'(x_0)}{\tilde{c}-x_0} = f''(c)$, for some t between x_0 and $\tilde{c}$ (which forces $\tilde{c}$ to also be between x_0 and x). Hence

$$\tfrac{f(x)-f(x_0)-f'(x_0)(x-x_0)}{(x-x_0)^2} = \tfrac{1}{2}f''(c)$$

which is exactly (1_1).

At this stage, we know that (1_n) applies to all (sufficiently differentiable) functions for $n = 0$ and $n = 1$. To prove it for general n, we proceed by induction. That is, we assume that we already know that (1_n) applies to $n = k - 1$ for some k (as is the case for $k = 1, 2$) and that we wish to prove that it also applies to $n = k$. We apply the generalized mean–value theorem with $F(x) = E_k(x)$, $G(x) = (x - x_0)^{k+1}$, $a = x_0$ and $b = x$. Then $F(a) = G(a) = 0$, so that

$$\tfrac{F(b)}{G(b)} = \tfrac{F'(\tilde{c})}{G'(\tilde{c})} \;\Rightarrow\; \tfrac{E_k(x)}{(x-x_0)^{k+1}} = \tfrac{E_k'(\tilde{c})}{(k+1)(\tilde{c}-x_0)^k}$$

But

$$\begin{aligned} E_k'(\tilde{c}) &= \tfrac{d}{dx}\Big[f(x) - f(x_0) - f'(x_0) - \cdots - \tfrac{1}{k!}f^{(k)}(x_0)(x - x_0)^k\Big]_{x=\tilde{c}} \\ &= \Big[f'(x) - f'(x_0) - \cdots - \tfrac{1}{(k-1)!}f^{(k)}(x_0)(x - x_0)^{k-1}\Big]_{x=\tilde{c}} \\ &= f'(\tilde{c}) - f'(x_0) - \cdots - \tfrac{1}{(k-1)!}f^{(k)}(x_0)(\tilde{c} - x_0)^{k-1} \end{aligned}$$

The last expression is exactly the definition of $E_{k-1}(\tilde{c})$, but for the function $f'(x)$, instead of the function $f(x)$. But we already know that (1_{k-1}) is true, so we already know that the last expression equals

$$E_k'(\tilde{c}) = E_{k-1}(\tilde{c}) = \tfrac{1}{(k-1+1)!}\big(f'\big)^{(k-1+1)}(c)(\tilde{c}-x_0)^{k-1+1} = \tfrac{1}{k!}f^{(k+1)}(c)(\tilde{c}-x_0)^k$$

for some c between x_0 and $\tilde{c}$. Subbing this in

$$\tfrac{E_k(x)}{(x-x_0)^{k+1}} = \tfrac{E_k'(\tilde{c})}{(k+1)(\tilde{c}-x_0)^k} = \tfrac{1}{(k+1)!}f^{(k+1)}(c)$$

which is exactly (1_k). Repeating this for $k = 2, 3, 4, \cdots$ gives (1_k) for all k. ■

Proof of $(\mathbf{2_n})$: We again proceed by induction. That is, we assume that we already know that (2_n) applies to $n = k-1$ for some k (as is the case for $k=1$) and we then prove that it also applies to $n = k$. So we are assuming that

$$E_{k-1}(x) = \tfrac{1}{(k-1)!}\int_{x_0}^{x}(x-t)^{k-1}f^{(k)}(t)\ dt$$

Integrate by parts with $u(t) = f^{(k)}(t)$ and $v'(t)\,dt = \frac{(x-t)^{k-1}}{(k-1)!}\,dt$. Note that t is now the integration variable and x is just some constant. So $u'(t)\,dt = f^{(k+1)}(t)\,dt$ and we may take $v(t) = -\frac{1}{k!}(x-t)^k$. This gives

$$\begin{aligned} E_{k-1}(x) &= -\tfrac{1}{k!}(x-t)^k f^{(k)}(t)\Big|_{t=x_0}^{t=x} + \tfrac{1}{k!}\int_{x_0}^{x}(x-t)^k f^{(k+1)}(t)\ dt \\ &= \tfrac{1}{k!}(x-x_0)^k f^{(k)}(x_0) + \tfrac{1}{k!}\int_{x_0}^{x}(x-t)^k f^{(k+1)}(t)\ dt \end{aligned}$$

Since

$$\begin{aligned} E_k(x) &= f(x) - P_k(x) = f(x) - P_{k-1}(x) - \tfrac{1}{k!}(x-x_0)^k f^{(k)}(x_0) \\ &= E_{k-1}(x) - \tfrac{1}{k!}(x-x_0)^k f^{(k)}(x_0) \end{aligned}$$

we have

$$E_k(x) = \tfrac{1}{k!}\int_{x_0}^{x}(x-z)^k f^{(k+1)}(t)\ dt$$

which is exactly (2_k). Repeating this for $k = 2, 3, 4, \cdots$ gives (2_k) for all k. ■