# Evaluating Limits Using Taylor Expansions

Taylor polynomials provide a good way to understand the behaviour of a function near a specified point and so are useful for evaluating complicated limits. We'll see examples of this later in these notes.

We'll just start by recalling that if, for some natural number $n$, the function $f(x)$ has $n+1$ derivatives near the point $x_0$, then

$$f(x) = f(x_0) + f'(x_0)\,(x - x_0) + \cdots + \tfrac{1}{n!} f^{(n)}(x_0)\,(x - x_0)^n + E_n(x)$$

where

$$P_n(x) = f(x_0) + f'(x_0)\,(x - x_0) + \cdots + \tfrac{1}{n!} f^{(n)}(x_0)\,(x - x_0)^n$$

is the Taylor polynomial of degree $n$ for the function $f(x)$ and expansion point $x_0$ and

$$E_n(x) = f(x) - P_n(x) = \tfrac{1}{(n+1)!} f^{(n+1)}(c)\,(x - x_0)^{n+1}$$

is the error introduced when we approximate $f(x)$ by the polynomial $P_n(x)$. Here $c$ is some unknown number between $x_0$ and $x$. As $c$ is not known, we do not know exactly what the error $E_n(x)$ is. But that is usually not a problem. In taking the limit $x \to x_0$, we are only interested in $x$'s that are very close to $x_0$, and when $x$ is very close $x_0$, $c$ must also be very close to $x_0$. As long as $f^{(n+1)}(x)$ is continuous at $x_0$, $f^{(n+1)}(c)$ must approach $f^{(n)}(x_0)$ as $x \to x_0$. In particular there must be constants $M, D > 0$ such that $\big|f^{(n+1)}(c)\big| \le M$ for all $c$'s within a distance $D$ of $x_0$. If so, there is another constant $C$ (namely $\frac{M}{(n+1)!}$) such that

$$\big|E_n(x)\big| \le C|x - x_0|^{n+1} \qquad \text{whenever } |x - x_0| \le D$$

There is some notation for this behaviour.

**Definition 1** **(Big O)** We say "$F(x)$ is of order $|x - x_0|^m$ near $x_0$" and we write $F(x) = O\big(|x - x_0|^m\big)$ if there exist constants $C, D > 0$ such that

$$\big|F(x)\big| \le C|x - x_0|^m \qquad \text{whenever } |x - x_0| \le D \tag{1}$$

Whenever $O\big(|x - x_0|^m\big)$ appears within an algebraic expression, it just stands for some (unknown) function $F(x)$ that obeys (1). This is called "big O" notation. Here are some examples.

**Example 2** Let $f(x) = \sin x$ and $x_0 = 0$. Then

$$\begin{array}{llllll} f(x) = \sin x & f'(x) = \cos x & f''(x) = -\sin x & f^{(3)}(x) = -\cos x & f^{(4)}(x) = \sin x & \cdots \\ f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f^{(3)}(0) = -1 & f^{(4)}(0) = 0 & \cdots \end{array}$$

and the pattern repeats. Thus $\left|f^{(n+1)}(c)\right| \le 1$ for all $c$. So the Taylor polynomial of, for example, degree 4 and its error term are

$$\begin{aligned} \sin x &= x - \tfrac{1}{3!}x^3 + \tfrac{\cos c}{5!}x^5 \\ &= x - \tfrac{1}{3!}x^3 + O(|x|^5) \end{aligned}$$

under Definition 1, with $C = \frac{1}{5!}$ and any $D > 0$. Similarly, for any natural number $n$,

$$\begin{aligned} \sin x &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots + (-1)^n \tfrac{1}{(2n+1)!}x^{2n+1} + O\big(|x|^{2n+3}\big) \\ \cos x &= 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots + (-1)^n \tfrac{1}{(2n)!}x^{2n} + O\big(|x|^{2n+2}\big) \end{aligned}$$

**Example 3** Let $n$ be any natural number. We have seen that, since $\frac{d^m}{dx^m}e^x = e^x$ for every integer $m \ge 0$,

$$e^x = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots + \tfrac{x^n}{n!} + \tfrac{e^c}{(n+1)!}x^{n+1}$$

for some $c$ between 0 and $x$. If, for example, $|x| \le 1$, then $|e^c| \le e$, so that the error term

$$\left|\tfrac{e^c}{(n+1)!}x^{n+1}\right| \le C|x|^{n+1} \qquad \text{with } C = \tfrac{e}{(n+1)!} \qquad \text{whenever } |x| \le 1$$

So, under Definition 1, with $C = \frac{e}{(n+1)!}$ and $D = 1$,

$$e^x = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots + \tfrac{x^n}{n!} + O\big(|x|^{n+1}\big)$$

**Example 4** Let $f(x) = \ln(1+x)$ and $x_0 = 0$. Then

$$\begin{array}{lllll} f'(x) = \frac{1}{1+x} & f''(x) = -\frac{1}{(1+x)^2} & f^{(3)}(x) = \frac{2}{(1+x)^3} & f^{(4)}(x) = -\frac{2\times 3}{(1+x)^4} & f^{(5)}(x) = \frac{2\times 3\times 4}{(1+x)^5} \\ f'(0) = 1 & f''(0) = -1 & f^{(3)}(0) = 2 & f^{(4)}(0) = -3! & f^{(5)}(0) = 4! \end{array}$$

For any natural number $n$,

$$f^{(n)}(x) = (-1)^{n-1}\tfrac{(n-1)!}{(1+x)^n} \qquad \tfrac{1}{n!}f^{(n)}(0)\,x^n = (-1)^{n-1}\tfrac{(n-1)!}{n!}x^n = (-1)^{n-1}\tfrac{x^n}{n}$$

so

$$\ln(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \cdots + (-1)^{n-1}\tfrac{x^n}{n} + E_n(x)$$

with

$$E_n(x) = \tfrac{1}{(n+1)!} f^{(n+1)}(c)\,(x-x_0)^{n+1} = (-1)^n \tfrac{1}{(n+1)(1+c)^{n+1}} x^{n+1}$$

If we choose, for example $D = \frac{1}{2}$, then for any $x$ obeying $|x| \le \frac{1}{2}$, we have $|c| \le \frac{1}{2}$ and $|1+c| \ge \frac{1}{2}$ so that

$$|E_n(x)| \le \tfrac{1}{(n+1)(1/2)^{n+1}} |x|^{n+1} = O\big(|x|^{n+1}\big)$$

under Definition 1, with $C = \frac{2^{n+1}}{n+1}$ and $D = 1$. Thus we may write

$$\ln(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \cdots + (-1)^{n-1}\tfrac{x^n}{n} + O\big(|x|^{n+1}\big) \tag{2}$$

**Example 5** In this example we'll use the Taylor polynomial of Example 4 to evaluate $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x}$ and $\lim\limits_{x\to 0}(1+x)^{a/x}$. The Taylor expansion (2) with $n = 1$ tells us that

$$\ln(1+x) = x + O(|x|^2)$$

That is, for small $x$, $\ln(1+x)$ is the same as $x$, up to an error that is bounded by some constant times $x^2$. So, dividing by $x$, $\frac{1}{x}\ln(1+x)$ is the same as 1, up to an error that is bounded by some constant times $|x|$. That is

$$\tfrac{1}{x}\ln(1+x) = 1 + O(|x|)$$

But any function that is bounded by some constant times $|x|$, for all $x$ smaller than some constant $D > 0$, necessarily tends to 0 as $x \to 0$. Thus

$$\lim_{x\to 0} \tfrac{\ln(1+x)}{x} = \lim_{x\to 0} \tfrac{x+O(|x|^2)}{x} = \lim_{x\to 0}\big[1 + O(|x|)\big] = 1$$

and

$$\lim_{x\to 0}(1+x)^{a/x} = \lim_{x\to 0} e^{\frac{a}{x}\ln(1+x)} = \lim_{x\to 0} e^{\frac{a}{x}[x+O(|x|^2)]} = \lim_{x\to 0} e^{a+O(|x|)} = e^a$$

Here we have used if $F(x) = O(|x|^2)$, that is if $|F(x)| \le C|x|^2$ for some constant $C$, then $\big|\frac{a}{x}F(x)\big| \le C'|x|$ for the new constant $C' = |a|C$, so that $F(x) = O(|x|)$.

**Remark 6** The big O notation has a few properties that are useful in computations and taking limits. All follow immediately from Definition 1.

(1) If $p > 0$, then $\lim\limits_{x\to 0} O(|x|^p) = 0$.

(2) For any real numbers $p$ and $q$, $O(|x|^p)O(|x|^q) = O(|x|^{p+q})$.
(This is just because $C|x|^p \times C'|x|^q = (CC')|x|^{p+q}$.)
In particular, $ax^m\, O(|x|^p) = O(|x|^{p+m})$, for any constant $a$ and any integer $m$.

(3) For any real numbers $p$ and $q$, $O(|x|^p) + O(|x|^q) = O(|x|^{\min\{p,q\}})$.
(For example, if $p = 2$ and $q = 5$, then $C|x|^2 + C'|x|^5 = \big(C + C'|x|^3\big)|x|^2 \le (C+C')|x|^2$ whenever $|x| \le 1$.)

(4) For any real numbers $p$ and $q$ with $p > q$, any function which is $O(|x|^p)$ is also $O(|x|^q)$ because $C|x|^p = C|x|^{p-q}|x|^q \le C|x|^q$ whenever $|x| \le 1$.

 

**Example 7** In this example we'll evaluate the harder limit

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x\sin x}{[\ln(1+x)]^4}$$

Using Examples 2 and 4,

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x\sin x}{[\ln(1+x)]^4} &= \lim_{x\to 0} \frac{\big[1-\frac{1}{2}x^2+\frac{1}{4!}x^4+O(x^6)\big]-1+\frac{1}{2}x\big[x-\frac{1}{3!}x^3+O(|x|^5)\big]}{[x+O(x^2)]^4} \\
&= \lim_{x\to 0} \frac{(\frac{1}{4!}-\frac{1}{2\times 3!})x^4+O(x^6)+\frac{x}{2}\,O(|x|^5)}{[x+O(x^2)]^4} \\
&= \lim_{x\to 0} \frac{(\frac{1}{4!}-\frac{1}{2\times 3!})x^4+O(x^6)+O(x^6)}{[x+O(x^2)]^4} \qquad \text{by Remark 6, part (2)} \\
&= \lim_{x\to 0} \frac{(\frac{1}{4!}-\frac{1}{2\times 3!})x^4+O(x^6)}{[x+x\,O(|x|)]^4} \qquad \text{by Remark 6, parts (2), (3)} \\
&= \lim_{x\to 0} \frac{(\frac{1}{4!}-\frac{1}{2\times 3!})x^4+x^4O(x^2)}{x^4[1+O(|x|)]^4} \qquad \text{by Remark 6, part (2)} \\
&= \lim_{x\to 0} \frac{(\frac{1}{4!}-\frac{1}{2\times 3!})+O(x^2)}{[1+O(|x|)]^4} \\
&= \frac{1}{4!}-\frac{1}{2\times 3!} \qquad \text{by Remark 6, part (1)} \\
&= \frac{1}{3!}\Big(\frac{1}{4}-\frac{1}{2}\Big) = -\frac{1}{4!}
\end{aligned}$$